PROJECT DESCRIPTION – PROJECT PROPOSALS – ANR DFG 2024 NLE

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AKE PACT

The AKE Philosophy: from Arithmetic Geometry to Classification Theory

Project description.

1 Starting point. State of the art and preliminary work. The model theory of valued fields started with A. Robinson's work on algebraically closed valued fields (ACVF) in the 1950's [Rob56]. The celebrated work of Ax and Kochen [AK65] and Ershov [Erš65] on the model theory of henselian valued fields in the 1960's led to fruitful interactions between model theory and number theory and was used to lay the foundations for motivic integration. Since then, model-theoretic techniques have proved to be at the heart of solutions to many central problems of arithmetic geometry, including transfer theorems between fields of different characteristic, Hilbert's 10th problem and Hrushovski's proof of the Mordell-Lang conjecture for function fields of positive characteristic [Hru96].

The Ax–Kochen–Ershov (AKE) theorem asserts that two unramified henselian valued fields with perfect residue field are elementarily equivalent in the (first-order) language of valued fields $\mathcal{L}_{\mathrm{val}} = \mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ if and only if their residue fields and their value groups are elementarily equivalent (respectively, in the language of rings $\mathcal{L}_{\mathrm{ring}} = \{0, 1, +, \cdot\}$ and the language of ordered abelian groups $\mathcal{L}_{\mathrm{oag}} = \{0, +, <\}$). Here, two $\mathcal{L}$-structures are *elementarily equivalent* if the same $\mathcal{L}$-theorems hold in both, i.e., that the structures cannot be distinguished by their first-order properties. What can be phrased as an $\mathcal{L}$-theorem of course very much depends on the language. The AKE theorem allows one to prove *transfer* results for completeness and decidability: for a theory of unramified henselian valued fields with perfect residue field, each of these properties is equivalent to the corresponding one holding for both the theories of residue field and value group. In particular, Ax, Kochen, and Ershov obtained that a field K is $\mathcal{L}_{\mathrm{ring}}$-elementarily equivalent to the field of p-adic numbers $\mathbb{Q}_p$ if and only if it is p-adically closed, i.e., $\mathrm{char}(K) = 0$ and K admits a henselian valuation v with residue field $\mathbb{F}_p$ and value group $\mathcal{L}_{\mathrm{oag}}$-elementarily equivalent to $\mathbb{Z}$ with $v(p)$ minimum positive. Morally speaking, whenever the model theory of a henselian valued field is well understood, this is due to an **AKE-type transfer theorem**.

The proof of the AKE theorem goes via the coarsening technique: Every unramified henselian valued field can be decomposed into a henselian valuation of equicharacteristic 0 and a mixed characteristic henselian valuation taking values in $\mathbb{Z}$. As a stepping stone, they prove a version of their theorem for henselian fields of equicharacteristic 0. By taking ultraproducts (where p varies), this then allows the transfer of results between henselian valued fields of mixed characteristic and positive characteristic (for $p \gg 0$), e.g., $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$. However, despite much effort over the past 60 years, very little is known about the theory of $\mathbb{F}_p((t))$, for fixed p.

Very recently, Anscombe and Jahnke generalized the classical AKE result to unramified mixed characteristic henselian valued fields with imperfect residue field [AJ22]. This result was not expected by the community (cf the discussions in [Dri14], pp. 144 and 153). The key difference to the setting with perfect residue field is that the Witt ring $W(k)$ over an imperfect field k is not necessarily a valuation ring. We use Cohen rings instead of Witt rings, for which the algebra (and hence the model theory) is considerably more complicated.

This is one instance of a variety of AKE-type transfer theorems proven since Ax, Kochen and Ershov: those exist for (separably) algebraically maximal Kaplansky valued fields [Del82]; for tame and separably tame valued fields [Kuh16; KP16]; and for many key cases of henselian valued fields enriched with further structure [Sca00; BMS07; Rid17]. An obstacle to proving AKE-type theorems is defect: this is a measure of the existence of proper immediate finite extensions, i.e. finite extensions of valued fields in which the residue field and value group do not grow. If a valued field admits proper immediate finite extensions, reducing its theory to the theories of its residue field and value group is not feasible. A prominent case where defect may occur are perfectoid fields, introduced by Scholze [Sch12]. Here, the technique of tilting is a powerful tool that gives different type of transfer of results between the mixed characteristic and the positive characteristic setting. However, even without an AKE-type theorem, Kartas [Kar21] was able to prove that for certain perfectoid fields, decidability is preserved by the tilting correspondence.

In joint work of Jahnke and Kartas [JK23], for perfectoid fields and, more generally, many deeply ramified fields an AKE principle down to value group and an infinitesimal thickening of the residue field is proven. This approach builds on Kuhlmann's AKE principles for tame valued fields [Kuh16]. Jahnke and Kartas prove that tilting preserves elementary equivalence, decidability, and existential decidability (as well as suitable versions for untilting). A striking corollary is that the embedding of $((\mathbb{F}_p(t)^{\mathrm{hens}})^{\mathrm{perf}}, v_t)$ into $(\mathbb{F}_p((t))^{\mathrm{perf}}, v_t)$ is elementary. This is a "perfected" version of a prominent open question in the **model theory of valued fields of positive characteristic**, namely whether or not the same holds for the extension $(\mathbb{F}_p(t)^{\mathrm{hens}}, v_t) \subseteq (\mathbb{F}_p((t)), v_t)$.

Henselian valuations, especially those with complete discrete valuation rings, are fundamental in number theory. From a number-theoretic perspective, henselian valued fields of equicharacteristic p are usually easier to study than their mixed characteristic analogues, one reason being that the residue field embeds as a subfield (as long as it is separably generated over its prime field, e.g., any finitely generated residue field). This imbalance between the characteristics is one of the key features which makes the technique of tilting important in arithmetic geometry. Tilting allows one to reduce questions about henselian valued fields of mixed characteristic to positive characteristic, for example in the calculation of cohomological dimension. From the model-theoretic point of view, the situation is somewhat opposite: the theory of $\mathbb{Q}_p$ is decidable, while that of $\mathbb{F}_p((t))$ remains out of reach. To date, the part of $\mathrm{Th}(\mathbb{F}_p((t)))$ which we understand best is its existential fragment (also known as Hilbert's 10th problem for $\mathbb{F}_p((t))$).

In its original form, **Hilbert's 10th problem** asked for an algorithm to determine when a polynomial with integer coefficients has integer solutions. In [Mat70], building on work of Davis, Putnam, and J. Robinson, Matiyasevich gave a negative solution, i.e. he proved that no such algorithm exists. Immediately, the analogous question arises over other rings and fields, which has been the subject of much research ever since. The most famous open instance is for $\mathbb{Q}$, which would be solved by finding a diophantine definition of $\mathbb{Z}$ in $\mathbb{Q}$. Strikingly, by work of Koenigsmann (building on [Poo09]), the complement of $\mathbb{Z}$ is diophantine in $\mathbb{Q}$ [Koe16]. Definitive results exist for many other notable fields, including global fields of positive characteristic: for $\mathbb{R}(t)$ and $\mathbb{F}_q(t)$, there is no algorithm (for polynomials with coefficients in $\mathbb{Q}[t]$ and $\mathbb{F}_q[t]$ respectively) [Den78; Phe91], and for finite extensions of $\mathbb{F}_p(t)$, the same holds [Shl92; Eis03]. Turning to henselian valued fields, it is a consequence of the classical AKE theorem that in equicharacteristic 0 (and similarly in unramified henselian fields with perfect residue field) such an algorithm exists if and only if one exists for the residue field. More generally, it follows from [AJ22] that the same holds for all unramified henselian valued fields. This fails in the finitely ramified case [Dit23]; however, by identifying the precise structure induced on the residue field, Anscombe, Dittmann, and Jahnke prove that such an algorithm exists if and only if one exists for the residue field with this induced structure [ADJ23]. In equal positive characteristic, it was shown by Denef and Schoutens in [DS03] that, by assuming the conjectural resolution of singularities in positive charactersitic, there is such an algorithm for $\mathbb{F}_p((t))$ for polynomials with coefficients in $\mathbb{F}_p[t]$. A refinement of this was shown recently by Anscombe, Dittmann, and Fehm, in [ADF23]: such an algorithm exists conditionally on a certain consequence of Local Uniformization (itself a consequence of resolution of singularities) in positive characteristic. For any henselian valued field of positive characteristic, but for polynomials with coefficients in the prime field, the existence of an algorithm again reduces to the residue field, as shown by Anscombe and Fehm [AF16], building on Kuhlmann's work on tame valued fields [Kuh16]. In particular the existential theory of $\mathbb{F}_p((t))$ in $\mathcal{L}_{\mathrm{ring}}$ is decidable unconditionally.

A different area of research following the AKE philosophy is **motivic integration**, first introduced in a lecture by Kontsevich in 1995. Roughly speaking, it generalizes p-adic integration to henselian settings where the residue field need not be finite and might not even admit a translation-invariant measure. This allows both for a uniform approach to p-adic integration via ultraproducts, as well as to consider integrals in $\mathbb{C}((t))$ and other henselian valued fields. The model-theoretic approach to motivic integration, first developed by Denef and Loeser [DL01] and later refined by Cluckers and Loeser [CL08; CL10; CL15], gives rise to the transfer of integral equations between different fields and characteristics. It is an AKE-type machinery going beyond the transfer of first-order properties. Perhaps the most spectacular application of motivic integration is the transfer of the Fundamental Lemma of the Langlands programme [CHL11] from positive to mixed characteristic (for $p \gg 0$). Depending on the precise setup, motivic integration works in different classes of fields. The case of complete discrete valued fields of mixed characteristic with perfect residue field is also treated in [LS03; Seb04; CL11]. The Hrushovski-Kazhdan approach [HK06] is designed for algebraically closed fields of equicharacteristic 0 but specializes to the p-adic setting [CH21b].

For a long time, the model theory of henselian valued fields was considered solely as an area of "applied" model theory, and was rather disjoint from the more combinatorial aspect of "pure" model theory, at the heart of which is classification theory. In the past few years, the model theory of henselian valued fields has found applications within classification theory, most strikingly in Johnson's characterizations of dp-minimal and dp-finite fields [Joh23; Joh20c].

Classification theory provides a structural framework for first-order theories which do not allow the encoding of certain combinatorial patterns in their formulae (e.g., stable theories, simple theories, NIP theories; [She78]). This framework is often of a geometric nature, generalizing ideas of dimensions and connected components, which is encapsulated in the slogan *model theory is algebraic geometry minus fields* [Hod97, p. vii]. The question of whether these purely combinatorial conditions correspond to well-known algebraic properties when applied to groups and fields has been a leitmotif of model-theoretic research over the past 50 years. For groups, the central question is the long-standing *Algebraicity Conjecture* ("every simple group of finite Morley rank is an algebraic group", [Zil77; Che79]). For fields, the case of finite Morley rank (and even ω-stability) was settled by Macintyre [Mac71]: all infinite such fields are algebraically closed and generalized to infinite superstable fields by Cherlin and Shelah [CS80]. However, despite much effort over the past decades, the *Stable Fields Conjecture* ("every infinite stable field is separably closed") remains open.

Many of the crucial methods of stability theory were extended to simple and to NIP theories. On the one hand, there is the *Simple Fields Conjecture* ("every infinite simple field is pseudo algebraically closed and bounded", [Cha99]) which by [Dur80] generalizes the Stable Fields Conjecture. On the other hand, the Stable Fields Conjecture was extended by Shelah to the realm of NIP theories [She14], which first appeared as a preprint in 2005. The **Shelah Conjecture** states that every infinite NIP field is either separably closed, real closed or admits a nontrivial henselian valuation — bringing henselian fields into the picture. Essentially, the Shelah Conjecture expresses the idea that "model-theoretically tame" fields are exactly those for which the applications of model theory have been most spectacular: algebraically closed fields (first studied by Tarski, [Tar48]), real closed fields (Tarski–Seidenberg, [Tar48; Sei54]) algebraically closed valued fields (A. Robinson, [Rob56]) and certain henselian valued fields (starting with Ax, Kochen, and Ershov, [AK65; Erš65]). By [AJ24], the Shelah Conjecture is in fact equivalent to a purely algebraic classification of NIP fields. Hence, a proof of the Shelah Conjecture would indeed show that NIP is not just a combinatorial dividing line but rather an entirely natural concept when applied to fields. Moreover, the Shelah Conjecture has several striking consequences: it implies both the Stable Fields Conjecture by [JK15b] as well as the **Henselianity Conjecture** ("every NIP valued field is henselian"), see [HHJ20].

To date, the Shelah Conjecture is known in several special cases [JSW17; Joh23; Joh20c], the most spectacular being Johnson's proof for dp-finite fields which spans (at least) 7 preprints and papers of more than 400 pages in total [Joh21a; Joh21b; Joh19a; Joh19b; Joh20a; Joh20b; Joh20c] and was the subject of a Bourbaki seminar given by Anscombe in 2021 [Ans22]. In his proof, Johnson developed two different topological machineries in order to find a V-topology on a sufficiently saturated infinite dp-finite field: he defined the canonical topology (a far-reaching adaptation of his approach in [Joh18]) as well as W_n-topologies, a generalization of V-topologies. Moreover, Johnson proved the Henselianity Conjecture not only for dp-finite fields [Joh20c], but for all NIP valued fields of positive characteristic [Joh21b]. By [HHJ19], Johnson's results give rise to an algebraic classification of all dp-finite fields.

In most cases, AKE-type theorems do not only allow the transfer of completeness and decidability but even of classification-theoretic properties. The first such result was shown by Delon [Del81]: the theory of a henselian valued field of equicharacteristic 0 is NIP (as an $\mathcal{L}_{\mathrm{val}}$-structure) if and only if that of its residue field is NIP (as an $\mathcal{L}_{\mathrm{ring}}$-structure). Here, the value group does not occur since the $\mathcal{L}_{\mathrm{oag}}$-theory of any ordered abelian group is NIP [GS84]. In all henselian valued fields for which an AKE-type theorem holds, **NIP transfer theorems** have been shown [Bél99; JS20; AJ22]. While the earliest of these theorems were shown using relative quantifier elimination, Jahnke and Simon employed a new technique [JS20], based on a strategy developed by Chernikov and Hils in the NTP_2 context [CH14]. Conversely, building on Kaplan, Scanlon, and Wagner [KSW11], Anscombe and Jahnke showed that henselian NIP fields satisfy an AKE-type transfer theorem, by classifying their $\mathcal{L}_{\mathrm{val}}$-theories modulo the $\mathcal{L}_{\mathrm{ring}}$-theories of their residue fields [AJ24].

A natural simultaneous generalization of the classes of simple and of NIP theories is that of *NTP_2 theories* [Che14]. A formula $\varphi(x, y)$ has TP_2 (*Tree Property of the second kind*) if there is an array of parameters $(a_{i,j})_{i,j<\omega}$ such that each row of formulae $(\varphi(x, a_{i,j}))_{j<\omega}$ is inconsistent, and, for every function $f : \omega \to \omega$, the "vertical path" $(\varphi(x, a_{i,f(i)}))_{i<\omega}$ is consistent. A theory is NTP_2 if no formula has TP_2 (in any model). Key examples of NTP_2 fields which are neither simple nor NIP include ultraproducts over the p-adic numbers (where p varies) [Che14], again established via an AKE-type transfer theorem. Dp-rank, a key notion of dimension in NIP theories, generalizes to burden in NTP_2-theories, with the one-dimensional case being **inp-minimality** (which specializes to dp-minimality in the NIP setting). In fact, there is also an AKE-type transfer theorem for inp-minimality [CS19], implying that ultraproducts of p-adic are in fact inp-minimal. Pseudo real closed (PRC) and pseudo p-adically closed fields (PpC) of finite burden are classified in [Mon17]. Although NTP_2 fields of positive characteristic may admit Artin-Schreier extensions, they only admit boundedly many such [CKS15]. Hence, NTP_2 valued fields of positive characteristic are deeply ramified and in particular only admit independent defect [Kuh22].

Introducing the *étale open topology* on the set of K-rational points of K-varieties, Johnson, Tran, Walsberg, and Ye [Joh+23] were recently able to prove that the Stable Fields Conjecture holds for large stable fields: large stable fields are separably closed. The étale open topology on $V(K)$ is defined by taking the K-points of images of étale maps as a basis, and it is nondiscrete on $\mathbb{A}_1(K)$ if and only K is large. On separably closed, real closed or p-adically closed fields, it coincides with the usual topology (namely the Zariski, resp. euclidean, resp. p-adic topology). Moreover, as long as the field is not separably closed, the étale open topology is a V-topology if and only if it is t-henselian (i.e., in a saturated model, there is a henselian valuation inducing the topology).

2 Objectives and work programme.

2.1 Anticipated total duration of the project. The project will run for three years.

2.2 Objectives. The aim of this proposal is to make significant advances in the model theory of henselian valued fields, resolving questions from both arithmetic geometry and classification theory. We expect that working on problems from both sides simultaneously will create synergies and lead to new interactions. Over the duration of the project, we have two main objectives in each of the areas of arithmetic geometry and classification theory, but there is ample scope for the research to broaden over time. Moreover, we have a fifth objective in which we apply our work to motivic integration. We now introduce our objectives and give motivations.

On the arithmetic-geometric side, we will address the following:

EQUI AKE principles in positive characteristic.

EQUIa Separably tameable fields of rank 1: give examples and establish AKE principles.

EQUIb Prove AKE principles for separably tameable fields of higher rank.

EQUIc Prove existential decidability of $\mathbb{F}_p((t))^{\mathrm{perf}}$.

Very few AKE principles are known for fields of positive characteristic. Kuhlmann proves such results for tame fields, i.e. algebraically maximal valued fields (K, v) of characteristic p with p-divisible value group and perfect residue field [Kuh16]. Tame fields are in particular perfect and admit no defect. More recently, Kuhlmann and Rzepka have begun a model-theoretic study of deeply ramified fields [KR23], which are perfect fields that may admit *independent defect*. The strategy of Jahnke–Kartas is to "tame" certain deeply ramified fields, including all perfectoid fields. More precisely, Jahnke and Kartas consider the class of henselian valued fields (K, v) of residue characteristic $p > 0$ together with a distinguished element $t \in \mathfrak{m}_v$ such that $\mathcal{O}_v/(p)$ is semi-perfect and $\mathcal{O}_v[t^{-1}]$ is algebraically maximal. This class turns out to be elementary, which then allows them to prove AKE-type theorems for this class, by extending the machinery of tame fields.

However, as soon as $\mathcal{O}_v/(p)$ is semi-perfect, the field K is necessarily perfect. Our first two subobjectives are to use separably tame fields instead of tame fields to obtain the first AKE-type theorems for imperfect fields which are not separably tame. A first step is to consider rank-1 valued fields (**EQUIa**) before considering the general case (**EQUIb**).

For any field k, the Artin-Schreier closure of $k((t))$ is an example of a rank-1 field which is not separably tame but which we expect to be "separably tameable", in analogy to [JK23]. This will yield AKE principles for the Artin-Schreier closures of $\mathbb{F}_p((t))$ and $\mathbb{F}_p^{\mathrm{alg}}((t))$. The conjunction of (**EQUIa**) and (**EQUIb**) is suitable as a **PhD project**.

The field $(\mathbb{F}_p((t))^{\mathrm{perf}}, v_t)$ fits into the framework of Jahnke–Kartas: it is deeply ramified, of rank 1, but not tame. Thus, its complete theory is Turing-reducible to that of $\mathbb{F}_p[t^{p^{-\infty}}]/(t)$. The existential theory of $\mathbb{F}_p[t^{p^{-\infty}}]/(t)$ is decidable. The aim of **EQUIc** is to combine these insights with new embedding lemmas to prove that Hilbert's 10th problem has a positive solution over $\mathbb{F}_p((t))^{\mathrm{perf}}$: there is an algorithm which decides whether polynomials with coefficients in $\mathbb{F}_p[t]$ have a solution. The key is that our strategy does not rely on resolution of singularities or local uniformization. By [Kar21], **EQUIc** implies decidability of the existential theories of $\mathbb{Q}_p(p^{p^{-\infty}})$ and of $\mathbb{Q}_p(\zeta_{p^\infty})$.

MIX Hilbert's Tenth Problem in mixed characteristic.

- **MIXa** Study H10 in compositions of finitely ramified and equicharacteristic p henselian valuations.
- **MIXb** Axiomatize existential theories of tame fields of mixed characteristic.
- **MIXc** Prove H10 relative to the residue field in mixed characteristic: the general case.

We plan to resolve Hilbert's 10th problem (H10) in mixed characteristic henselian valued fields, i.e., give a method to reduce the question of the existence of rational points on a variety to an analogous question over the residue field. Such methods are known to exist for henselian fields of equicharacteristic 0 (by [AK65]) and in positive characteristic for varieties defined over the prime field [AF16]. The key problem in mixed characteristic is that the parameter p is $\emptyset$-definable, and so (existential) decidability in $\mathcal{L}_{\mathrm{ring}}$ in mixed characteristic corresponds to (existential) decidability in positive characteristic with a parameter t (i.e., for varieties over $\mathbb{F}_p[t]$). For unramified henselian fields, H10 can be reduced to the residue field, by work of Anscombe and Jahnke [AJ22]. For finitely ramified henselian fields, an expansion of the residue field is necessary [Dit23]. Anscombe, Dittmann, and Jahnke prove a transfer of H10 for the expanded structure [ADJ23].

By the standard decomposition method (see **CORE**), a mixed characteristic henselian valuation decomposes into an equicharacteristic 0 valuation, a rank-1 valuation of mixed characteristic, and an equicharacteristic p component. Assuming sufficient saturation, the rank-1 part is either finitely ramified or defectless with value group $\mathbb{R}$. We treat these two cases separately. First, we will combine our previous results in mixed characteristic [AJ22; ADJ23] with results from positive characteristic [AF16] to obtain a relative H10 principle for henselian valued fields admitting a finitely ramified coarsening (**MIXa**). Secondly, for tame valued fields of mixed characteristic, we aim to reduce solvability of H10 down to Kv (**MIXb**). As a consequence, we expect a transfer of H10 down to $\mathcal{O}/p\mathcal{O}$, assuming $\mathcal{O}/p\mathcal{O}$ is semiperfect: under this assumption, the rank-1 part in the standard decomposition is tame or unramified. This task is already very ambitious and might require further assumptions. The ultimate task is to combine **MIXa** and **MIXb** (and new ideas developed in **EL**) to resolve H10 for henselian fields of mixed characteristic in full generality (**MIXc**).

Working towards a solution of the Shelah Conjecture, we will address the following problems:

$\mathbf{HC_0}$ Prove the Henselianity Conjecture in characteristic 0.

- **$\mathbf{HC_0a}$** Solve the Inverse Galois Problem over NIP fields: show that the absolute Galois group of any NIP field is pro-solvable.
- **$\mathbf{HC_0b}$** Prove the p-Henselianity Conjecture: every p-henselian NIP valued field is henselian.

The Henselianity Conjecture is a stepping stone for the Shelah Conjecture, but it seems much more within reach: It is known in positive characteristic, and it does not immediately imply the notoriously hard Stable Fields Conjecture. The two subobjectives are two independent approaches which we will take towards its solution.

In Johnson's proof of the Henselianity Conjecture in positive characteristic, the crucial ingredient is the result of Kaplan, Scanlon and Wagner that infinite NIP fields of positive characteristic are Artin-Schreier closed. A naïve attempt for an analogue of this result in characteristic 0 could be to aim to prove the same for Kummer extensions. However, NIP fields of characteristic 0 can have unboundedly many Kummer extensions, an example being the generalized power series field

$$K = \mathbb{C}((\bigoplus_{i\in\omega}\mathbb{Z}))$$

where $\bigoplus_{i\in\omega}\mathbb{Z}$ is ordered inverse lexicographically. Here, $K^\times/(K^\times)^p$ is infinite for all primes p, and thus K (which contains all roots of unity) has unboundedly many Kummer extensions for all p.

Nonetheless, the Shelah Conjecture has Galois-theoretic implications for NIP fields of characteristic 0: it implies that every NIP field has a pro-solvable absolute Galois group via its reformulation as a conjectural classification of NIP fields [AJ24]. Establishing **HC$_0$a** (not even known in positive characteristic) would significantly improve our general understanding of NIP fields and allow us to study them further with Galois-theoretic machinery. Moreover, if a henselian field has a pro-solvable (or even just nonuniversal) absolute Galois group, it admits a nontrivial definable henselian valuation [JK15a]. Hence, **HC$_0$a** implies that the only examples of nonhenselian NIP fields that admit a t-henselian V-topology (in the sense of [PZ78]) are separably closed or real closed.

A valuation is called *p-henselian* if it extends uniquely to every Galois extension of the field of p-power degree. Thus, the p-Henselianity Conjecture is a special case of the Henselianity Conjecture. A solution to **HC$_0$b** would at the very least also imply the existence of nontrivial definable henselian valuations on henselian NIP fields [JK15b]. Moreover, it could even give rise to a way to detect henselian valuations on NIP fields: in order to determine whether a field admits a nontrivial p-henselian valuation, it suffices to check whether the collection of sets

$$((a(K^\times)^p - b) \cap (c(K^\times)^p - d))_{a,b,c,d\in K, a,c\neq 0}$$

forms the basis of a V-topology [Koe95] (assuming $\mathrm{char}(K) \neq p$ and K contains a primitive pth root of unity). By [DHK19], this reduces to verifying four seemingly straightforward axioms.

INP Characterize inp-minimal fields.

- **INPa** Show that inp-minimal fields admit only finitely many Galois extensions of degree n for every $n \in \mathbb{N}$, i.e., inp-minimal fields are bounded.
- **INPb** Prove the V-topological conjecture for inp-minimal fields.
- **INPc** Show that henselian expansions of NTP_2 fields are NTP_2.

Inp-minimality is a natural generalization of dp-minimality. Prominent examples of non-dp-minimal inp-minimal fields are perfect bounded pseudo algebraically closed fields (which are not algebraically closed; in particular pseudofinite fields), as well as bounded pseudo real closed (resp. pseudo p-adically closed) fields that admit a unique ordering (resp. a unique p-adic valuation) [Mon17]. By [PP95], supersimple fields are bounded and perfect. Johnson's classification of dp-minimal fields [Joh23] implies that dp-minimal fields (i.e. inp-minimal fields which are NIP) are bounded and perfect. The first subobjective is to show boundedness for all inp-minimal fields (**INPa**). This topic, combined with part of **GAL**, has ample scope for a **PhD thesis**.

Unlike dp-minimal fields, inp-minimal fields may admit definable nonhenselian valuations, e.g., any p-adic valuation on a pseudo p-adically closed field which is not p-adically closed. Here, a field K is *pseudo real closed* (resp. *pseudo p-adically closed*) if for every absolutely irreducible variety V defined over K, if V has a simple rational point in every real closure (resp. p-adic closure) of K, then V has a K-rational point. So far, all known examples of inp-minimal fields are constructed via an AKE-type theorem: they are all henselian fields with residue field PAC, PRC, or PpC (where the henselian valuation may be trivial). However, we expect that more examples exist. We conjecture that every unstable inp-minimal field admits a unique definable V-topology (**INPb**), i.e., that the V-topology conjecture (cf [Ans22]) holds for inp-minimal fields. In recent work, Montenegro and Rideau-Kikuchi [MR23] introduce pseudo T-closedness: for a theory of large fields T, a field K

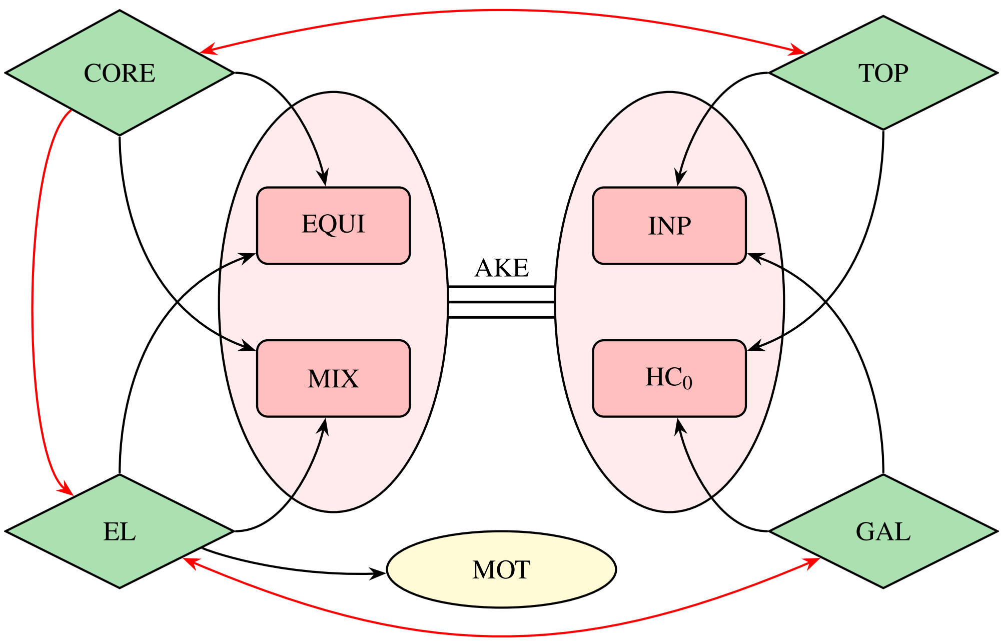
is *pseudo T-closed* if V has a K-rational point whenever it has a smooth rational point in every model of T extending K, for every absolutely irreducible V defined over K. We conjecture that any inp-minimal valued field is pseudo T-closed, where T is the theory of its henselization. The feasibility of these conjectures will be explored in subobjective **INPb** and beyond.

Inp-minimal fields are a first test case for how much of the machinery developed for NIP fields still applies in the NTP_2-setting. By [Jah23], henselian expansions of NIP fields are NIP; we seek a generalization to the NTP_2 setting (**INPc**). This will further our understanding of what one should expect a general NTP_2 field to look like.

In our fifth main objective, we aim to create connections between our research and motivic integration, intentionally seeking to go beyond the core of our current expertise.

MOT Motivic Integration with imperfect residue fields via Cohen rings.

In motivic integration on schemes over complete discrete valued fields in mixed characteristic, measurable sets are subsets of the Greenberg transform [NS08; Har19], a construction that only works well for perfect residue fields [Gre61]. For certain schemes and in the case of residue fields of finite degree of imperfection, Bertapelle and Suzuki introduced a variant of the Greenberg transform in [BS20]. We aim for a different approach, namely replacing Witt rings in the construction of the Greenberg transform by Cohen rings. Moreover, a deep obstruction to many applications of motivic integration in mixed and positive characteristic is that resolution of singularities is not known in positive characteristic. This can sometimes be overcome by the use of weak Néron models [LS03]. However, weak Néron models can only be used to compute motivic integrals in case the residue field is perfect. We aim to find an adaptation of weak Néron models which can take inseparable residue field extensions into account by once again exploiting the model theory and algebra of Cohen rings.



Schematic overview of research (arrows indicate the use of *newly developed* tools)

Tools. The tools we will develop are both algebraic and model-theoretic, and are detailed in the work programme. They can be divided into standard decompositions of valuations via ultraproducts **CORE**, embedding lemmas **EL**, Galois-theoretic methods **GAL**, and studying fields via their topologies **TOP**. These tools are interlinked: there are close connections between **GAL** and **EL**, as well as between **CORE** and **TOP**. Importantly, the two more algebraic tools **GAL** and **TOP** are crucial to the classification theoretic side, whereas the two more model theoretic tools **CORE** and **EL** are key ingredients on the arithmetic geometric side.

2.3 Work programme including proposed research methods. In the work programme, we indicate the roles of the researchers on the project and those with whom we plan to cooperate. We also highlight for each component where it lies on the scale *feasible* < *risk* < *high risk*.

Tool box. We start by explaining the four sets of tools which we plan to develop further.

CORE The core valuation. *Researchers*: SA, FJ, Ketelsen

This tool will be developed in joint work of Anscombe, Jahnke, and Ketelsen. It revolves around the "Standard Decomposition" (cf [AJ24, Remark 2.6]): beginning with an $\aleph_1$-saturated valued field (K, v) and a nonzero element t in the maximal ideal of v, we denote by Δ_{t+} (resp., Δ_{t-}) the smallest (resp., largest) convex subgroup of the value group vK of v that contains (resp., does not contain) t. Denoting by v_{t+} (resp., v_{t-}) the coarsening of v corresponding to Δ_{t+} (resp., Δ_{t-}), the group Δ_{t+}/Δ_{t-} is archimedean, and $(Kv_{t+}, \bar{v}_{t-})$ is a rank-1 valued field. Since (K, v) was assumed to be $\aleph_1$-saturated, $(Kv_{t+}, \bar{v}_{t-})$ is complete and maximal, moreover Δ_{t+}/Δ_{t-} is isomorphic to $\mathbb{Z}$ or $\mathbb{R}$ [AK16, Section 4]. We may decompose v as a composition of $\bar{v}$, $\bar{v}_{t-}$, and v_{t+}, as illustrated in the following diagram, where arrows represent the associated places.

$$K \xrightarrow{vK/\Delta_{t+}} Kv_{t+} \xrightarrow{\Delta_{t+}/\Delta_{t-}} Kv_{t-} \xrightarrow{\Delta_{t-}} Kv$$

This powerful method goes back at least to the original theorems of Ax–Kochen–Ershov, and it continues to be used regularly in this spirit ([AK16; HHJ19; Joh23; ADJ23; AF23]). Moreover in [JK23], it is used in the "taming" process. In [AJ24] this decomposition is used in the proof of the NIP transfer theorem to argue that (under suitable algebraic conditions) NIP transfers from Kv to Kv_{t-}, then to Kv_{t+}, and then finally to K itself, each equipped with the composed valuations.

It is naturally applied in mixed characteristic, where usually we take $t = p$, but it can also be applied in equal characteristic. We aim to study the theory of the core field $(Kv_{t+}, \bar{v}_{t-})$ for $\aleph_1$-saturated models (K, v, t) of theories of henselian valued fields with parameter t. Under the hypothesis that $\mathcal{O}/t\mathcal{O}$ is semiperfect, Kv_{t-} is perfect, and so in the case that Δ_{t+}/Δ_{t-} is not discrete, the core field is tame. Note that tameness really relies on $\aleph_1$-saturation, e.g., when we take an analogous decomposition of a rank-1 field, what corresponds to the core is just the field itself (which might not be tame). We aim to show that the theory of the core field is an invariant of the theory of (K, v) (*feasible*). The key is to distinguish whether, for some model of the theory, Kv_{t-} admits a tame valuation with divisible value group – such fields were studied in [AJ18].

In a similar vein, we aim to show that the existential theory of (K, v) is determined 'monotonically' by that of the core field, in which case the former is Turing reducible to the latter [AF23] (*risk*). We expect the converse direction to be *feasible*. Such a reduction will open up a new route towards understanding existential theories of henselian fields both in the mixed and the equicharacteristic setting, for **EQUI**, **MIX** and beyond.

EL Embedding lemmas. *Researchers*: SA, FJ

Embedding lemmas lie at the heart of all Ax-Kochen/Ershov type theorems. They give conditions on how to lift embeddings of residue fields and value groups to embeddings of valued fields and are used to transfer model-theoretic properties (like decidability or NIP). Embedding lemmas can also be used to prove relative quantifier elimination statements which are key ingredients both to motivic integration and classification results. For tame, separably tame and finitely ramified henselian fields, embedding theorems are known [Kuh16; KP16; AJ22; ADJ23].

We plan to strengthen Kuhlmann's (and Kuhlmann–Pal's) embedding lemmas for tame and separably tame fields to get control over parameters, both in the equicharacteristic (*risk*) and the mixed characteristic (*risk* to *high risk*, depending on the surrounding theory) settings. In order to lay the foundations for this project, we are currently jointly running a reading group between our working groups on tame fields. Crucially, our strategy relies on proving refinements of Kuhlmann's theorems "Henselian Rationality" [Kuh19] and "Generalized Stability" [Kuh10]. We aim to proceed in a similar way to the techniques developed in [ADJ23], where we (with Dittmann) prove such a strengthened embedding theorem for finitely ramified fields. For applications to **MIX** and **EQUI**, we thus plan to extend known embedding theorems for tame valued fields over tame subfields, to allow arbitrary subfields, by combining insights developed from **CORE**. For example, for **MIX** it then suffices to consider the composition of two embedding problems in equal characteristic and one embedding problem in mixed characteristic – in the latter problem the common subfield is now tame! Last but not least, we will extend the embedding lemmas proven in [JK23] from tameable fields to separably tameable fields (*feasible*). Embedding lemmas for unramified henselian valued fields are developed in [AJ22; ADJ23]. Studying the family of suitable systems of representatives (so-called "S-maps") we will prove powerful new embedding lemmas, giving greater control over the representatives of elements from the residue field (*feasible*). This is key to applications in **MOT**.

GAL Galois-theoretic methods. *Researchers*: SA, FJ, PhD

Studying fields via their absolute Galois groups is a common method in arithmetic geometry. From a classification perspective, the most important Galois-theoretic result is that infinite NIP fields of positive characteristic admit no Artin-Schreier extensions [KSW11]. This was applied in Johnson's proof of the Henselianity Conjecture in positive characteristic [Joh21b], as well as in most other works on henselian valuations on NIP fields [JSW17; HH19; Jah23; JS20; Asc+22; AJ24].

We plan to detect IP-patterns via Galois extensions of degree n with Galois group S_n (*feasible* in positive characteristic, *risk* in characteristic 0) as well as inp-patterns in ordered and valued fields via infinitely many distinct Galois extensions of degree n for some n (*risk*). There are many occurrences of polynomial IP patterns to guide our study. The first such was isolated by Duret in his proof that non-separably closed PAC fields have the independence property [Dur80]. The fact that infinite NIP fields of positive characteristic are Artin-Schreier closed [KSW11] can also be traced back to a polynomial pattern [Boi21]. Polynomial IP patterns also play a crucial role in Johnson's work [Joh21b; Joh16, Chapter 11], as well as in work of Chernikov and Hempel [CH21a].

For application in **INPa**, we consider fields with unbounded absolute Galois group. One consequence of boundedness for a perfect field K is that for all finite extensions L and all primes p, the index $L^\times/(L^\times)^p$ is finite (although this is strictly weaker than boundedness, cf [FJ17]). If K is an inp-minimal field, then there is at most one prime such that $K^\times/(K^\times)^p$ is infinite [CKS15]. If K is an inp-minimal ordered field, then $K^\times/(K^\times)^p$ is finite for all p. However, since it is not known whether finite extensions of inp-minimal fields are inp-minimal, this does not lift to finite extensions. In [Mon17], Montenegro shows that pseudo real closed and pseudo p-adically closed NTP_2 fields are bounded. We will combine Montenegro's machinery with ideas from Johnson [Joh23].

Building on [ADJ23], we aim to understand defectless real-valued fields of mixed characteristic via sequences of Eisenstein polynomials over Cohen rings over the residue field (*feasible*). More precisely, given a tame real-valued field (K, v) of mixed characteristic, we will identify an elementary subfield that is the directed union of its finitely ramified subfields, analyzing theory of (K, v) by applying the tools developed in [ADJ23] to the finitely ramified subfields of (K, v). This is closely connected to **EL**. To reduce the general case to this setting, we will apply **CORE** (*risk*).

TOP Studying fields via their topologies. *Researchers*: SA, FJ, Soto Moreno

Every non-trivial valuation induces a V-topology on the field [PZ78]. Conversely, every V-topology is induced by a valuation or an ordering ([KD53]). If the valuation ring is definable, then the corresponding V-topology has a uniformly definable basis. Conversely, if the V-topology has a uniformly definable basis, then the valuation ring is externally definable [HHJ20]. The relationship between henselian valuations and their topologies is often key to defining valuations [PZ78; AJ18]. In their proof of the Stable Fields Conjecture for large fields, Johnson et al. introduce and study the étale-open topology [Joh+23] which is a V-topology if and only if the valuation is henselian. En route to his characterization of dp-finite fields [Joh20a; Joh20b], Johnson initiated the study of W_n-topologies, generalizations of V-topologies generated by intersections of n valuation rings. A third powerful topological tool crucial for studying valued fields of mixed characteristic is to dissect them via the standard decomposition of their nonstandard models, as studied in **CORE**.

Anscombe and Soto Moreno plan to develop a structure theory for W_n-topologies (*risk*), beginning with a detailed study of [Joh20a; Joh20b]. Do they form a tree on a given field? What is the join of two W_n-rings? Is the union of a chain of W_n-rings again a W_n-ring? When do two W_n-rings induce the same W_n-topology? The $n = 2$ case is currently investigated by Soto Moreno in his PhD under Anscombe's supervision. We plan to build on Soto Moreno's PhD project after completion.

Anscombe and Jahnke plan to construct and study the p-étale-open topology, such that on large fields, the p-étale topology is a V-topology iff the valuation is p-henselian (*high risk*). Our candidate for the basis of the p-étale topology is given by considering only those étale images $f(W(K))$ where the degree of $f : W \to V$ is a power of p and f is moreover a Galois cover. We will apply this in particular to investigate the question whether all p-henselian fields are large.

Building on these tools, we now describe the work programme for each of our objectives.

EQUI AKE principles in positive characteristic. *Researchers*: SA, FJ, Kartas, PhD

Key recent ingredients: [ADF23; JK23; Lis21; Lis23]

An important part of [JK23] is the axiomatization of the class of valued fields (K, v) with distinguished $t \in \mathfrak{m}_v \setminus \{0\}$ for which $\mathcal{O}_v/p\mathcal{O}_v$ is semiperfect and $\mathcal{O}_v[t^{-1}]$ is the valuation ring of an algebraically maximal valuation. To models of this theory Jahnke and Kartas are able to apply the taming method: that is, using the model theory of the roughly tame valuation corresponding to $\mathcal{O}_v[t^{-1}]$, they obtain AKE principles for this class. This heavily involves the method **CORE**.

Our two subobjectives **EQUIa** and **EQUIb** both involve extending this theory to include the case that the valuation associated to $\mathcal{O}_v[t^{-1}]$ is only assumed to be separably algebraically maximal and $\mathcal{O}_v/(t)$ is semi-perfect. The crux is to show that separable defectless of $\mathcal{O}_v[t^{-1}]$ implies the same in elementary extensions $(F, w, t) \succ (K, v, t)$, allowing us to then employ the methods from **CORE**. We first explain the case of a rank-1 valued field (K, v, t), where the valuation ring $\mathcal{O}_v[t^{-1}]$ is trivial and hence separably algebraically maximal. As (K, v) has rank 1, v has no nontrivial proper coarsenings. If $\mathcal{O}_w[t^{-1}]$ was not separably algebraically maximal, one can deduce from [KR23] that it has independent defect. By applying work in progress of our PhD candidates Ketelsen, Ramello (both supervised by Jahnke), and Szewczyk (supervised by Anscombe), this independent defect gives rise to a definable coarsening of w. By quantifying over the parameters, this yields a proper nontrivial coarsening of v, which cannot exist. Thus, we may study $\aleph_1$-saturated models using **CORE**, which makes this part of **EQUIa** *feasible*. Potential examples to which this strategy will apply are the Artin-Schreier closures of $\mathbb{F}_p((t))$ and $\mathbb{F}_p^{\mathrm{alg}}((t))$, but we expect there to be even more examples of arithmetic interest. The aim of subobjective **EQUIb** is to generalize this method to higher rank. Here, the definability argument needs to be refined, as higher rank valuations may admit proper nontrivial (definable) coarsenings. However, if $\mathcal{O}_v[t^{-1}]$ is separably defectless, it has no coarsening with separable defect. We expect to be able to show that the Ketelsen–Ramello–Szewczyk definition gives rise to a (definable) coarsening of v with separable defect. An alternative approach is to generalize the results in [JK23, Sections 4.1 and 4.2], where the approach is purely algebraic and does not rely on definability of valuations. Subobjective **EQUIb** is between *feasible* and *risk*, thus this pair of subobjectives is very suitable for the basis of a PhD thesis.

A further feature of the model-theoretic approach to perfectoid fields in [JK23] is the care required over the type of the pseudouniformizer t. This necessitates a deeper knowledge of the model theory of tame valued fields with an extra parameter. In this direction are two recent preprints of Lisinski [Lis21; Lis23] which study tame valued fields $F((t^\Gamma))$ of equal characteristic in the language $\mathcal{L}_t$ of valued fields expanded by a constant for t, for a perfect field F of characteristic $p > 0$ and Γ a p-divisible value group. He proves an AKE principle for decidability: the theory of such a valued field is decidable if and only if the $\mathcal{L}_{\mathrm{ring}}$-theory of F and the $\mathcal{L}_{\mathrm{oag}}$-theory of Γ are decidable.

As an intermediate goal, we will build on Lisinski's work in the context of existential theories to resolve H10 for tame valued fields in $\mathcal{L}_t$. This is *feasible* and will use both **CORE** and **EL**. Another approach is to reprove Lisinski's work by explicitly axiomatising these theories *à la* [ADF23]. This is analogous to the difference in the approaches to decidability in [DS03] and in [AF16; ADF23].

The final subobjective **EQUIc** is noticably harder, and is the subject of joint work of Anscombe and Jahnke with Kartas: since $\mathbb{F}_p((t))^{\mathrm{perf}}$ is perfect and not separably defectless, all of its defect is independent. The method will combine what is known about $\mathbb{F}_p[t^{p^{-\infty}}]/t$ with consequences of Lisinski's theory for existential theories of tame valued fields, the standard decomposition (**CORE**), and new embedding lemmas (**EL**). We consider this an exciting but ambitious goal (*high risk*).

MIX Hilbert's 10th problem in mixed characteristic *Researchers*: SA, Dittmann, FJ

Key recent ingredients: [ADJ23; AJ22]

The entire objective is joint work of Anscombe, Dittmann, and Jahnke. Any henselian valuation v of mixed characteristic $(0, p)$ on a field K admits a finest coarsening of mixed characteristic, which we denote by v_p (cf **CORE**). Whether v_p is finitely ramified or not is encoded in the theory of (K, v), and is an important case distiction for studying valued fields of mixed characteristic via their decompositions. Subobjective **MIXa** studies H10 for (K, v) when v_p is finitely ramified. By [ADJ23], Hilbert's 10th problem for (K, v_p) is solvable just in case the existential theory of a certain expansion of its residue field Kv_p is decidable. By [AF16], H10 for the induced valuation $(Kv_p, \bar{v})$ is solvable if and only if H10 is solvable for Kv. We plan to combine these results with a view towards obtaining a H10 principle for (K, v), i.e. the composition v of v_p and $\bar{v}$, down to an appropriate expansion of Kv, which reflects the necessary expansion of Kv_p from [ADJ23]. As a first step, we will study the situation when (K, v_p) is tamely ramified, where the induced structure on Kv_p just involves naming an appropriate class of nth powers (*feasible*). We expect the intuition from the tamely ramified case to guide us in the general case (*feasible* to *risk*).

The second subobjective is to axiomatize existential theories of tame valued fields of mixed characteristic relative to the existential residue field theory (**MIXb**, *risk*). In order to show that two tame valued fields (K, v) and (L, w) of mixed characteristic have the same existential theory, one needs to verify that (L, w) embeds into a sufficiently saturated elementary extension of (L, w) and vice versa. Note that the complete theory of a tame valued field is not axiomatized by the theory of its residue field, the theory of its value group and its algebraic part [AK16, Theorem 1.5]. Thus, we expect such embeddings not to exist in general, even on the assumption that Kv and Lw have the same existential theory and the algebraic parts coincide, and that the study of the existence of such embeddings requires us to identify suitable additional structure on the residue fields of the core fields, in analogy with [ADJ23]. Applying **CORE** and [AF16], we can reduce to an embedding problem of maximal real-valued fields of mixed characteristic. We aim to solve this embedding problem with the techniques developed in **EL**.

Once we have understood existential theories of tame valued fields of mixed characteristic, we may use **CORE** to show that in any $\aleph_1$-saturated henselian valued field (K, v) of mixed characteristic with $\mathcal{O}_v/(p)$ semiperfect, the existential theory of (K, v) reduces to that of K_{p-} (*risk*). Thus, assuming that $\mathcal{O}_v/(p)$ is semiperfect, H10 relative to the residue field in mixed characteristic (**MIXc**) will be resolved with the composition method, in analogy to **MIXa**. Without the assumption of semi-perfectness, an even stronger embedding lemma is needed (*high risk*).

$\mathbf{HC_0}$ Prove the Henselianity Conjecture in characteristic 0. *Researchers*: SA, FJ, Postdoc
Key recent ingredients: [AJ24; Joh21b; Joh+23]

The Henselianity Conjecture is a heuristic for the Shelah Conjecture: in joint work of Halevi, Hasson and Jahnke, they show that the Shelah Conjecture implies the Henselianity Conjecture [HHJ20]. Recently, Johnson showed that every NIP valued field of positive characteristic is henselian [Joh21b]. Our goal is to prove the analogous result for fields of characteristic 0. The two subobjectives detailed below both give possible routes to a solution of the Henselianity Conjecture in characteristic 0, but they are entirely disjoint. This will be joint work of Anscombe, Jahnke, and likely the postdoc.

The first route is Galois-theoretic (cf **GAL**): we plan to solve the inverse Galois problem over NIP fields, showing in particular that absolute Galois groups of NIP fields are pro-solvable **$\mathbf{HC_0}$a** (*risk to high risk*). As a first step, we will show that not every symmetric group occurs as a Galois group over an NIP field, i.e., for an NIP field K there is some $n > 2$ such that no Galois extension L/K has Galois group $\mathrm{Gal}(L/K) \cong S_n$. This will imply that an NIP field cannot be Hilbertian, since any symmetric group occurs as a Galois group over a Hilbertian field ([FJ08, Chapter 16]). The next step is to show that in fact, there is an n such that S_n does not occur as a Galois group over any finite extension of K (*risk*), i.e. the absolute Galois group is *nonuniversal*. Equivalent to nonuniversality is the existence of a finite group G such that for every tower $M/L/K$, with M/K finite and Galois, we have $\mathrm{Gal}(M/L) \ncong G$. For an infinite NIP field K of characteristic $p > 0$, the Artin-Schreier closedness of all finite extensions of K [KSW11] implies that the symmetric group S_m does not occur as a Galois group over K, nor over any finite extension of K, for any $m \geq p$. Thus, nonuniversality of the absolute Galois group generalizes the field being Artin-Schreier closed. Once we have achieved nonuniversality, we will approach pro-solvability (*high risk*), which is not even known for NIP fields of positive characteristic. This new understanding of the absolute Galois group of NIP fields may allow the transfer of known results for NIP fields of positive characteristic (where proof techniques rely on Artin-Schreier closedness), most prominently [Joh21b], to characteristic 0.

The second route we propose is via p-henselianity, a version of henselianity where Hensel's Lemma is only required to hold for polynomials whose splitting field has p-power degree. Like for henselianity, every field admits a canonical p-henselian valuation, but they are easier to study because the canonical p-henselian valuation is usually definable [JK15b]. We aim to develop a p-version of the étale open topology (cf **TOP**) to prove the p-Henselianity Conjecture: every p-henselian NIP valued field is henselian **HC$_0$b** (*high risk*). The key to proving the p-Henselianity Conjecture is to show that the p-étale open topology coincides with the étale open topology. This then implies that the étale open topology is a V-topology and hence is induced by a henselian valuation [Joh+23]. The method we will use is a (yet-to-be-proven) topological version of an approximation argument on curves that was introduced in [Joh22] and refined in [HHJ20] in order to detect IP patterns via independent valuations.

INP Characterize inp-minimal fields. *Researchers*: SA, Boissonneau, FJ, Rideau-Kikuchi, PhD
Key recent ingredients: [AJ24; Boi21; Joh21a; Joh20b; Kuh22]

An important generalization of NIP is NTP_2 (Not the Tree Property of the second kind), which includes pseudofinite fields and ultraproducts of the p-adics (where p varies). Unlike for NIP fields, there is not even a conjectural description of NTP_2 fields. Moreover, NTP_2 fields need not be Artin-Schreier closed, but they only admit finitely many Artin-Schreier extensions [CKS15]. In analogy to dp-minimality, inp-minimality is the 1-dimensional version of NTP_2.

We aim to study inp-minimal fields and give a conjectural classification. Firstly, we plan to show that inp-minimal fields have bounded absolute Galois group **INPa** (cf **GAL**). This topic, combined with part of **GAL**, has ample scope for a PhD thesis. The test case (*feasible*) is to study ordered inp-minimal fields for which more technology exists [CKS15; Mon17]. The general case (*high risk*) requires new ideas even in the dp-minimal setting, where boundedness only follows a posteriori from Johnson's classification. Moreover, we plan to show that every unstable inp-minimal field admits a unique definable V-topology **INPb** (*high risk*). That an inp-minimal valued field admits at most one definable V-topology can be deduced from the approximation theorem for V-topologies [PZ78]. We will approach the problem of finding a definable V-topology on an inp-minimal field by studying both Johnson's canonical topology [Joh18; Joh21a] as well as W-topologies (**TOP**, [Joh20b]) on inp-minimal fields. The feasibility of these conjectures will be explored jointly by Anscombe, Jahnke, and Rideau-Kikuchi, in subobjective **INPb** and beyond. The work on our two subobjectives on inp-minimality will go hand in hand, but may be begun independently. The next objective is the subject of joint work of Jahnke and Boissonneau. Since NTP_2 valued fields only admit independent defect [Kuh22], we aim to understand henselian expansions of NTP_2 fields **INPc** (*feasible* for tame expansions, *high risk* in general). The key question is whether coarsenings of NTP_2 henselian valuations are again NTP_2. Since any coarsening w of a valuation v is externally definable in the $\mathcal{L}_{\mathrm{val}}$-structure of (K, v), if (K, v) is NIP then so is (K, w) (applying Shelah's expansion theorem [She09]). This coarsening argument is crucial in the solution of the NIP analogue of **INPc** [Jah23]. The Shelah expansion theorem fails in NTP_2 structures, witnessed for example by the Shelah expansion on the random graph. However, in case the value group of an NTP_2 henselian field (K, v) is stably embedded (with the induced structure being NIP, e.g., as a pure ordered abelian group), then any coarsening w of v is in fact NTP_2 by work of Montenegro, Onshuus and Simon [MOS20]. For an NIP henselian field, the value group is always stably embedded: this follows essentially because every NIP henselian field can be decomposed such that each part fits into the context of an AKE type theorem. Hence, we will explore the precise structure of NTP_2 henselian fields, building on [Boi21], with a view towards understanding the structure on the value group.

MOT Motivic Integration with imperfect residue fields. *Researchers*: SA, FJ, Postdoc
Key recent ingredient: [AJ22; BS20]

Motivic integration assigns a measure to (a reasonable class of) subsets of the arc space of an algebraic variety over a field K, taking values in the Grothendieck ring of algebraic varieties over K. In [CL15], Cluckers and Loeser introduce an axiomatic framework for motivic integration in mixed characteristic. There is another approach to motivic integration for schemes in the mixed characteristic setting, where motivic integrals take values in the Greenberg transform of a scheme [LS03; Har19]. The Greenberg transform allows the transformation of schemes over a complete unramified mixed characteristic henselian field with value group $\mathbb{Z}$ and perfect residue field into schemes over the residue field. It only works for perfect residue fields since its definition relies crucially on the functoriality of Witt vectors [Gre61]. The construction of Cohen rings is known not to be functorial: for example, an isomorphism between imperfect residue fields lifts non-uniquely to an isomorphism of the corresponding Cohen rings. Nevertheless, applying the machinery developed in [AJ22], we expect it to become functorial once we redefine a morphism between Cohen rings to be a consistent family of ring homomorphisms respecting the parameterized family of S-maps. This definition gives a canonical lifting of any isomorphism between residue fields to a unique morphism. Thus, with this notion of morphism, the Cohen construction is functorial on the level of Cohen rings themselves. We plan to extend this functoriality to schemes, which will in particular allow us to obtain an imperfect version of the Greenberg transform. Our functorial approach may also help to study analogues of weak Néron models (applied to great effect, e.g., in [Nic09; Nic11]) which take imperfect extensions of the residue field into account (**MOT**, *high risk*).

We are certain that our research plan is both timely and ambitious, and that we are in the best possible position to achieve these objectives. We are convinced that through hard work and intuition, combined with mathematical exchange with our broad network of excellent researchers ranging over all career stages, we will realize many of our most ambitious goals.

2.4 Handling of research data. The preprints of all results obtained in the context of this project will be made publicly available through ArXiv and other preprint servers (e.g. Modnet, see `http://www.logique.jussieu.fr/modnet/Publications/Preprintserver/`).

2.5 Relevance of sex, gender and/or diversity. As a team of female researchers, we are very aware of how the dynamics of any research group changes when it is diverse rather than homogeneous. We value diversity and are passionate about developing a diverse next generation. For example, Franziska Jahnke is an organizer of the upcoming conference "Young Women in Model Theory and Applications" (`https://sites.google.com/view/ywmta`) which is designed for early-career female and non-binary researchers in Model Theory. Sylvy Anscombe is an invited speaker at this conference. Moreover, we plan to fill at least one of the two positions with a female researcher.

The Logic group in Münster currently comprises 9 women (3 of them professors) and 11 men (2 of them professors). Moreover, the Cluster of Excellence "Mathematics Münster" identifies equal opportunity as one of its three major structural support areas, and in particular offers a family friendly environment. More details can be found on the Cluster web page (`https://www.uni-muenster.de/MathematicsMuenster/careers/diversity.shtml`).

The Logique team in the Institut de Mathématiques de Jussieu-Paris Rive Gauche has 30 members (15 permanent and four emeritus), of whom nine are women (three permanent and two emeritus). Université Paris Cité has a Mission dedicated to equality, diversity, and inclusion (`https://u-paris.fr/mission-egalites/`), and the IMJ-PRG has a [Sex/Gender] Parity committee (`https://www.imj-prg.fr/comite-parite/`).

Another focus is diversity along the lines of ethnic and national background: we are aware of how overwhelmingly white the mathematical research community of Europe is at present. The postdoctoral and PhD positions that are part of this proposal will be advertised as widely as we can, by distributing through international networks, aiming to attract the best possible candidates regardless of ethnic background or national origin. In particular we will reach out to the Masters students at the African Institute for Mathematical Sciences (AIMS). Sylvy Anscombe is already putting together an application with Gareth Boxall (Stellenbosch) for the program *Chaire 'Pays Du Sud'* (https://www.cirm-math.com/programmes.html) at CIRM (Marseille) for Summer 2024. This is a program designed to strengthen research collaborations between African and European mathematicians, and especially to provide opportunities for young African masters and PhD students. If that application is successful, one of the themes of the research school and subsequent workshop would be model theory of valued fields, and this would be an excellent opportunity to encourage suitable Master's students, and to build links with potential future collaborators.

2.6 Added value of the French-German scientific cooperation. The two applicants are already a close knit team with a long-standing research collaboration dating back a number of years. We have three joint publications and one joint preprint so far and regularly exchange on a number of scientific questions. We combine working remotely via email and zoom with regular visits. between Amsterdam/Münster and Paris. In 2020 we co-organized the seminar 'Valuation Theory Seminar' as part of the semester 'Decidability, definability and computability in number theory' at the MSRI in Berkeley. We are currently running the 'Working group on tame fields' (`https://www.uni-muenster.de/IVV5WS/WebHop/user/sramello/tame/`) via zoom, with attendees from Münster, Paris, and Amsterdam.

The Logic teams in Paris and Münster are two of Europe's largest centres of model theory, and have had a range of joint projects in the past. For example there was the recent, highly successful ANR-DFG project 'GeoMod' (AAPG2019), of which both applicants were members. In Autum 2024, Sylvy Anscombe will make a sabbatical visit to work with Franziska Jahnke in Münster as a Junior Fellow of the Cluster of Excellence "Mathematics Münster". Both of Franziska Jahnke's current PhD students, Simone Ramello and Margarete Ketelsen, have recently spent time visiting Paris to develop their mathematical collaborations. Sylvy Anscombe's PhD student Piotr Szewczyk (Dresden, co-supervised with Arno Fehm) made an extended visit to Münster in 2022, and her other PhD student Paulo Soto Moreno will also be visiting Münster for an extended period in Autumn 2024. The current proposal will make the ties between our research groups even stronger, and will particularly benefit the early career researchers involved. More broadly it will strengthen the international cooperation between German and French model theory communities.

3 Project- and subject-related list of publications. Our ten most relevant publications are **highlighted**.

[Ans22] Sylvy Anscombe. "Shelah's conjecture and Johnson's theorem [*after* Will Johnson]". In: *Astérisque* 438 (2022), pp. 247–279. DOI: 10.24033/ast.1187.

[ADF23] Sylvy Anscombe, Philip Dittmann, and Arno Fehm. "Axiomatizing the existential theory of $\mathbb{F}_q((t))$". In: *Algebra & Number Theory* 17.11 (2023), pp. 2013–2032. DOI: 10.2140/ant.2023.17.2013.

[ADJ23] Sylvy Anscombe, Philip Dittmann, and Franziska Jahnke. *Ax–Kochen–Ershov principles in finitely ramified henselian fields*. 2023. arXiv: 2305.12145 [math.LO].

[AF16] Sylvy Anscombe and Arno Fehm. "The existential theory of equicharacteristic henselian valued fields". In: *Algebra & Number Theory* 10.3 (2016), pp. 665–683. DOI: 10.2140/ant.2016.10.665.

[AF23] Sylvy Anscombe and Arno Fehm. *Interpretations of syntactic fragments of theories of fields*. 2023. arXiv: 2312.17616 [math.LO].

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[AJ24] Sylvy Anscombe and Franziska Jahnke. *Characterizing NIP henselian fields*. To appear in Journal of the London Mathematical Society. 2024. arXiv: 1911.00309v2 [math.LO].

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[BS20] Alessandra Bertapelle and Takashi Suzuki. *The relatively perfect Greenberg transform and cycle class maps*. 2020. arXiv: 2009.05084 [math.NT].

[Boi21] Blaise Boissonneau. *Artin-Schreier extensions and combinatorial complexity in henselian valued fields*. 2021. arXiv: 2108.12678 [math.LO].

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[CH21a] Artem Chernikov and Nadja Hempel. "On n-dependent groups and fields II". In: *Forum Math. Sigma* 9 (2021), Paper No. e38, 51. DOI: 10.1017/fms.2021.35.

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4 Supplementary information on the research context.

4.1 Ethical and/or legal aspects of the project. None.

4.1.1 General ethical aspects. None.

4.1.2 Descriptions of proposed investigations involving humans, human materials or identifiable data. None.

4.1.3 Descriptions of proposed investigations involving experiments on animals. None.

4.1.4 Descriptions of projects involving genetic resources (or associated traditional knowledge) from a foreign country. None.

4.1.5 Explanations regarding any possible safety-related aspects ("Dual Use Research of Concern; foreign trade law). None.

4.2 Employment status information.

- Anscombe, Sylvy: Maîtresse de conférences (tenured) at the Université Paris Cité.
- Jahnke, Franziska: W2 professor (tenured) at the University of Münster, from September 2024 onwards. Currently (09/2023–08/2024) on leave from Münster and employed as a Universitair hoofddocent (Associate Professor, tenured) at the University of Amsterdam.

4.3 First-time proposal data. None.

4.4 Composition of the project group. At the University of Münster:

- Franziska Jahnke (W2 professor)
- Margarete Ketelsen (PhD student)
- Simone Ramello (PhD student)

Due to Franziska Jahnke's recent appointment as a scout in the Henriette-Herz-programme of the Alexander von Humboldt foundation, two postdoctoral researchers funded by the Humboldt foundation are expected to join her working group at the University of Münster by the time the project starts. The first Humboldt Fellow proposed by Franziska Jahnke in her role as a scout will be Mariana Vicaria (UCLA).

At the Université Paris Cité:

- Sylvy Anscombe (Maîtresse de conférences)
- Vincent Bagayoko (Postdoc)
- Paulo Soto Moreno (PhD student)

4.5 Researchers in Germany with whom you have agreed to cooperate on this project. Margarete Ketelsen (Münster) **CORE**, Philip Dittmann (TU Dresden)

4.6 Researchers abroad with whom you have agreed to cooperate on this project. Paulo Soto Moreno (IMJ-PRG, Paris Cité), Konstantinos Kartas (IMJ-PRG, Sorbonne), Blaise Boissonneau, Silvain Rideau-Kikuchi (ENS PSL)

4.7 Researchers with whom you have collaborated scientifically within the past three years. Franziska Jahnke has scientifically collaborated in the past three years with:

- Sylvy Anscombe (Université Paris Cité)
- Philip Dittmann (TU Dresden)
- Konstantinos Kartas (Sorbonne Université)
- Sebastian Krapp (University of Konstanz)
- Salma Kuhlmann (University of Konstanz)

She is also currently writing a monograph on the "Model Theory of Valued Fields" with Martin Hils (University of Münster) and Silvain Rideau-Kikuchi (ENS PSL).

Sylvy Anscombe has scientifically collaborated in the past three years with:

- Franziska Jahnke (University of Münster, University of Amsterdam)
- Philip Dittmann (TU Dresden)

- Arno Fehm (TU Dresden)
- Valentijn Karemaker (Utrecht University)
- Zeynep Kisakürek (HHU Düsseldorf)
- Vlerë Mehmeti (Sorbonne Université)
- Margherita Pagano (Leiden University)
- Laura Paladino (University of Calabria)

4.8 Project-relevant cooperation with commercial enterprises. None.

4.9 Project-relevant participation in commercial enterprises. None.

4.10 Scientific equipment. No larger scientific equipment sought.

4.11 Other submissions.

- `HI 2004/2-1` Martin Hils (University of Münster) and Franziska Jahnke have a joint DFG project *Model theory of valued fields with endomorphism* ("Sachbeihilfe") which finances a PhD student (Simone Ramello) who is currently working on his doctorate under their joint supervision at the University of Münster. While the subject of Simone's dissertation is also on the model theory of valued fields, none of the objectives of `HI 2004/2-1` lie in the scope of the project proposed here. The project is scheduled to end in 02/2025.
- Sylvy Anscombe is a co-applicant (25%) of the project "Model theory of algebraic structures/ théorie des modèles des structures algébriques" (MAS), submitted in response to the call AAPG2024 (PRC) of the ANR. The project MAS is coordinated by Frank O. Wagner. There is no overlap between the objectives of MAS and the objectives of the current proposal. The subject of Soto Moreno's doctoral studies is related to – but not overlapping with – the objectives of this proposal.

4.12 Other information. Both the PhD candidate and the postdoctoral researcher will be able to enjoy the infrastructure of the Cluster of Excellence 'Mathematics Münster'. In particular, the PhD student will be a member of the 'Mathematics Münster Graduate School' (MMGS, see `https://www.uni-muenster.de/MathematicsMuenster/GraduateSchool/`). The MMGS is the joint graduate program of the Cluster of Excellence Mathematics Münster and the mathematical institutes of the university of Münster. It offers an active and demanding research environment for talented and motivated doctoral researchers aiming to become the next generation of leading researchers. In addition to an academic supervisor, MMGS students must choose a mentor who offers support in various matters, e.g. enabling a transparent progress review. The MMGS has a broad variety of advanced lectures, seminars and courses on recent and fundamental research topics to broaden the horizon regarding neighbouring mathematical fields. Furthermore, there are a wide range of informal events specially for doctoral students. The MM Connect is an event format where new MMGS members can present themselves and their research project (MM Arrival), say goodbye when the next career step is due (MM Departure) and exchange basic ideas and concepts (MM Common Ground). Moreover, there is a Welcome Event at the beginning of the academic year and an annual retreat for its student members. In addition, any member of the MMGS can profit from the general infrastructure of the Cluster and apply for a variety of measures, including the opportunity to apply for travel support (for short and long-term trips) or to get involved in conference organisation (in particular as part of the Young Mathematicians Conference Network).

Though the PhD student will be based principally in Münster, they will also be registered at the Institut de Mathématiques de Jussieu–Paris Rive Gauche, under the École Doctorale de Mathématiques de Paris-Centre, via a *thèse en cotutelle* arrangement. Thus they will benefit from the support of the IMJ-PRG and the Équipe de Logique (including being assigned a mentor from among the faculty), and in the end they will obtain a double diploma. The postdoc will work in the IMJ-PRG, where there is a very vibrant and supportive community of young researchers. For example, Vincent Bagayoko and Konstantinos Kartas are currently postdocs in the IMJ-PRG, both of whom will cooperate with this project. Since the Paris mathematical community is so large, it is a great city in which to be a postdoctoral researcher, with dozens of weekly seminars and plentiful teaching and career-development opportunities.

5 Requested modules/funds

GERMANY. On the German side, all modules and funds are requested by: Jahnke, Franziska.

5.1 Basic Module.

5.1.1 Funding for Staff. One position (TV-L 13, 75%) for a PhD student for the duration of three years. This position will be located at the University of Münster. The preferred starting date is January 2025. Excellent possible candidates include Tianyiwa Xie (currently a Part III student at the University of Cambridge) and Jonas van der Schaaf (MSc student at University of Amsterdam).

It is expected that the PhD student supported by the project will be based in Münster but that they will be spending time in Paris (preferably 12-18 months in total). This does however depend on the family circumstances of the student. The PhD student will be jointly supervised by Sylvy Anscombe and Franziska Jahnke, with regular virtual meetings between the three of them, as well as one-on-one discussions.

5.1.2 Direct Project Cost.

5.1.2.1 Equipment up to € 10,000, Software and Consumables. We apply for funds for one laptop computer as well as one tablet for the PhD student, to make virtual supervisions as effective as possible. In total, we apply for the German node for 2.000 EUR.

We also plan to subscribe to the overleaf group plan to facilitate optimal working conditions between the two working groups. This is priced at 690 EUR/year. Thus, we apply for 2.070 EUR for an overleaf group license, for all members of the project group, managed from Germany. Overleaf is already the main tool for collaborative writing of the two applicants.

5.1.2.2 Travel Expenses. In order to foster exchange between the two groups and in particular the long-term stays of the PhD student in Paris, we apply for 7.000 EUR travel money per year for the German side. In total, we thus apply for 21.000 EUR for the German side. While this sum is very high, this will be used to cover the travel of members of the Münster node to Paris, and in particular the PhD students' stays in Paris where accommodation is very expensive. The price for single accommodation in the Villa Louis Pasteur, which offers temporary lodging to international researchers, is around 1.200 EUR per month. This is very cheap by Parisian standards, and consequently very sought-after. Even with optimal foresight and planning, we might have to explore more expensive alternatives on occasion. The travel money will also be used for workshop and conference attendance of the early career researchers, to enable them to integrate into the larger scientific community and present the work on the project.

5.1.2.3 Visiting Researchers (excluding Mercator Fellows). We apply for 1.000 EUR per year for inviting visitors to come to Münster, totalling 3.000 EUR for the German side.

5.1.2.4 Expenses for Laboratory Animals. None.

5.1.2.5 Other Costs. None.

5.1.2.6 Project-related Publication Expenses. None.

5.1.3 Instrumentation. None.

5.1.3.1 Equipment exceeding € 10,000. None.

5.1.3.2 Major Instrumentation exceeding € 50,000. None.

5.2 Module Temporary Position for Principal Investigator. None.

5.3 Module Replacement Funding. None.

5.4 Module Temporary Clinician Substitute. None.

5.5 Module Mercator Fellows. None.

5.6 Module Workshop Funding. We plan to hold a conference and a workshop. We plan to organize a conference taking place in France (CIRM or Paris), and a mini workshop taking place at the Landhaus Rothenberge of the University of Münster. Details for the mini workshop are given in this section, details for the conference can be found in section 5.17 below.

The idea of the mini workshop is foremost to bring everybody involved in the project together for an intense week focusing on the exchange of ideas, and to invite 3 or 4 special guests to also get inspiration from the wider community. Possible special guests include:

- Zoé Chatzidakis (Université Paris Cité)
- Artem Chernikov (Maryland)
- Philip Dittmann (TU Dresden)
- Nadja Valentin (HHU Düsseldorf)
- Will Johnson (Fudan University)
- Itay Kaplan (Hebrew University)
- Samaria Montenegro (University of Costa Rica)
- Silvain Rideau-Kikuchi (ENS PSL)
- Anna Rzepka (University of Silesia in Katowice)
- Mariana Vicaria (UCLA)
- Jinhe Ye (University of Oxford)

The format of the mini workshops will be modelled on American Institute of Mathematical Sciences workshops, with a select number of talks and long discussion sessions in small groups. We expect to hold the mini workshop within the first year of the project. We apply for 4.000 EUR for the mini workshop, to cover accommodation for all participants at the Landhaus Rothenberge, as well as travel expenses of the special guests.

5.7 Module Public Relations Funding. None.

5.8 Module Standard Allowance for Gender Equality Measures. First and foremost, if one of the early career researchers is the main carer of a baby during (part of) the project duration, we plan to support their attending conferences or making research visits. This may be by allocating extra funds to take both the baby and a carer to the events, but other flexible uses are also possible (in line with the DFG guidelines). Currently, the Cluster of Excellence "Mathematics Münster" provides such support, and if the Cluster is renewed, we would use the Cluster infrastructure if possible.

Secondly, we believe that the challenges – often through implicit biases – faced by female researchers as well as minorities in academia can only be overcome by raising awareness and developing suitable reactions. Thus, we plan to hold a half-day workshop on these challenges and possible resolutions during our mini-workshop at the Landhaus Rothenberge for all members of the project. A potential topic is "The bystander effect and effective interventions". Budget permitting, we will in addition offer individual coaching sessions to the female early career researchers involved in the project.

For these measures, we apply for 1.000 EUR per year, thus 3.000 EUR in total.

FRANCE. On the French side, all modules and funds are requested by: Anscombe, Sylvy.

5.10 French side – requested funds (from ANR).

5.12 Staff expenses. We apply for one position for a postdoctoral researcher for the duration of 24 months who will be based at Université Paris Cité. We expect the postdoc to visit the group in Münster regularly, and to spend at least 6 months in total in Münster (depending on family responsibilities). The position is expected to start in summer 2025. Depending on the expertise of the postdoc, they might be working on different parts of the proposal, e.g., $\mathbf{HC}_0$ or **MOT**. Potential candidates include:

- Leo Gitin (currently a PhD candidate at Oxford University)
- Sarah-Tanja Hess (currently a PhD candidate at University of Konstanz)
- Stefan Ludwig (currently a PhD candidate at ENS PSL)

- Anna de Mase (currently a postdoctoral researcher at University of Caserta)
- Margherita Pagano (currently a postdoctoral research at the University of Leiden)
- Floris Vermeulen (currently a PhD candidate at KU Leuven)

5.14 Instruments and material costs. We apply for funds for one laptop computer as well as one tablet for the postdoctoral researcher, to make virtual discussions as effective as possible. In total, we apply for the French node for 2.000 EUR.

5.15 Building and ground costs. None.

5.16 Outsourcing / subcontracting. None sought from the French side.

5.17 Overhead costs. We apply for 7.000 EUR of travel money per year from the ANR. This will be used in three ways: to cover the trips of the French members of the project to Germany, in particular for the postdoc's long-term stay(s) in Münster, for conference and workshop attendance to enable the early career researchers to integrate into the larger scientific community and to disseminate the results of the project, and for inviting other researchers for scientific collaboration. In total, we thus apply for 21.000 EUR for the French side.

Moreover, we plan to organize a conference around the topics of the project, with applications of model theory as its main focus. This conference will take place in France, preferably at the Centre International de Rencontres Mathématiques (CIRM). We will apply to the CIRM to organize a conference in 2026 as soon as the next call opens. The aim of the conference will be to bring together researchers working in all branches of model theory and related fields, but with a specific view towards new applications to and interactions with arithmetic geometry. We expect around 70 to 80 participants. The conference will in particular give the early career researchers involved in the project an opportunity to present their results and make an impact on the model theory community. Such opportunities are crucial for a successful career in academia.

Both applicants have experience in organizing international conferences: Franziska Jahnke co-organized large conferences on "Model Theory and Groups" in Münster in 2023, on "Model Theory of Valued Fields" at the CIRM in 2023, and one on "Model Theory of Valued Fields and Applications" in Münster in 2019. Sylvy Anscombe co-organized "Logique à Paris" in 2023, and "From Permutation Groups to Model Theory" at the ICMS Edinburgh in 2018. For the planned event, Jonathan Pila will be a third co-organizer, and he has agreed to join the scientific committee. Jonathan Pila's world-leading expertise on number-theoretic applications of model theory is complementary to ours.

CIRM part-funds their conferences, covering the accommodation costs for up to 40 participants, and we will apply for further funding from our home institutions to pay for speakers' travel. However, in order to support the participation of early career researchers, we apply for an additional 5.000 EUR from the French side. With this funding, we would be able to fully support 10 early career researchers.

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