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Hyperseries and surreal numbers

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Introduction

How fast can real functions grow at infinity? In order to make this question more precise, we will focus on germs of univariate real-valued functions at infinity.

Consider real-valued functions whose domain contains an interval $(a, +\infty)$ for some $a \in \mathbb{R}$. Identifying two such functions whenever they coincide on a common subinterval $(b, +\infty)$ of their respective domains, one obtains an equivalence relation whose classes are called germs (at $+\infty$). We say that the germ of f is strictly smaller than that of g if $f(t) < g(t)$ for sufficiently large $t \in \mathbb{R}$.

At the simplest level, we have germs of constant functions $0, 1, 2, \ldots$ as well as the germ x of the identity function, which is larger than all constant functions. Germs can be added and multiplied pointwise. This allows us to obtain the following growth rates

$$x+1 < x+2 < \cdots < 2\,x < 2\,x+1 < \ldots < x^2 < x^2+1 < \cdots.$$

The ordering on germs is only partial, because of the oscillatory behavior of some functions. For instance, the sine function sin cannot be compared with the constant function 0, whereas $\sin \cdot \exp(x)$ cannot be compared with x. Many operations on germs, such as addition and multiplication, are compatible with the partial ordering. For instance, the growth rates $x-1$, $\frac{x^2+1/2}{x-1}$, $\sqrt{2}\,x$ or $\sqrt{x}$ are all ordered as

$$0 < 1 < \cdots < \sqrt{x} < x-1 < x < x < \frac{x^2+1/2}{x-1} < x+1 < x+2 < \cdots < \sqrt{2}\,x < 2\,x.$$

If $f\colon (a, +\infty) \longrightarrow \mathbb{R}$ is differentiable, then the germ of its derivative only depends on the germ of f. Thus germs of differentiable functions can be differentiated as well. Finally, if g tends to $+\infty$ at $+\infty$, and $f\colon (a, +\infty) \longrightarrow \mathbb{R}$ is a function, then the germ of $f \circ g$ only depends on that of f and g, thus yielding a partial composition law on germs.

In order to avoid pathological growth rates of arbitrary functions, we turn to the notion of regular growth rates, i.e. we look for classes of germs of functions that are regular with respect to the structure on germs: continuity, differentiability, smoothness... In the context of real geometry, one may distinguish various types of regularity: continuous functions, analytic functions, quasi-analytic classes, germs in Hausdorff fields, Hardy fields, definable maps in tame expansions of $\mathbb{R}$...

The crudest level of regularity is that of continuous functions. An ordered field of germs of continuous functions is called a Hausdorff field. They include the field $\mathbb{R}(x)$ of (germs of) rational functions with real coefficients, as well as $\mathbb{R}(\log \log x)$. Germs in Hausdorff fields cannot oscillate at $+\infty$ (for instance the germ of the cosine function does not lie in a Hausdorff field) because the intermediate value theorem implies that oscillating functions have zeroes at $+\infty$ and thus cannot have a multiplicative inverse. Thus Hausdorff fields give us a first notion of regularity of the growth order of a function. Note however that $\mathbb{R}(x + 2\sin(x))$ is a Hausdorff field, which is isomorphic as an ordered field to $\mathbb{R}(x)$, despite the fact that $x + 2\sin(x)$ is not the germ of a monotone function.

A higher degree of regularity can be achieved with Bourbaki's notion of Hardy field [26]. A Hardy field is a field of continuously differentiable germs at $+\infty$ which is closed under derivation of germs. This excludes oscillatory behavior in a stronger sense. For instance, although the function $x^2 + \sin$ does not oscillate at $+\infty$, its second derivative does, so the germ of this function at infinity does not belong to a Hardy field. In particular, germs in Hardy fields and all their derivatives are monotone. Hardy considered *logarithmico-exponential functions*, or *L-functions*, as functions constructed from the identity function x and the real numbers using the field operations, exponentiation, and logarithms. He showed that the set $\mathcal{H}_{\mathrm{LE}}$ of germs of L-functions is a Hardy field. The regularity of germs in a Hardy field $\mathcal{H}$ can also be stated as a form of regularity of its derivation operator $f \mapsto f'$ with respect to the dominance relation, for instance, we have

$$\forall f \in \mathcal{H}, (f > \mathbb{N} \Longrightarrow f' > 0).$$

Even more regular growth rates can be obtained by imposing that the corresponding functions be definable in tame expansions of the real ordered field. The theory of o-minimality (see [32]) provides many such examples. Indeed, a first-order expansion $\mathcal{R} = (\mathbb{R}, +, \times, <, \ldots)$ of the real ordered field is o-minimal if and only if the set $\mathcal{H}_{\mathcal{R}}$ of germs of unary functions that are definable in $\mathcal{R}$ is a Hardy field. Furthermore $\mathcal{H}_{\mathcal{R}}$ is closed under composition of composable germs as well as under multivariate operations coming from the first-order language. In particular, taking $\mathcal{R}$ as the real exponential field $\mathbb{R}_{\exp} = (\mathbb{R}, +, \times, <, \exp)$ (resp. the real exponential field with restricted analytic functions), one obtains a large Hardy field $\mathcal{H}_{\exp}$ (resp. $\mathcal{H}_{\mathrm{an},\exp}$) containing $\mathcal{H}_{\mathrm{LE}}$ which is closed under composition.

One difficulty entailed by the study of regular growth rates is that of their representation. We have seen with L-functions how to obtain regular growth rates using algebraic operations and exp and log. However, there are gaps among regular growth rates that can be expressed in this form. For instance, writing $\exp_2(x) = \exp(\exp(x))$, $\exp_3 = \exp(\exp_2(x))$ and so on, Boshernitzan showed [25] that there are Hardy fields that contain germs f with

$$\exp(x) < \exp_2(x) < \exp_3(x) < \cdots < f. \tag{1}$$

Any such germ cannot be expressed in terms of elementary functions such as exp and log and algebraic operations. How can one represent such growth rates as f in a consistent way? Can a calculus of such growth rates, similar to that of L-functions, be proposed?

Working with the notions of regularity above also leads to difficulties of an analytic nature when trying to account for all possible regular growth rates. Many important problems in the field of tame geometry over the past few decades can be stated as a question of whether it is possible to close certain types of tame structure under certain operations. For instance, it is known [23] that Hardy fields can be extended into larger Hardy fields by integrals, exponentials, logarithms of germs, or in general by Pfaffian differential equations of order 1. Likewise, an o-minimal expansion of the real ordered field can be expanded with the real exponential function, or more generally with solutions of generalized Pfaffian equations, while remaining o-minimal [94]. Similar problems however remain difficult open questions. For instance, a long open question is whether there exists an o-minimal expansion of $(\mathbb{R}, +, \times, <)$ which defines a transexponential germ f as in (1). It is also unknown whether so-called Dulac germs, connected to Dulac's problem [38], can be defined in an o-minimal expansion of $(\mathbb{R}, +, \times, <)$. It was conjectured [5] that Hardy fields can be extended with solutions of all differential algebraic equations of odd degree, and a proof by the same authors is in preparation [8].

In this thesis, we will follow an alternative formal approach. Instead of growth rates of real-valued regular functions, we will consider formal objects, designed explicitly to be both closed under many operations and equations, and also to be expressible in terms of a definite list of "regular" operators such as exp, log, and field operations. This is the realm of generalized power series. Defining derivations $\partial: \mathbb{S} \longrightarrow \mathbb{S}$ and composition laws $\circ: \mathbb{S} \times \mathbb{S}^{>\mathbb{R}} \longrightarrow \mathbb{S}$ on an ordered field $\mathbb{S}$ of formal series containing the reals is a way to let those series act as regular, infinitely differentiable functions on $\mathbb{S}^{>\mathbb{R}}$.

Transseries, introduced independently by Dahn-Göring [30] and Écalle [39], are generalized series based on operators exp, log and arithmetic operations. They form a natural generalization of regular growth rates. As van der Hoeven's PhD thesis [60] illustrates, transseries are at the same time naturally closed under many operations and equations while being amenable to formal and algorithmic methods for solving equations or more general problems. The closure properties of transseries, e.g. under derivation and composition, are inherited from the regularity of the operators exp and log and the arithmetic operations. By extending this list of operators, one could in theory extend the list of formal growth rates amenable to a formal calculus, while retaining the properties of regular growth rates of real-valued functions. Hyperseries, as developed successively by Écalle [39], van der Hoeven [60], Schmeling [92] and van der Hoeven, van den Dries and Kaplan [33] are such extensions of transseries with so-called transfinite iterators $\exp_\alpha, \log_\alpha$ of exp and log for ordinals α, which have even stronger closure properties such as closure under conjugation (see [40, 10]). For instance the first transfinite iterators $\exp_\omega$ and $\log_\omega$ satisfy the equations

$$\begin{aligned} \exp_\omega \circ (x+1) &= \exp \circ \exp_\omega, \\ \log_\omega \circ \log &= \log_\omega - 1. \end{aligned}$$

Compared to regular growth rates of real-valued functions, hyperseries can be extended with new growth rates in a relatively simple and uniform way. In fact, van der Hoeven conjectured [60, Section 2.7] (see also [5, p. 14]) the existence of a large field **Hy** of hyperseries equipped with a derivation ∂ and a composition law $\circ$ that should be closed under many operations and equations considered when working with growth rates of regular real-valued functions. In particular, he conjectured that given a unary term $t(\cdot)$ in the first-order language of $(\mathbf{Hy}, +, \times, \partial, \circ)$ and $f < h$ in **Hy** such that $t(f)$ and $t(h)$ are defined, the following intermediate value property would hold:

$$t(f)\,t(h) < 0 \Longrightarrow \exists g\,(f < g < h \wedge t(g) = 0).$$

Such a field **Hy** would be an ultimate "field-with-no-escape". It is tempting to take all hyperseries in **Hy** to coherently subsume "all regular growth rates".

One remarkable aspect of the theories of transseries and hyperseries is that they form a far-reaching extension of the calculus of real numbers with infinite and infinitesimal quantities, and closure under exponentiation, logarithm, and transfinite summation. Even more remarkably, a similar calculus was proposed in a different area by Conway [28]. He introduced the class **No** of surreal numbers which extends the reals with infinitely large and infinitesimal quantities. Just as transseries, surreal numbers form a real-closed field with a notion of infinite summation. Moreover, Gonshor defined [55] an exponential function $\exp\colon \mathbf{No} \longrightarrow \mathbf{No}^{>0}$ on surreal numbers, illuminating to their similarity to transseries. Cantor's class **On** of ordinals $\omega, \omega+1, \omega^{\omega}, \ldots, \varepsilon_0, \ldots$ is naturally contained in **No**, which implies that seemingly exotic quantities such as

$$\frac{\omega_1^{\sqrt{2\omega+1}} - \omega - \omega^{1/3} - \omega^{1/5} - \cdots}{\exp(\sqrt{\varepsilon_0 - \omega})} \tag{2}$$

can be made sense of in **No**.

Surreal numbers possess two interesting and defining features. Firstly, they come with a linear ordering $<$ for which the surreal line **No** is set-wise saturated: given two sets L, R of surreal numbers with $L < R$, there is always a number $\{L \mid R\} \in \mathbf{No}$ with $L < \{L \mid R\} < R$. In other words, any gap in the surreal line can be filled by a surreal number. This form of completeness, which is difficult to obtain in rings of growth rates of real-valued functions or formal series, is a particularly desirable feature that suggests that all orders of infinity can be accounted for in **No**. Secondly, the number $\{L \mid R\}$ can be chosen "simplest", in an abstract sense, to lie between L and R. This yields a well-defined function $(L, R) \mapsto \{L \mid R\}$ which allows one to select simplest ways to fill holes between numbers in a manner that is coherent with a given algebraic structure on **No**. This allowed Gonshor to define his exponential function in a natural way. In this thesis, we will exploit this phenomenon to construct transfinite iterators of exp and log on **No**. It is remarkable [15] that in defining a distinguished function E on surreal numbers so as to satisfy $a < \exp(a) < \exp(\exp(a)) < \cdots < E(a)$ for large enough $a \in \mathbf{No}$, one also incidentally obtains a solution to the functional equation

$$E(a+1) = \exp(E(a))$$

of $\exp_{\omega}$, for all sufficiently large $a \in \mathbf{No}$. This suggests that surreal numbers and hyperseries are naturally related, and that the two ways of regular extending growth rates by solutions of equations and by filling gaps are connected in a beautiful manner over **No**.

During the past two decades, the class of surreal numbers became a prominent universal domain for several types of ordered algebraic structures, including ordered groups, ordered fields and models of the real exponential field. Van der Hoeven conjectured [63, p. 16] (see also [5, Conjecture 5.5] for a more precise statement) that surreal numbers are canonically isomorphic to **Hy**, the isomorphism being an evaluation map sending each hyperseries $f \in \mathbf{Hy}$ to its "value at ω" $f(\omega) \in \mathbf{Hy}(\omega) = \mathbf{No}$.

The main goal of this thesis is to prove this last conjecture. We do this by representing surreal numbers as hyperseries, i.e. as formal expressions involving a definite set of operations, of which we will provide a solid understanding. On the class of surreal numbers, we will see how to define (besides usual arithmetic operations) an infinite arity summation operator $\sum$, exponentials exp and logarithms log and transfinite iterators $\exp_\alpha$ and $\log_\alpha$ thereof, where α ranges in the class **On** of ordinals. We will see that such expressions as

$$\exp_{\omega^2}\left(\frac{\sqrt{2\,\omega^3} - \log(\omega) - \log_2(\omega) - \cdots}{\log_{\omega^\omega}(\omega)}\right) - 5\exp\left(\sqrt{\log\omega}\right)\omega$$

are legitimate surreal numbers, and that representing surreal numbers such as (2) in this manner is a good way to establish a link between surreal numbers and hyperseries and to allow one to compute with numbers as if they were regular growth rates. In establishing this representation, we will construct a field $\mathbf{Hy}(\omega)$ of hyperseries in ω, thus answering van der Hoeven's conjecture in the positive.

1 Toward an algebra of all regular growth rates

First systematic investigations of regular growth rates were made by Hardy [57, 58], based on earlier ideas by du Bois-Reymond [20, 21, 22]. Hardy proved that the set $\mathcal{H}_{\mathrm{LE}}$ of L-functions was a Hardy field and observed [57, p. 16] that "The only scales of infinity that are of any practical importance in analysis are those which may be constructed by means of the logarithmic and exponential functions.". In other words, Hardy suggested that the framework of L-functions not only allows for the development of a systematic asymptotic calculus, but that this framework is also sufficient for all "practical" purposes. However, as was later suspected by Hardy himself, the set $\mathcal{H}_{\mathrm{LE}}$ is not sufficient for all practical purposes, e.g. it is not closed under functional inversion [35, 60]. Part of Hardy's work can be interpreted as a search for an elusive algebra, which we denote $\mathbf{\Omega}$, of regular real-valued functions that would encompass all instances of prominent regular functions appearing in the literature.

We will discuss several possible instantiations of $\mathbf{\Omega}$ in the realm of growth rates by considering ways to extend Hardy fields with solutions of equations or with elements that fill certain gaps.

1.1 Hardy fields and algebraic differential equations

Many simple examples of Hardy fields, such as subfields of $\mathbb{R}$, the field $\mathbb{R}(x, \exp)$ generated by the germs of the identity function and the exponential function, and the set $\mathcal{H}_{\mathrm{LE}}$ of L-functions, share the property of being differentially algebraic [23, Lemma 3.7]. This means that each germ $f \in \mathcal{H}_{\mathrm{LE}}$ satisfies an algebraic differential equation

$$P(f, f', f'', \ldots, f^{(n)}) = 0,$$

where $P \in \mathbb{R}[X_0, \ldots, X_n]$ is a non-zero polynomial with real coefficients. On the other end of the spectrum, Zorn's lemma implies the existence of Hardy fields, called maximal Hardy fields, which are not properly contained in any larger Hardy field. Aschenbrenner, van den Dries and van der Hoeven conjectured [5] that all maximal Hardy fields satisfy an intermediate value theorem for differential polynomials: if $P \in \mathcal{H}[X_0, \ldots, X_n]^{\neq}$ and f, g are germs in a maximal Hardy field $\mathcal{H}$ such that

$$P(f, f', \ldots, f^{(n)})\, P(g, g', \ldots, g^{(n)}) < 0,$$

then there is a germ h in $\mathcal{H}$ lying between f and g, such that $P(h, h', \ldots, h^{(n)}) = 0$. Non maximal Hardy fields satisfying this property exist that can be embedded into fields of transseries [64]. Although maximal Hardy fields need not be closed under composition, this intermediate value property would make them good candidates for $\mathbf{\Omega}$ from the standpoint of asymptotic differential algebra.

1.2 Filling gaps in Hardy fields

One of du Bois-Reymond, Hardy and Hausdorff's goals was to fill gaps between sets of germs. The problem of filling gaps can be stated in the setting of Hardy fields as follows: given a Hardy field and countable subsets A and B of with $A < B$, is there a larger hardy field which contains a germ h with $A < h < B$? This question was recently answered in the positive by Aschenbrenner, van den Dries and van der Hoeven in an article in preparation [7]. One readily sees that this process of filling gaps invokes germs whose growth is very dissimilar to that of elementary functions such as L-functions. Let us single out three particular cases. Taking

$$(A,B) = (\varnothing, \{\log_n(x) : n \in \mathbb{N}\}),$$

one obtains a *sublogarithmic* solution, i.e. a germ h whose growth rate is smaller than that of any finite iterate of the logarithm. Taking

$$(A,B) = (\{\exp_n(x) : n \in \mathbb{N}\}, \varnothing),$$

one obtains *transexponential* solution, i.e. a germ h whose growth rate is greater than any finite iterate of the exponential function. Finally, taking

$$(A,B) = \left(\left\{ \sqrt{x}, \sqrt{x} + \mathrm{e}^{\sqrt{\log(x)}}, \sqrt{x} + \mathrm{e}^{\sqrt{\log(x) + \mathrm{e}^{\sqrt{\log_2(x)}}}}, \ldots \right\}, \left\{ 2\sqrt{x}, \sqrt{x} + \mathrm{e}^{2\sqrt{\log(x)}}, \sqrt{x} + \mathrm{e}^{\sqrt{\log(x) + \mathrm{e}^{2\sqrt{\log_2(x)}}}}, \ldots \right\} \right)$$

one obtains a "nested" solution h whose growth cannot be precisely approximated using finite combinations of exp and log. We will see when studying surreal numbers that filling these three types of gaps and generalizations thereof in the case of hyperseries is sufficient in order to obtain a full copy of the surreal numbers. In fact this can also be seen in the realm of germs in the work of Aschenbrenner, van den Dries and van der Hoeven [7].

1.3 Hyperexponentials and hyperlogarithms

In order to study and represent transexponential or sublogarithmic germs, it is convenient to single out specific such germs that would play the role of exp and log as building blocks for more complicated germs. Since transexponential and sublogarithmic germs do not appear as solutions of differential equations over elementary functions [24, Section 12], one must turn to more general functional equations in order to find examples. The simplest difference equation which generates a transexponential germ is Abel's equation for the exponential function, which is the following equation in y

$$y \circ (x+1) = \exp \circ\, y. \tag{1.1}$$

Kneser defined [66] an analytic and monotone function $\exp_\omega : \mathbb{R}^{\geqslant 0} \longrightarrow \mathbb{R}^{\geqslant 1}$ with $\exp_\omega(0) = 1$ and $\exp_\omega(r+1) = \exp(\exp_\omega(r))$ for all $r \in \mathbb{R}^{\geqslant 0}$, hence the germ of $\exp_\omega$ is a solution of (1.1). The function $\exp_\omega$ can be construed as a transfinite iterator of the exponential function, with $\exp_\omega(n) = \exp_n(1)$ for all $n \in \mathbb{N}$.

Conversely, the difference equation

$$y \circ \log = y - 1 \tag{1.2}$$

which can be obtained by formally inverting (1.1), generates a sublogarithmic germ. In fact the functional inverse $\log_\omega : \mathbb{R}^{\geqslant 1} \longrightarrow \mathbb{R}^{\geqslant 0}$ of $\exp_\omega$ is a solution of (1.2).

Boshernitzan showed [25] that $\mathbb{R}(\exp_\omega(x), \exp_\omega'(x), \exp_\omega''(x), \ldots)$ is a Hardy field. It is unknown whether the expansion $(\mathbb{R}, +, \times, <, \exp_\omega)$ (extending $\exp_\omega$ to be zero on $\mathbb{R}^{<0}$) of the real ordered field is o-minimal. A first step toward establishing a positive answer to this question is done in Padgett's thesis [83] where she constructed a Hardy field $\mathcal{H}$ containing x, closed under exp and log, as well as under $\exp_\omega$ and $\log_\omega$.

Extending Kneser's method, Schmeling constructed [92, Appendix A] analytic solutions $\exp_{\omega^2}$, $\exp_{\omega^3}$, *etc*. to the following equations:

$$\begin{aligned} \exp_{\omega^2} \circ (x+1) &= \exp_\omega \circ \exp_{\omega^2} && (1.3) \\ \exp_{\omega^3} \circ (x+1) &= \exp_{\omega^2}(\exp_{\omega^3}(x)) && (1.4) \\ &\vdots \end{aligned}$$

The fast growing germs $\exp_\omega(x)$, $\exp_{\omega^2}(x)$, ... are called *hyperexponentials*. Their functional inverses $\log_\omega(x)$, $\log_{\omega^2}(x)$, ... are called *hyperlogarithms* and they grow extremely slowly. All these germs are contained in a Hardy field.

Écalle studied analytic properties of very fast and slowly growing germs such as the transfinite iterators of exp and log. He introduced a systematic technique for the construction of quasi-analytic solutions to these and more general iteration equations [39]. After having introduced a subclass of **Hy** in the form of his "Grand Cantor" [39, Chapter 8], he proposed [40] an instantiation of $\mathbf{\Omega}$ in the form of his "natural growth scale". This is a group under composition of positive infinite germs, including transfinite iterators and exp and log, in which many functional equations involving compositions only should have solutions.

1.4 Levels

One important property of $\mathcal{H}_{\mathrm{LE}}$ or $\mathcal{H}_{\mathrm{an,exp}}$ is the classification of their *levels*. Levels were introduced by Rosenlicht [89, Section 2] in the case of Hardy fields of finite rank, by Écalle [39] in the case of transseries. They were later studied by Marker and Miller [79] in the case of fields $\mathcal{H}_{\mathcal{R}}$ for o-minimal expansions $\mathcal{R}$ of real-closed fields which define an exponential function (the definition of $\mathcal{H}_{\mathcal{R}}$ being similar as in the real case). We introduce them in the form of exp-log classes. Let us fix an ordered field $\mathbf{F} \supsetneq \mathbb{R}$ equipped with an isomorphism $\exp: \mathbf{F} \longrightarrow \mathbf{F}^{>}$ such that $(\mathbf{F}, +, \times, <, \exp)$ embeds into an elementary extension of $\mathbb{R}_{\exp}$. This means that the inclusion $\mathbb{R} \longrightarrow \mathbf{F}$ preserves all first-order properties in the language of ordered exponential rings. Then the exp-log *class* $\mathrm{EL}(a)$ of $a \in \mathbf{F}^{>\mathbb{R}}$ is its equivalence class $\mathrm{EL}(a) \subseteq \mathbf{F}^{>\mathbb{R}}$ for the relation

$$a \asymp^L b \Longleftrightarrow \exists n \in \mathbb{N}, (\log_n(a) - \log_n(b) \prec 1).$$

Writing $\mathcal{E}$ for the set $\{\exp_n \circ (\log_n \pm 1) : n \in \mathbb{N}\}$ of strictly increasing bijections $\mathbf{F}^{>\mathbb{R}} \longrightarrow \mathbf{F}^{>\mathbb{R}}$, we see that each $\mathrm{EL}(a)$ for $a \in \mathbf{F}^{>\mathbb{R}}$ is the convex hull in $(\mathbf{F}^{>\mathbb{R}}, <)$ of the class

$$\mathcal{E}a := \{g(a) : g \in \mathcal{E}\}.$$

Exp-log classes are linearly ordered by universal comparison $\mathrm{EL}(a) < \mathrm{EL}(b)$. For $a, b \in \mathbf{F}^{>\mathbb{R}}$, we have

$$\mathrm{EL}(a) < \mathrm{EL}(b) \Longleftrightarrow \mathcal{E}a < \mathcal{E}b \Longleftrightarrow \mathcal{E}a < b \Longleftrightarrow a < \mathcal{E}b.$$

This type of inequalities concerning convex equivalence relations frequently appears when studying germs, hyperseries, and especially surreal numbers, which are ideally suited to investigate them (see in particular Chapters 10 and 11).

Consider a Hardy field $\mathcal{H}$ containing x, closed under exp and log, as well as under $\exp_\omega$ and $\log_\omega$. Write $\lambda_n := \mathrm{EL}(\exp_n(x))$ and $\lambda_{-n} := \mathrm{EL}(\log_n(x))$ for all $n \in \mathbb{N}$. Note that the set $\mathcal{E}$ is contained in $\mathcal{H}^{>\mathbb{R}}$. The exp-log classes of $\exp_\omega(x)$ and $\log_\omega(x)$, respectively, suggestively denoted λ_ω and $\lambda_{-\omega}$, satisfy

$$\lambda_{-\omega} < \lambda_n < \lambda_\omega$$

for all $n \in \mathbb{Z}$.

Further "transfinite" exp-log classes $\lambda_{\omega-1} := \mathrm{EL}(\log(\exp_\omega(x)))$, $\lambda_{\varepsilon_0} := \mathrm{EL}(\exp_\omega(\exp_\omega(x)))$, $\lambda_{-\omega 2} := \mathrm{EL}(\log_\omega(\log_\omega(x)))$, *etc*, can be defined within $\mathcal{H}$.

Given $r \in \mathbb{R}$, the exp-log class λ_r of the function $\exp^{[r]}(x) := \exp_\omega(\log_\omega(x) + r)$ satisfies

$$\forall r, s \in \mathbb{R}, r < s \Longleftrightarrow \lambda_r < \lambda_s.$$

Indeed, one can see that $(\mathbb{R}, +, 0, <) \longrightarrow (\mathcal{H}^{>\mathbb{R}}, \circ, x, <); r \mapsto \exp^{[r]}(x)$ is an embedding of ordered monoids with $\exp^{[1]}(x) = \exp(x)$, and in particular that all $\exp^{[r]}(x)$ for $r \in \mathbb{R}$ commute. We can readily construct other intermediate levels in $\mathcal{H}$ by noting that given $\varphi, \psi \in \mathcal{H}^{>\mathbb{R}}$ with

$$\varphi - \psi \succ \frac{1}{\log_n(\exp_\omega(\varphi))} \quad \text{for all } n \in \mathbb{N},$$

the exp-log classes of $\exp_\omega \circ \varphi$ and $\exp_\omega \circ \psi$ are distinct. Indeed one can show that the sets

$$\varphi \pm \frac{1}{\log_n(\exp_\omega(\varphi))} \qquad \text{and} \qquad (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \varphi$$

have the same convex hull. Those relations, which follow from the asymptotics of Abel's equation and basic properties of smooth functions allow us to construct "infinitesimal" exp-log classes such as

$$\lambda_{1/\omega} := \mathrm{EL}\left(\exp_\omega\left(\log_\omega(x) + \frac{1}{\log_\omega(x)}\right)\right).$$

which indeed satisfies $\lambda_0 < \lambda_{1/\omega} < \lambda_r$ for all $r \in \mathbb{R}^>$. To this date, there is no known o-minimal expansion of $(\mathbb{R}, +, \times, <)$ which defines a germ $f > \mathbb{R}$ with non-integer level [79]. One can therefore say that there is no known o-minimal instantiation of $\mathbf{\Omega}$. In contrast, as we will see, it is not too difficult to construct fields of hyperseries which account for infinite or infinitesimal levels.

2 Formal series

Formal series can be used as formal asymptotic expansions of certain germs or as easy-to-construct models of certain first-order theories. The particular kind of series which is relevant in our case are well-based series or Hahn series as per [56], over $\mathbb{R}$, because well-based series are subject to many formal operators and algorithms [82, 92, 62] which can be used to define operations and solve equations that appear in the context of Hardy fields.

Given a multiplicatively denoted Abelian, linearly ordered group $(\mathfrak{M}, \times, 1, \prec)$, the field $\mathbb{R}[[\mathfrak{M}]]$ of well-based series over $\mathbb{R}$ with monomial group $\mathfrak{M}$ is the set of functions $f: \mathfrak{M} \longrightarrow \mathbb{R}$ whose support

$$\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{M} : f_\mathfrak{m} \neq 0\}$$

is well-ordered for the reverse ordering $\succ$ on $\mathfrak{M}$ (see Chapter 1 for more details and an extension of the definition for class-sized monomial groups $\mathfrak{M}$). Each function $f \in \mathbb{R}[[\mathfrak{M}]]$ is represented as a formal sum

$$f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_\mathfrak{m}\, \mathfrak{m}$$

where $f_\mathfrak{m} := f(\mathfrak{m}) \in \mathbb{R}$ and the field operations as defined by Hahn are as the pointwise sum

$$f + g := \sum_{\mathfrak{m} \in \mathfrak{M}} (f_\mathfrak{m} + g_\mathfrak{m})\, \mathfrak{m},$$

and the Cauchy product

$$f + g := \sum_{\mathfrak{u}, \mathfrak{v} \in \mathfrak{M}} (f_\mathfrak{u}\, g_\mathfrak{v})\, \mathfrak{u}\, \mathfrak{v}.$$

Fields of well-based series come equipped with a dominance relation

$$f \prec g \Longleftrightarrow \max \operatorname{supp} f \prec \max \operatorname{supp} g$$

and an asymptotic equivalence

$$f \asymp g \Longleftrightarrow \max \operatorname{supp} f = \max \operatorname{supp} g$$

for non-zero $f, g \in \mathbb{R}[[\mathfrak{M}]]$. An important feature of fields of well-based series for us is that one can define a notion of transfinite sum of certain families of series [82, 59]. That is, given a field of well-based series $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ and a family $(f_i)_{i \in I} \in \mathbb{T}^I$, we can sometimes define its sum $\sum_{i \in I} f_i$ as a well-based series in $\mathbb{T}$, which is useful in defining operations on $\mathbb{T}$. An $\mathbb{R}$-linear function $\Psi: \mathbb{T} \longrightarrow \mathbb{T}$ which preserves summability and commutes with the summation operator $\sum: (f_i)_{i \in I} \mapsto \sum_{i \in I} f_i$ is said to be *strongly linear*. Let us see how well-based series over $\mathbb{R}$ relate to the investigation of regular growth rates by describing partial constructions of $\mathbf{Hy}$ in the literature.

2.1 Transseries

Elementary germs, such as L-functions, are nothing but finite combinations of exp, log, and polynomial or semi-algebraic functions. Accordingly there should exist a representation of such germs as formal combinations of terms e^x, $\log x$ and algebraic expressions. Dahn and Göring [30] and Écalle [39] introduced formal series, called transseries, that serve this purpose. Transseries are specific types of well-based series with real coefficients. Each transseries is constructed from a variable x taken as a generic positive infinite element and from the real numbers, using exponentiation, logarithms, and *infinite* sums. One example is

$$\mathrm{e}^{\mathrm{e}^x+\mathrm{e}^{x/2}+\mathrm{e}^{x/3}+\cdots}-3\,\mathrm{e}^{x^2}+5\,(\log x)^{\pi}+42+x^{-1}+2\,x^{-2}+6\,x^{-3}+24\,x^{-4}+\cdots+\mathrm{e}^{-x}.$$

Transseries form an ordered, valued, exponential field $(\mathbb{T}_{\mathrm{LE}},+,\times,<,\prec,\exp)$ with extra structure called the field of logarithmic-exponential (or log-exp) transseries. The field $\mathbb{T}_{\mathrm{LE}}$ was first defined by Dahn and Göring [30] in an effort to study Tarski's problem on the real ordered exponential field $\mathbb{R}_{\exp}=(\mathbb{R},+,\times,<,\exp)$: is the elementary theory of $\mathbb{R}_{\exp}$ decidable? Tarski's problem is still open, and has been reduced [93] to certain open (and considered very difficult) number theoretic conjectures; in particular the *Weak Schanuel's Conjecture* [76].

Écalle proposed [39] a wide ranging analytic theory of *accelero-summation* which relates transseries to large classes of quasi-analytic real-valued functions called *analyzable functions*. This lead in particular to his proof of Dulac's conjecture about the finiteness of limit cycles of polynomial vector fields.

The subfield $\mathbb{T}_g$ of $\mathbb{T}_{\mathrm{LE}}$ of so-called grid-based transseries, among others, was studied by Joris van der Hoeven, after Écalle [60], in order to set a framework that is rich in methods to analyze the asymptotic behaviors of real-valued functions, such as differentially algebraic functions, that naturally occur in geometry and analysis. This work allowed for a weaker but simpler method than Écalle's accelero-summation in order to embed fields of transseries into Hardy fields [64]. Van der Hoeven later showed that $\mathbb{T}_g$ satisfies the intermediate value theorem for differential polynomials [63, Theorem 9.33].

Transseries are interesting in particular for their rich structure besides the field operations and exponentiation. Indeed the field $\mathbb{T}_{\mathrm{LE}}$ is equipped with a canonical derivation $\partial\colon\mathbb{T}_{\mathrm{LE}}\longrightarrow\mathbb{T}_{\mathrm{LE}}$ [36, 60], as well as with a composition law $\circ\colon\mathbb{T}_{\mathrm{LE}}\times\mathbb{T}_{\mathrm{LE}}^{>\mathbb{R}}\longrightarrow\mathbb{T}_{\mathrm{LE}}$ [36, 60], which extends the composition of rational functions in the variable x and shares a few first-order properties with the composition law on Hardy fields that are closed under composition. In fact, there is a natural embedding of $\mathcal{H}_{\exp,\mathrm{an}}$ into $\mathbb{T}_{\mathrm{LE}}$ that sends the germ x of the identity to x and preserves the derivations and composition laws [36, Corollaries 3.12 and 6.30]. The image of the same embedding is contained in the subfield $\mathbb{T}_g$. As such those fields of transseries can be used as formal counterparts to certain Hardy fields, with the advantage of lending themselves to formal methods and algorithms to solve equations and inequations.

However, transseries are by their definition insufficient to describe the asymptotic behavior of functions involving hyperexponentials and hyperlogarithms. In fact, the exp-log classes in $\mathbb{T}_g$ and $\mathbb{T}_{\mathrm{LE}}$ are also parametrized by integers, and thus transseries very broadly fail to approximate germs whose levels are infinite, fractional and infinitesimal as we mentioned above.

2.2 Model theory of transseries

In a more model theoretic vein, van den Dries, Macintyre and Marker took interest [36] in transseries as a natural non-standard model for certain expansions of the real ordered field, including $\mathbb{R}_{\exp}$ and $\mathbb{R}_{\mathrm{an},\exp}$. Indeed it is a by-product of their previous work [34] on the theory of $\mathbb{R}_{\mathrm{an},\exp}$ that $\mathbb{R}_{\mathrm{an},\exp}$ is an elementary submodel of any field of well-based series equipped with a well-behaved exponential function.

With Aschenbrenner, van den Dries studied the elementary properties of H-fields, which are ordered, valued, differential fields designed to abstractly represent Hardy fields. They focused in particular on the existence of closures of such models under certain differentially algebraic equations [2, 3]. Since transseries are naturally closed under many operations, they are a prominent instance of H-fields.

The encounter between Aschenbrenner, van den Dries and van der Hoeven lead to a fruitful cooperation, combining formal analytic methods and model theory. They set a research program [5] toward conjunctively understanding the elementary theory T_{LE} of $(\mathbb{T}_{\mathrm{LE}}, +, \times, <, \prec, \partial)$ and that of maximal Hardy fields. This motivates many recent projects and already yielded the major results of recursive axiomatization of T_{LE}, model completeness, quantifier elimination in an extended language [4].

The model theory of $(\mathbb{T}_{\mathrm{LE}}, +, \times, \circ)$ is much less tame. This stems from the fact that many simple functional equations in $\mathbb{T}_{\mathrm{LE}}$ lack transseries solutions. An important example of an equation without solution in $\mathbb{T}_{\mathrm{LE}}$ is the functional equation in f

$$f = \sqrt{x} + \mathrm{e}^{f \circ \log x} \qquad \text{with the condition } f \sim \sqrt{x}. \tag{2.1}$$

One can conceive [60] natural "syntactic" solutions

$$f_s = \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}} \tag{2.2}$$

to (2.1). It is plausible [61, Section 1.4] that the equation have quasi-analytic solutions that could be partially described by the nested expansion (2.2). In order to understand this type of object and formal expansion, van der Hoeven introduced the notion of abstract transseries [61].

2.3 Abstract transseries

Going beyond these results requires tools to be able to produce formal, transserial models for various fields of real-valued functions beyond the spectrum of exponential-logarithmic-analytic germs. It became clear that this implies extending the range $\{\mathbb{T}_{\mathrm{LE}}, \mathbb{T}_g\}$ of admissible fields of transseries to more abstract fields of transseries as defined by van der Hoeven [61]. This was accomplished by Schmeling in his thesis [92]. Transseries fields are fields of well-based series $\mathbb{T}$ equipped with a logarithm log which is a morphism $(\mathbb{T}^{>0}, \times, <) \longrightarrow (\mathbb{T}, +, <)$ that shares certain key features with the logarithm $\log = \exp^{\mathrm{inv}}$ on $\mathbb{T}_{\mathrm{LE}}$. One specific restriction, denoted axiom **T4** in [92, Definition 2.2.1] (see also Section 11.2.2), pertains to nested expansions such as (2.2). In particular, Schmeling showed how to to construct transseries fields containing f_s.

Transseries fields can be extended so that the logarithm becomes surjective. Such fields $\mathbb{T}$ are equipped with an external composition law $\circ : \mathbb{T}_{\mathrm{LE}} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$. Schmeling showed how to define derivations and compositions on transseries fields, and how to extend them when closing under exponentials. These results are related to S. Kuhlmann's work on fields of generalized series. Indeed Kuhlmann and her co-authors F-V. Kuhlmann, Matusinski, Shelah, and Tressl, showed how to construct fields of well-based series equipped with logarithms and exponentials [68, 67], how to distinguish between several types of generalized transseries [72], and how to extend derivations when closing such fields under exponentials [70].

2.4 Hyperseries

Despite the excellent closure properties of transseries for the resolution of differential equations, the functional equation (1.1) does not have a transseries solution. In order to establish a universal formal framework for asymptotic calculus, we therefore need to incorporate extremely fast growing formal counterparts E_ω, E_{ω^2}, $E_{\omega^3}, \ldots$ to the functions $\exp_\omega, \exp_{\omega^2}, \exp_{\omega^3}, \ldots$ The first construction of a field of generalized transseries that is closed under E_{ω^n} and L_{ω^n} for all $n \in \mathbb{N}$ was accomplished in [92]. The hyperlogarithms L_ω, L_{ω^2}, *etc.* satisfy the functional equations

$$\begin{aligned} L_\omega(L_1(x)) &= L_\omega(x) - 1 \\ L_{\omega^2}(L_\omega(x)) &= L_{\omega^2}(x) - 1 \\ L_{\omega^3}(L_{\omega^2}(x)) &= L_{\omega^3}(x) - 1 \\ &\vdots \end{aligned} \tag{2.3}$$

In addition, we have a simple formula for their derivatives

$$L_\alpha(x)' = \prod_{\beta<\alpha} \frac{1}{L_\alpha(x)}, \qquad (2.4)$$

where $\alpha \in \{1, \omega, \omega^2, \ldots\}$ and

$$L_{\omega^n k_n + \cdots + \omega k_1 + k_0}(x) := L_1^{\circ k_0}(L_\omega^{\circ k_1}(\cdots (L_{\omega^n}^{\circ k_n}(x)) \cdots))$$

for all $n \in \mathbb{N}$ and $k_0, \ldots, k_n \in \mathbb{N}$.

A second construction of such a field, for $n=1$ was done in Padgett's thesis [83], where she defines fields M containing well-based series, which are closed under exp, log, L_ω, E_ω, as well as under a natural derivation $\partial: M \longrightarrow M$. Padgett's framework is distinct from ours in that she sometimes only allows finite sums in her series constructions, however we expect that our methods are compatible, and that her fields should be embeddable in the fields of hyperseries we will mention later.

The formula (2.4) is eligible for generalization to arbitrary ordinals α. Taking $\alpha = \omega^\omega$, we note that the function L_{ω^ω} does not satisfy any functional equation. Yet any truly universal formal framework for asymptotic calculus should accommodate functions such as L_{ω^ω} for the simple reason that it is possible to construct models with good properties in which they exist. For instance, by [25, 7], there exist Hardy fields with infinitely large functions that grow more slowly than L_{ω^n} for all $n \in \mathbb{N}$.

An advantage of hyperseries over their geometric counterparts is that the formal setting (and as we will see the surreal setting) allows one to single out "simplest" operators in each growth class of operators on hyperseries. For instance, in stark contrast with the vast number of possible solutions of Abel's equations (1.1-1.2), there is one distinguished hyperexponential function L_ω of strength ω, and other solutions $s \mapsto L_\omega(s) + r, r \in \mathbb{R}$ of (2.3) in hyperseries are expressed using this simplest one.

2.5 Logarithmic hyperseries

The construction of the field $\mathbb{L}$ of *logarithmic hyperseries* in [33] was the first step toward the incorporation of hyperlogarithms L_α with arbitrary α. The field $\mathbb{L}$ is the smallest non-trivial field of generalized power series over $\mathbb{R}$ that is closed under all hyperlogarithms L_α and infinite real power products. It turns out that $\mathbb{L}$ is a proper class and that $\mathbb{L}$ is closed under differentiation, integration, and composition. One remarkable feature of $\mathbb{L}$ is that its construction is relatively simple: it is simply a field of well-based series $\mathbb{L} = \mathbb{R}[[\mathfrak{L}]]$ where $\mathfrak{L}$ is the group under pointwise multiplication of formal expressions

$$\mathfrak{l} = \prod_{\gamma<\rho} \ell_\gamma^{\mathfrak{l}_\gamma}, (\mathfrak{l}_\gamma)_{\gamma<\rho} \in \mathbb{R}^\rho$$

for $\rho \in \mathbf{On}$. The terms ℓ_γ for $\gamma \in \mathbf{On}$ correspond to $L_\gamma(x)$. Possibly transfinite sums as above are required so as to allow within the relation

$$\ell_\rho' = \prod_{\gamma<\rho} \ell_\gamma^{-1}$$

which extends (2.4).

The derivation $\partial: \mathbb{L} \longrightarrow \mathbb{L}; f \mapsto f'$ on $\mathbb{L}$ is defined on monomials *via* an infinite Leibniz rule and extended to $\mathbb{R}[[\mathfrak{L}]]$ by strong linearity. This gives a surjective derivation for which $(\mathbb{L}, \partial)$ is an H-field in the sense of [2].

Although $\mathbb{L}$ contains no formal functional inverses for hyperlogarithms ℓ_γ, certain equations $y \circ \ell_\gamma = f$ for given $f \in \mathbb{L}$ have unique solutions in $\mathbb{L}$ that are then denoted $f^{\uparrow \gamma}$ (see Section 4.1.4). The composition law $\circ: \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ is characterized by imposing the functional equations

$$\ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} - 1$$

for $\mu \in \mathbf{On}$ and fixing the simplest values of $\ell_\rho^{\uparrow \gamma}$ for $\gamma \leqslant \rho \in \mathbf{On}$. As for the interaction between the derivation and the composition law, we have the chain rule

$$\forall f \in \mathbb{L}, \forall g \in \mathbb{L}^{>\mathbb{R}}, (f \circ g)' = g' \cdot (f' \circ g),$$

and formal Taylor expansions around each point $s \in \mathbb{L}^{>\mathbb{R}}$ for each $f \in \mathbb{L}$. That is, for all δ in $\mathbb{L}$ with $\delta \prec s$, we have

$$f \circ (s + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ s}{k!} \delta^k$$

where $f^{(k)} = \partial^k(f)$ is the k-th derivative of f and $\sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ s}{k!} \delta^k$ is the sum of the corresponding summable family.

A first step in extending the work on logarithmic hyperseries would be to define a derivation and a composition law on an extension $\tilde{\mathbb{L}}$ of $\mathbb{L}$ which also takes hyperexponentials into account. Such work goes beyond the scope of the thesis but has been separately accomplished [10].

3 Numbers

The standard notion of number in order to represent geometric magnitudes is that of real number. The standard notion of number in order to represent set-theoretic or order-theoretic magnitudes is Cantor's notion of ordinal number. Whereas real numbers are the domain of the finite, ordinal numbers are designed to account for infinite quantities.

Dating to Antiphon-Eudoxus-Archimedes with the method of exhaustion, through Leibniz' infinitesimal calculus and Newton's fluxions, the inclusion of infinitesimal or infinite quantities in real calculus has posed several problems and paradoxes. The first rigorous treatment of finite, infinite and infinitesimal quantities in a unified context was Robinson's non-standard analysis where infinitesimals and infinite elements are introduced using ultrafilters [86, 87], and later Nelson's internal set theory [81] (see [43] for a more detailed discussion of the history of infinite and infinitesimal quantities in mathematics). These extensions $^*\mathbb{R}$ of $\mathbb{R}$ in non-standard analysis are called fields of hyperreal numbers. A crucial feature of hyperreal numbers is that they satisfy Robinson's transfer principle, a far reaching generalization and formalisation of Leibniz' law of continuity (see [27, Section 4.4]).

Using a simple generalization of both Dedekind's definition of real numbers and von Neumann's presentation of ordinal numbers, Conway proposed the unified setting of surreal numbers in his monograph *On Numbers and Games* [28]. The class **No** of surreal numbers encompasses both real and ordinal numbers, while allowing for ways to distinguish between quantities in an elegant way because of the way each number can be given a specific name or presentation, in various ways. Although surreal numbers do not enjoy an explicit transfer principle as strong as Robinson's, we will see that they do elementarily extend important first-order structures with tame properties, such as the real ordered field, the real exponential field, and the ordered valued differential field $\mathbb{T}_{\mathrm{LE}}$ of log-exp transseries. Crucially, the fact that one can name each surreal number is what will make it possible to intrinsically define distinguished hyperexponentials and hyperlogarithms on **No**, where doing so on $^*\mathbb{R}$ would, it seems, entail arbitrary choice or some reliance on a choice of corresponding germs in $\mathbb{R}$.

Let us explain how the field of surreal numbers came into prominence as a universal domain for transseries and hyperseries.

3.1 Surreal forms

Surreal numbers are abstract quantities, containing an unfathomably wide array of magnitudes that are amenable to a large number of surreal operations. This sentiment is summed up by Conway's phrase, that surreal numbers contain "All numbers great and small". Conway defines surreal numbers as abstract forms $\{L \mid R\}$ construed as the "simplest" objects lying between sets L and R of previously defined surreal numbers, with $L < R$. More precisely, Conway's definition is inductive in essence, and relies on a mutual inductive definition of numbers and their ordering $<$, according to which each number is characterized by the gap $\{L \mid R\}$ it fills in the class **No** of surreal numbers. Thus surreal numbers arise from playing a simple game of filling gaps inductively, starting with the empty setting $(L, R) = (\varnothing, \varnothing)$. Quite remarkably, the operational structure which, as we will see throughout the thesis, emerges from this construction, is very rich and closely related to the task of constructing large fields of formal series with compositions and derivations.

3.2 The field of surreal numbers

The main elements of the theory of surreal numbers were established by Conway. Conway showed how to define a sum and product of surreal numbers in a very simple way using the inductive definition of surreal numbers as gaps $\{L \mid R\}$. Indeed Conway's definition of the sum of numbers $x = \{L_x \mid R_x\}$ and $y = \{L_y \mid R_y\}$ is

$$x + y = \{L_x + y, x + L_y \mid x + R_y, R_x + y\} \tag{3.1}$$

where the operations involved between the brackets only involve strictly "simpler" surreal numbers, and are thus warranted by induction. One gets used to manipulating this type of inductive definition, and soon discovers that many interesting functions can be defined using the right inductive definitions. The operation defined by (3.1) turns **No** into a divisible ordered Abelian group. In a similar fashion, Conway defined a product on **No** and showed that **No** is a real-closed field which caonnically contains the real ordered field $\mathbb{R}$ as well as the ordered semi-group **On** of ordinals under their commutative Hessenberg operations.

The first enquiries about surreal numbers after Conway were centered around the representation of numbers as well-based series. Indeed Conway [28] showed (see also Gonshor [55]) that the ordered field of surreal numbers is canonically isomorphic to a field $\mathbb{R}[[\mathbf{Mo}]]$ of well-based series over $\mathbb{R}$, whose group of monomials **Mo** is a subgroup of $(\mathbf{No}^{>0}, \times, <)$. This gives a representation of each surreal number a as a formal series

$$a = \sum_{\mathfrak{m} \in \mathbf{Mo}} a_{\mathfrak{m}} \, \mathfrak{m}.$$

The series representation also allows one to use valuation theory to define embeddings of certain ordered algebraic structures into **No**. For instance, Ehrlich characterized over the years [42, 45, 46] the type of ordered algebraic structures that could be embedded into **No** while preserving the inductive definition of operations (i.e. as so-called initial subclasses of **No**).

3.3 Exponentiation on surreal numbers and transseries

Kruskal got interested in leveraging the expressive nature of surreal numbers in order to make sense of asymptotic expansions of certain regular functions, such as germs in Hardy fields. For instance, identifying the ordinal $\omega \in \mathbf{No}$ with the germ of the identity, any possibly divergent Laurent series $\sum_{n \geqslant N} r_n \, z^n$ with real coefficients gives rise to a well-defined surreal number $\sum_{n \geqslant N} r_n \, \omega^{-n} \in \mathbf{No}$. The reader will note that certain fast or slowly growing functions such as the real exponential and logarithm cannot be approximated by Laurent series in z^{-1}. Thus Kruskal's project requires at least the existence of an exponential function and a logarithm on surreal numbers. Using hints from Kruskal, Gonshor defined [55, Chapter 10] an exponential function exp which is an isomorphism $(\mathbf{No}, +, <) \longrightarrow (\mathbf{No}^{>0}, \times, <)$, and it was later shown by van den Dries and Ehrlich [96] that $(\mathbb{R}, +, \times, \exp, <)$ is an elementary substructure of $(\mathbf{No}, +, \times, \exp, <)$.

The existence of a well-behaved logarithm function $\log = \exp^{\mathrm{inv}}$ on **No** and the identification $\mathbf{No} = \mathbb{R}[[\mathbf{Mo}]]$ make **No** an ideal candidate for a large transseries field as per [92]. The question of the precise structural properties of $(\mathbb{R}[[\mathbf{Mo}]], \log)$ was first investigated by Kuhlmann and Matusinski [71]. One of the important steps of this task is to identify the class $\mathbf{Mo}_\omega$ of log-atomic elements of $(\mathbb{R}[[\mathbf{Mo}]], \log)$. Those were introduced by van der Hoeven in the case of transseries, and they can be defined in **No** as infinite monomials $\mathfrak{a} \in \mathbf{Mo}^{>1}$ such that $\log_n(\mathfrak{a}) \in \mathbf{Mo}$ for all $n \in \mathbb{N}$. Such numbers include ω, $\log_n(\omega)$ and $\exp_n(\omega)$ for all $n \in \mathbb{N}$, but also many other numbers which accordingly cannot be expressed purely as combinations of exponentials and logarithms, and algebraic expressions in ω. Berarducci and Mantova later identified $\mathbf{Mo}_\omega$ and showed that there was a canonical strictly increasing function

$$\lambda \colon \mathbf{No} \longrightarrow \mathbf{Mo}_\omega; z \mapsto \lambda_z$$

such that each positive infinite surreal number $a\in\mathbf{No}^{>\mathbb{R}}$ lies in the exp-log class of a unique λ_z for $z\in\mathbf{No}$. Note that this gives a tentative conclusion to our discussion on levels, by showing that the set of levels in an ordered exponential field can be as large as the class **No** itself. A similar result for a coarser notion of rank than that of levels was obtain by S. Kuhlmann and M. Matusinski [71], instantiating a previous result of Kuhlmann [69, Theorem 5.12]. Using this, Berarducci and Mantova were able to prove that $(\mathbb{R}[[\mathbf{Mo}]],\log)$ is a transseries field as per [92] by showing that it satisfies the axiom **T4** [18, Theorem 8.4]. Finally, Ehrlich and Kaplan showed [46, Theorem 8.1] that all transseries fields in the sense of [92] can be embedded into **No**, thus showing that **No** is the ultimate field of transseries.

3.4 Derivations and compositions on surreal numbers

If surreal numbers are to be isomorphic to the field **Hy**, they should be amenable to basic operations that can be performed on growth rates. In particular, there should exist a derivation $\partial\colon\mathbf{No}\longrightarrow\mathbf{No}$ and a composition law $\circ\colon\mathbf{No}\times\mathbf{No}^{>\mathbb{R}}\longrightarrow\mathbf{No}$ with respect to which numbers could be seen as infinitely differentiable surreal-valued functions. The derivation should be strongly linear with kernel $\mathbb{R}$, satisfy Leibniz' rule

$$\forall a,b\in\mathbf{No},\partial(a\,b)=\partial(a)\,b+a\,\partial(b),$$

and be compatible with the exponential in the sense that we have

$$\forall a\in\mathbf{No},\partial(\exp(a))=\partial(a)\exp(a).$$

Since ω is taken to represent the identity function or the generic variable x in transseries, the function ∂ should be a derivation with respect to ω, i.e. $\partial(\omega)=1$. The composition law should be associative

$$\forall a\in\mathbf{No},\forall b,c\in\mathbf{No}^{>\mathbb{R}},a\circ(b\circ c)=(a\circ b)\circ c,$$

positive infinite numbers $a\in\mathbf{No}^{>\mathbb{R}}$ should give rise to strictly increasing surreal valued functions

$$\forall a\in\mathbf{No}^{>\mathbb{R}},\forall b,c\in\mathbf{No}^{>\mathbb{R}},b<c\Longrightarrow a\circ b<a\circ c,$$

for each $b\in\mathbf{No}^{>\mathbb{R}}$, the function $\mathbf{No}\longrightarrow\mathbf{No};a\mapsto a\circ b$ should be a strongly linear morphism of rings, and we should have a chain rule

$$\forall a\in\mathbf{No},\forall b\in\mathbf{No}^{>\mathbb{R}},\partial(a\circ b)=\partial(b)\times(\partial(a)\circ b)$$

with respect to the derivation ∂. One expects that a sound definition of $(\partial,\circ)$ on **No** will yield a structure with good first-order properties, and in particular the intermediate value theorem conjectured for **Hy**.

Using their characterization of **No** as a transseries field, and Schmeling's method for defining derivations on transseries fields, Berarducci and Mantova defined [18] a derivation ∂ with respect to ω on **No**, in such a way that it is the "simplest" (see [18, Theorem 9.6]) derivation such that $(\mathbf{No},+,\times,<,\prec,\partial)$ is an H-field. In fact this derivation has good model theoretic properties, since $(\mathbf{No},+,\times,<,\prec,\partial)$ is an elementary extension of $\mathbb{T}_{\mathrm{LE}}$ [6].

However, Berarducci and Mantova showed [19, Theorem 8.4] that there is no composition law on **No** that is compatible with ∂. In our view [5], this is due to the fact that the definition of ∂ is irrespective of the natural structure of field of hyperseries that we seek to define on **No**. In particular, it does *not* satisfy $\partial(E_\omega(a))=\partial(a)\,E_\omega'(a)$ for all $a\in\mathbf{No}^{>\mathbb{R}}$, for our definition of the first hyperexponential E_ω on $\mathbf{No}^{>\mathbb{R}}$. In fact, we have $\partial(E_\omega(E_\omega(\omega)))=E_\omega'(E_\omega(\omega))\neq\partial(E_\omega(\omega))\,E_\omega'(E_\omega(\omega))$.

4 Overview of the thesis

Let us now give an overview of the results in the thesis. The thesis is split into four parts, each focused on a particular theme, and each beginning with a thematic introduction which describes in an informal way the main ideas and stakes at play. Parts I and III mainly contain known results, with the exceptions of Chapters 2, 9 and 10. Most our new results are to be found in Parts II and IV.

4.1 Well-based series

Part I mainly introduces known facts regarding well-based series, transseries, and operations on those series. In Chapter 1, we recall properties of the summation operator and strongly linear functions.

In Chapter 2, we study a notion of (formal) analytic functions on fields of well-based series. Analytic functions are functions which can be locally expressed by power series. The main result (Proposition 2.3.6) is that an analytic function is infinitely differentiable and has a Taylor expansion in terms of its iterated derivatives. We develop a short list of results regarding analytic functions which are very useful when working with the type of hyperserial calculus we introduce in Part II. Indeed, the hyperseries acting as functions on hyperserial fields or surreal numbers will always be analytic.

Finally, in Chapter 3, we give a short overview of Schmeling's work [92, Chapter 2] on transseries fields by introducing a slight generalization of those fields.

4.2 Hyperseries

In Part II, we introduce the setting of hyperserial fields, in which we will be able to compute with hyperseries. This setting is inspired by Schmeling's notion of transseries field, which it specifies by including hyperlogarithms along with the logarithm. Our constructions of fields of hyperseries rely on properties of the field $\mathbb{L}$ of logarithmic hyperseries. Intuitively speaking, the reason is that the derivative of E_α can be expressed as the composition of a logarithmic hyperseries with E_α:

$$E_\alpha'(a) \;=\; \frac{1}{\ell_\alpha' \circ (E_\alpha(a))}$$

and similarly for all higher derivatives. One key aspect of our approach is therefore to construct increasingly large fields $\mathbb{T}$ of hyperseries simultaneously with compositions

$$\circ \colon \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}^{>\mathbb{R}}.$$

Our definition of hyperserial fields involves a parameter $\boldsymbol{\nu}$ which for the sake of simplicity and exposition, we temporarily fix as $\boldsymbol{\nu} = \mathbf{On}$. Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series, where $\mathfrak{M}$ (and equivalently $\mathbb{T}$) can be a proeper class. Let $\circ \colon \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$ be a function. For $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}$, we define $\mathfrak{m}^r$ as follows: set $1^r := 1$, set $\mathfrak{m}^r := \ell_0^r \circ \mathfrak{m}$ if $\mathfrak{m} \succ 1$, and set $\mathfrak{m}^r := \ell_0^{-r} \circ \mathfrak{m}^{-1}$ if $\mathfrak{m} \prec 1$. So ℓ_0 acts as the identity variable x in transseries. For $\mu \in \mathbf{On}$, we define $\mathfrak{M}_{\in^\mu}$ to be the class of series $s \in \mathbb{T}^{>\mathbb{R}}$ with $\ell_\gamma \circ s \in \mathfrak{M}^{\succ 1}$ for all $\gamma < \omega^\mu$. Such series are said $L_{<\omega^\mu}$-atomic. We say that $(\mathbb{T}, \circ)$ is a *hyperserial field (of force* $\mathbf{On}$*)* if the following axioms are satisfied:

HF1. $\mathbb{L} \longrightarrow \mathbb{T}; f \mapsto f \circ s$ is a strongly linear morphism of ordered rings for all $s \in \mathbb{T}^{>\mathbb{R}}$.

HF2. $f \circ (g \circ s) = (f \circ g) \circ s$ for all $f \in \mathbb{L}$, $g \in \mathbb{L}^{>\mathbb{R}}$, and $s \in \mathbb{T}^{>\mathbb{R}}$.

HF3. $f \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $f \in \mathbb{L}$, $t \in \mathbb{T}^{>\mathbb{R}}$, and $\delta \in \mathbb{T}$ with $\delta \prec t$.

HF4. $\ell_{\omega^\mu}^{\uparrow \gamma} \circ s < \ell_{\omega^\mu}^{\uparrow \gamma} \circ t$ for all ordinals μ, all $\gamma < \omega^\mu$, and all $s, t \in \mathbb{T}^{>\mathbb{R}}$ with $s < t$.

HF5. The map $\mathbb{R} \times \mathfrak{M} \rightarrow \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$ is a law of ordered $\mathbb{R}$-vector space on $\mathfrak{M}$.

HF6. $\ell_1 \circ (s\, t) = \ell_1 \circ s + \ell_1 \circ t$ for all $s, t \in \mathbb{T}^{>\mathbb{R}}$.

HF7. $\operatorname{supp} \ell_1 \circ \mathfrak{m} \succ 1$ for all $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and
$\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ (\ell_\gamma \circ \mathfrak{a})^{-1}$ for all $\mu \geqslant 1$, $\gamma < \omega^\mu$, and $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

The first two axioms above impose compatibility between the composition laws on $\mathbb{L} \times \mathbb{L}^{>\mathbb{R}}$ and $\mathbb{L} \times \mathbb{T}^{>\mathbb{R}}$ and the structure of field of well-based series on $\mathbb{L}$. The series $\ell_{\omega^\mu}^{\uparrow \gamma}$ in the fourth axiom are defined as the unique solutions in $(\mathbb{L}, \circ)$ of

$$\ell_{\omega^\mu}^{\uparrow \gamma} \circ \ell_\gamma = \ell_{\omega^\mu}.$$

The other axioms are more technical and will make sense as the reader goes through Chapters 3 and 4.

The hyperserial field $(\mathbb{T},\circ)$ is *confluent* if $\mathfrak{M}\neq 1$ and for all $\mu\in\mathbf{On}$ and all $s\in\mathbb{T}^{>\mathbb{R}}$, there exist $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$ and $\gamma<\omega^\mu$ with

$$\ell_\gamma\circ s \asymp \ell_\gamma\circ\mathfrak{a}.$$

Confluent hyperserial fields of force $\mathbf{On}$ include $\mathbb{L}$ itself, and the axiomatic properties of $\circ\colon\mathbb{L}\times\mathbb{T}^{>\mathbb{R}}\longrightarrow\mathbb{T}$ can be seen as a generalization of those properties of the internal law $\circ\colon\mathbb{L}\times\mathbb{L}^{>\mathbb{R}}\longrightarrow\mathbb{L}$ that make sense when $\mathbb{T}$ itself is not equipped with a composition law or derivation.

Since it is difficult to define composition laws satisfying the axioms above, we will rely on a method whereby it is sufficient to consider a very restricted list of values of the law $\circ$. Given a confluent hyperserial field $(\mathbb{T},\circ)$, its skeleton is $\mathbb{T}$ together with the list of partial functions

$$\begin{aligned} L_{\omega^\mu}\colon\mathfrak{M}_{\omega^\mu} &\longrightarrow \mathbb{T}\\ \mathfrak{a} &\longmapsto \ell_{\omega^\mu}\circ\mathfrak{a}\end{aligned}$$

for all $\mu\in\mathbf{On}$. We will see that the skeleton completely determines the composition law, while being much easier to construct on certain fields of well-based series, and in particular on $\mathbf{No}$. The condition that $(\mathbb{T},\circ)$ be a confluent hyperserial field lets us isolate a few axiomatic properties that the skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ should satisfy. Let us temporarily write **Sk-Ax** for this list of properties (see Sections 4.2.1 and 4.2.2 for more details). Now consider a field $\mathbb{U}$ of well-based series over $\mathbb{R}$ together with a list of partial functions L_{ω^μ}, $\mu\in\mathbf{On}$ called *partial hyperlogarithms* that satisfy **Sk-Ax**. We call $(\mathbb{U},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ a *confluent hyperserial skeleton* (of force $\mathbf{On}$). An *embedding* of confluent hyperserial skeletons is a strongly linear morphism of ordered rings $\Psi\colon\mathbb{U}\longrightarrow\mathbb{V}$ which commutes with the partial hyperlogarithms. Our first main result is the following equivalence between confluent hyperserial fields and skeletons:

Theorem A. [Theorems 7.2.1 and 7.2.10] *If $(\mathbb{U},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ is a confluent hyperserial skeleton, then there is a unique function $\circ\colon\mathbb{L}\times\mathbb{U}^{>\mathbb{R}}\longrightarrow\mathbb{U}$ such that $(\mathbb{U},\circ)$ is a confluent hyperserial field with skeleton $(\mathbb{U},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$.*

Conversely, if $(\mathbb{T},\circ)$ is a confluent hyperserial field, then its skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ is a confluent hyperserial skeleton.

This correspondence will allow us to define the structure of hyperserial field on $\mathbf{No}$ in Part IV. In Chapter 4, we define hyperserial skeletons and show how to define a composition law $\circ$ on a confluent hyperserial skeleton (Theorem 4.3.1).

Consider a hyperserial skeleton $(\mathbb{U},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ and the corresponding composition law $\circ$ of Theorem 4.3.1. Each partial function L_{ω^μ} for $\mu>0$ extends to $\mathbb{U}^{>\mathbb{R}}$ by setting $L_{\omega^\mu}(s)=\ell_{\omega^\mu}\circ s$ for all $s\in\mathbb{U}^{>\mathbb{R}}$. We will see that this function is injective, whence it has a partially defined left inverse denoted E_{ω^μ}. In Chapter 5, we give a criterion on $(\mathbb{U},(L_{\omega^\mu})_{\mu\in\mathbf{On}})$ under which $L_{\omega^\mu}\colon\mathbb{U}^{>\mathbb{R}}\longrightarrow\mathbb{U}^{>\mathbb{R}}$ is surjective, i.e. under which $E_{\omega^\mu}\colon\mathbb{U}^{>\mathbb{R}}\longrightarrow\mathbb{U}^{>\mathbb{R}}$ is totally defined. This leads us to consider the notion of ω^μ-truncated series, which are series $\varphi\in\mathbb{U}^{>\mathbb{R}}$ with $\varphi>\ell_{\omega^\mu}^{\uparrow\gamma}\circ\mathfrak{m}^{-1}$ for all $\mathfrak{m}\in\operatorname{supp}\varphi$ with $\mathfrak{m}\prec 1$ and $\gamma<\omega^\mu$. We also define 1-truncated to be those positive infinite series whose support $\operatorname{supp}\varphi$ contains only infinite monomials. Writing $\mathbb{U}_{\succ,\omega^\eta}$ for the class of ω^η-truncated series for any $\eta\in\mathbf{On}$, we have the following criterion

Proposition. [Corollary 5.3.13] *Let $\mu\in\mathbf{On}$. If $L_{\omega^\mu}(\mathfrak{M}_{\omega^\eta})=\mathbb{U}_{\succ,\omega^\eta}$ for all $\eta\leqslant\mu$, then the function $L_{\omega^\mu}\colon\mathbb{U}^{>\mathbb{R}}\longrightarrow\mathbb{U}^{>\mathbb{R}}$ is surjective.*

We say that the confluent hyperserial skeleton $\mathbb{U}$ has force $(\mathbf{On},\mathbf{On})$ if each $L_{\omega^\mu}\colon\mathbb{U}^{>\mathbb{R}}\longrightarrow\mathbb{U}^{>\mathbb{R}}$ is surjective. Such a field is equipped both with hyperlogarithmic and hyperexponential functions. The skeleton of $\mathbb{L}$ itself is not at all of force $(\mathbf{On},\mathbf{On})$, since for instance no hyperexponential $E_{\omega^\mu}(x)$ for $\mu\in\mathbf{On}$ is defined. So it is necessary to find ways to close arbitrary confluent hyperserial skeletons under hyperexponentials. Using the previous criterion, we may define such extensions by adjoining $\mathbb{U}$ with formal hyperexponentials $\mathrm{e}_\alpha^\varphi$ of α-truncated series $\varphi\in\mathbb{U}$ which do not already lie in $L_\alpha(\mathbb{U}^{>\mathbb{R}})$. We will do this in Chapter 6 where we will prove the following other main result of Part II:

Theorem B. [Theorem 5.1.5] *Let* $(\mathbb{U}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ *be a confluent hyperserial skeleton. There is a confluent hyperserial skeleton* $\tilde{\mathbb{U}}$ *of force* $(\mathbf{On}, \mathbf{On})$ *and an embedding* $\Psi: \mathbb{U} \longrightarrow \mathbb{U}$ *with the following universal property: if* $\mathbb{V}$ *is a confluent hyperserial skeleton of force* $(\mathbf{On}, \mathbf{On})$ *and* $\Phi: \mathbb{U} \longrightarrow \mathbb{V}$ *is an embedding, then there is a unique embedding* $\Lambda: \tilde{\mathbb{U}} \longrightarrow \mathbb{V}$ *with* $\Phi = \Lambda \circ \Psi$.

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{\Psi} & \tilde{\mathbb{U}} \\ & {}_{\Phi}\searrow & \downarrow \exists! \Lambda \\ & & \mathbb{V} \end{array}$$

In Chapter 7, we define hyperserial fields, prove Theorem **A**, and study examples of hyperserial fields.

4.3 Surreal numbers

In Part II, we introduce the class **No** of surreal numbers, together with its linear ordering $<$ andits simplicity relation $\sqsubset$, and give a survey of its properties as they relate to the project of identifying numbers and hyperseries. In Chapter 8, we give Gonshor's formal definition of **No** and recall its properties as an ordered field of well-based series.

In Chapter 9 we introduce a type of subclass of **No** that plays an important role in Part IV. A surreal substructure is a subclass **S** of **No** such that there is an isomorphism $\Xi_{\mathbf{S}}: (\mathbf{No}, <, \sqsubset) \longrightarrow (\mathbf{S}, <, \sqsubset)$, which is then unique. It is known in particular that the class **Mo** of monomials and the class $\mathbf{Mo}_\omega$ of log-atomic surreal numbers are surreal substructures (see [55] and [18] respectively).

In Chapter 10 we introduce a convenient way to define surreal substructures that will turn out to subsume every surreal substructure studied in the last part of the thesis. More precisely, a convex partition $\mathbf{\Pi}$ of a surreal substructure is a partition of **S** whose members are convex subclasses of **S**. Given such a partition, the class $\mathbf{Smp}_{\mathbf{\Pi}}$ of numbers which are $\sqsubset$-minimal in each member of $\mathbf{\Pi}$ is a surreal substructure (Theorem 10.1.7).

In Chapter 11 we use surreal substructures and convex partitions in order to state relevant properties of the class **No** with Gonshor's exponential function. This is based on work of Gonshor [55], Berarducci and Mantova [18], Aschenbrenner, van den Dries and van der Hoeven [6], and Ehrlich and Kaplan [46]. In particular, we recall how **No** can be construed naturally as a universal transseries field.

4.4 Numbers as hyperseries

In Part IV, we introduce the hyperserial calculus on surreal numbers and use it to represent numbers as hyperseries. This entails in particular to construe the field of well-based series $\mathbf{No} = \mathbb{R}[[\mathbf{Mo}]]$ as a confluent hyperserial field of force $(\mathbf{On}, \mathbf{On})$, as will be our main task in Chapter 12. Thanks to Theorem A, this reduces to defining a confluent hyperserial skeleton $(\mathbf{No}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ of force $(\mathbf{On}, \mathbf{On})$. By the nature of axioms for hyperserial skeletons, we are to proceed by induction on $\mu \in \mathbf{On}$, whereas the definition of each partial hyperlogarithm $L_\alpha: \mathbf{Mo}_\alpha \longrightarrow \mathbf{No}$, $\alpha = \omega^\mu$ itself will be by well-founded induction on $(\mathbf{Mo}_\alpha, \sqsubset)$. The case $\mu = 0$ is already treated in essence in the literature, and recalled in Chapter 11 of Part III. The class $\mathbf{Mo}_\omega$ of log-atomic surreal numbers was already identified by Berarducci and Mantova as a surreal substructure. It will turn out in the inductive definition process that in general, each class $\mathbf{Mo}_\alpha$, $\alpha = \omega^\mu$ is a surreal substructure. This makes it possible to give an inductive definition for L_α:

$$\forall \mathfrak{a} \in \mathbf{Mo}_\alpha, L_\alpha \mathfrak{a} = \left\{ \mathbb{R}, L_\alpha \mathfrak{a}' + \frac{1}{L_\gamma \mathfrak{a}'} \mid L_\alpha \mathfrak{a}'' - \frac{1}{L_\gamma \mathfrak{a}''}, L_\gamma \mathfrak{a} \right\},$$

where $\mathfrak{a}'$, $\mathfrak{a}''$ respectively range among $L_{<\alpha}$-atomic numbers with $\mathfrak{a}', \mathfrak{a}'' \sqsubset \mathfrak{a}$ and $\mathfrak{a}' < \mathfrak{a} < \mathfrak{a}''$, and γ ranges in α. The class $\mathbf{No}_{\succ,\alpha}$ of α-truncated numbers also turns out to be a surreal substructure, and the hyperexponential E_α of strength α satisfies the following inductive equation

$$\forall \varphi \in \mathbf{No}_{\succ,\alpha}, E_\alpha \varphi = \{E_\gamma \varphi, E_\gamma(L_\gamma E_\alpha \varphi' + 1) \mid E_\gamma(L_\gamma E_\alpha \varphi' - 1)\},$$

where φ', φ'' respectively range in the class of α-truncated numbers with $\varphi', \varphi'' \sqsubset \varphi$ and $\varphi' < \varphi < \varphi''$, and γ ranges in α. We show that this definition works, i.e. that $(\mathbf{No}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ is indeed a hyperserial skeleton of force $(\mathbf{On}, \mathbf{On})$. This gives us the first main result of Part IV:

Theorem C. [Theorem 1] *There is a composition law* $\circ\colon \mathbb{L}\times\mathbf{No}^{>\mathbb{R}}\longrightarrow\mathbf{No}$ *such that* $(\mathbf{No},\circ)$ *is a confluent hyperserial field of force* $(\mathbf{On},\mathbf{On})$.

In order to explain the last results of the thesis, we need to describe in some detail how surreal numbers, and in particular monomials $\mathfrak{m}\in\mathbf{Mo}$ can be expanded using hyperexponentials and hyperlogarithms. In the hyperserial field $(\mathbf{No},\circ)$, every non-trivial monomial $\mathfrak{m}\in\mathbf{Mo}\setminus\{1\}$ admits a unique expansion of exactly one of the two following forms:

$$\mathfrak{m}=\mathrm{e}^{\psi}\,(L_{\beta}(\omega))^{\iota}, \tag{4.1}$$

where $\mathrm{e}^{\psi}\in\mathbf{Mo}$, $\iota\in\{-1,1\}$, and $\beta\in\mathbf{On}$, with $\operatorname{supp}\psi\succ\log(L_{\beta}(\omega))$; or

$$\mathfrak{m}=\mathrm{e}^{\psi}\,(L_{\beta}(E_{\alpha}(u)))^{\iota}, \tag{4.2}$$

where $\mathrm{e}^{\psi}\in\mathbf{Mo}$, $\iota\in\{-1,1\}$, $\beta\in\mathbf{On},\alpha\in\omega^{\mathbf{On}}$ with $\beta\omega<\alpha$, $\operatorname{supp}\psi\succ\log(L_{\beta}(E_{\alpha}(u)))$, and where $E_{\alpha}\,u$ lies in $\mathbf{Mo}_{\alpha}\setminus L_{<\alpha}\,\mathbf{Mo}_{\alpha\omega}$. Moreover, if $\alpha=1$ then it is imposed that $\psi=0$, $\iota=1$, and that u cannot be written as $u=\varphi+\varepsilon\,\mathfrak{b}$ where $\varphi\in\mathbf{No}$, $\varepsilon\in\{-1,1\}$ and $\mathfrak{b}\in\mathbf{Mo}_{\omega}$.
Note that we have two possible ways to further expand $\mathfrak{m}$:

i. If $\mathfrak{m}=(L_{\beta}(\omega))^{\iota}$ (i.e. $\psi=0$), then we need not expand $\mathfrak{m}$ further since $L_{\beta}(\omega)$ cannot be further simplified.

ii. If $\mathfrak{m}=\mathrm{e}^{\psi}\,(L_{\beta}(\omega))^{\iota}$ where $\psi\neq0$, then we may expand every monomial in $\operatorname{supp}\psi$ as in (4.1) or (4.2). We call this a left expansion.

iii. If $\mathfrak{m}=(L_{\beta}(E_{\alpha}(u)))^{\iota}$ (i.e. $\psi=0$), then we may expand any non-trivial monomial in $\operatorname{supp}u$ as in (4.1) or (4.2). We call this a right expansion.

iv. If $\mathfrak{m}=\mathrm{e}^{\psi}\,(L_{\beta}(E_{\alpha}(u)))^{\iota}$ where $\psi\neq0$, then we may expand $\mathfrak{m}$ on the right or on the left.

We adopt the notations

$$\begin{aligned} L_{\beta}\,E_{\alpha}^{u} &:= L_{\beta}(E_{\alpha}(u)) \text{ and}\\ \operatorname{term}a &:= \{a_{\mathfrak{m}}\,\mathfrak{m}\colon \mathfrak{m}\in\operatorname{supp}a\}.\end{aligned}$$

An infinite path $P=(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}}$ in $a\in\mathbf{No}$ is thus defined as a sequence of non-zero terms $P=(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}}\in(\mathbb{R}^{\neq}\,\mathbf{Mo}\setminus\{1,\omega\})^{\mathbb{N}}$ with

$$\forall i\in\mathbb{N}, r_i\,\mathfrak{m}_i\in\operatorname{term}\psi_i\cup\operatorname{term}u_i,$$

where $(u_0,\psi_0)=(a,0)$ and each $\mathfrak{m}_i$ expands as $\mathfrak{m}_i=\mathrm{e}^{\psi_{i+1}}\,(L_{\beta_i}\,\omega)^{\iota_i}$ or as $\mathfrak{m}_i=\mathrm{e}^{\psi_{i+1}}\,(L_{\beta_i}\,E_{\alpha_i}^{u_{i+1}})^{\iota_i}$.

For instance, here are the first terms of a path P in a which consists in a left, then right, then left expansion.

$$\begin{aligned} a &= \varphi_0+r_0\,\mathfrak{m}_0+\delta_0\\ &= \varphi_0+r_0\,\mathrm{e}^{\psi_1}(L_{\beta_0}\,E_{\alpha_0}^{u_1})^{\iota_0}+\delta_0\\ &= \varphi_0+r_0\,\mathrm{e}^{\varphi_1+r_1\mathrm{e}^{\psi_2}(L_{\beta_1}E_{\alpha_1}^{u_2})^{\iota_1}+\delta_1}\,(L_{\beta_0}\,E_{\alpha_0}^{u_1})^{\iota_0}+\delta_0\\ &= \varphi_0+r_0\,\mathrm{e}^{\varphi_1+r_1\mathrm{e}^{\psi_2}\left(L_{\beta_1}E_{\alpha_1}^{\varphi_2+r_2\mathrm{e}^{\psi_3}\left(L_{\beta_2}E_{\alpha_2}^{u_3}\right)^{\iota_2}+\delta_2}\right)^{\iota_1}+\delta_1}\,(L_{\beta_0}\,E_{\alpha_0}^{u_1})^{\iota_0}+\delta_0\\ &= \varphi_0+r_0\,\mathrm{e}^{\varphi_1+r_1\mathrm{e}^{\psi_2}\left(L_{\beta_1}E_{\alpha_1}^{\varphi_2+r_2\mathrm{e}^{\cdot^{\cdot^{\cdot}}}\left(L_{\beta_2}E_{\alpha_2}^{u_3}\right)^{\iota_2}+\delta_2}\right)^{\iota_1}+\delta_1}\,(L_{\beta_0}\,E_{\alpha_0}^{u_1})^{\iota_0}+\delta_0. \end{aligned} \tag{4.3}$$

To each ordered pair (P,a) corresponds the sequence $\Sigma=(\varphi_i,\psi_{i+1},r_i,\iota_i,\alpha_i,\beta_i,u_{i+1},\delta_i)_{i\in\mathbb{N}}$ of parameters which allows us to describe the path P within a. These expansions and corresponding paths raise several questions. What properties must the sequences Σ satisfy? Under what conditions on Σ does there exist infinite paths with Σ as a sequence of parameters? How many numbers share the same sequence of parameters? One additional problem is the possibility of infinite branching, i.e. of the alternation of left and right expansions. In order to study these questions, we consider the notion of good path.

We say that the path $P=(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}}$ in a is *good* if there is $i_0\in\mathbb{N}$ such that for all $i\geqslant i_0$, we have

$$\begin{aligned}\mathfrak{m}_{i+1} &\notin \operatorname{supp}\psi_{i+1},\\ r_i &\in \{-1,1\},\\ \beta_i &= 0, \text{ and}\\ \delta_i &= 0.\end{aligned}$$

This implies in particular that the branching phenomenon stops and that for $j=i_0+1$, we have

$$u_j=\varphi_j\pm\mathrm{e}^{\psi_{j+1}}\left(E_{\alpha_j}^{\varphi_{j+1}\pm\mathrm{e}^{\psi_{j+2}}\left(E_{\alpha_{j+1}}^{\cdot^{\cdot^{\cdot \varphi_{j+i}\pm\mathrm{e}^{\psi_{j+i+1}}\left(E_{\alpha_{j+i}}^{\cdot^{\cdot^{\cdot}}}\right)^{\iota_{j+i}}}}}\right)^{\iota_{j+1}}}\right)^{\iota_j}. \tag{4.4}$$

We say that a is *well-nested* if every path in a is good. Relying on Chapter 12 and a study of paths in $\mathbf{No}$, we prove the main result of Chapter 13:

Theorem D. [Theorem 13.2.7] *Every number is well-nested.*

In Chapter 14, we study the existence of numbers such as a_j above. Consider a sequence $\Sigma=:(\varphi_i,\varepsilon_i,\psi_i,\iota_i,\alpha_i)_{i\in\mathbb{N}}$ where for $i\in\mathbb{N}$, we have $\varphi_i\in\mathbf{No}$, $\varepsilon_i,\iota_i\in\{-1,1\}$, $\psi_i\in\mathbf{No}_{\succ}$ and $\alpha_i\in\omega^{\mathbf{On}}$ along with other technical conditions (see Definition 14.1.1). For $k\in\mathbb{N}$, consider the class $\mathbf{Ad}_{\nearrow k}$ of first terms a_k of a sequence $(a_{k+i})_{i\in\mathbb{N}}$ with

$$\begin{aligned}a_{k+i} &= \varphi_{k+i}+\varepsilon_{k+i}\,\mathrm{e}^{\psi_{k+i}}\,(E_{\alpha_{k+i}}\,a_{k+i+1})^{\iota_{k+i}},\\ \operatorname{supp}\varphi_{k+i} &\succ \mathrm{e}^{\psi_{k+i}}\,(E_{\alpha_{k+i}}\,a_{k+i+1})^{\iota_{k+i}},\\ \operatorname{supp}\psi_{k+i} &\succ \log E_{\alpha_{k+i}}\,a_{k+i+1}, \quad \text{and}\\ \varphi_{k+i+1}-a_{k+i+1} &\succ (L_{<\alpha_{k+i}}\,E_{\alpha_{k+i}}^{\varphi_{k+i}})^{-1},\end{aligned}$$

for all $i\in\mathbb{N}$. We say that Σ is *admissible* if $\mathbf{Ad}:=\mathbf{Ad}_{\nearrow 0}\neq\varnothing$, and that it is *nested* if it is admissible and

$$\mathbf{Ad}_{\nearrow k}=\varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_k}\,(E_{\alpha_k}(\mathbf{Ad}_{\nearrow k+1}))^{\iota_k}$$

for all $k\in\mathbb{N}$.

Write $\mathbf{Ne}$ for the class of numbers $a_0\in\mathbf{Ad}$ such that the corresponding sequence $(a_i)_{i\in\mathbb{N}}$ satisfies $E_{\alpha_i}^{a_{i+1}}\in\mathbf{Mo}_{\alpha_i}\setminus L_{<\alpha_i}\mathbf{Mo}_{\alpha_i\omega}$ for all $i\in\mathbb{N}$. In other words, $\mathbf{Ne}$ is the class of numbers $a_0\in\mathbf{No}$ which admit the nested expansion

$$a_0=\varphi_0+\varepsilon_0\,\mathrm{e}^{\psi_0}\left(E_{\alpha_0}^{\varphi_1+\varepsilon_1\mathrm{e}^{\psi_1}\left(E_{\alpha_1}^{\cdot^{\cdot^{\cdot \varphi_i+\varepsilon_i\mathrm{e}^{\psi_i}\left(E_{\alpha_i}^{\cdot^{\cdot^{\cdot}}}\right)^{\iota_i}}}}\right)^{\iota_1}}\right)^{\iota_0}.$$

Our second main result is the following generalization of [11, Theorem 8.8]:

Theorem E. [Theorem 14.2.4] *If* Σ *is nested, then* $(\mathbf{Ne},<,\sqsubset)$ *is isomorphic to* $(\mathbf{No},<,\sqsubset)$*, i.e.* $\mathbf{Ne}$ *is a surreal substructure.*

In Chapter 15, we give a presentation of numbers as hyperseries in ω, using trees labelled by real numbers, ordinal numbers and surreal numbers. We call such expressions hyperserial descriptions. The main result is the following:

Theorem F. [Theorem 15.3.1] *Every surreal number has a unique hyperserial description. Two numbers with the same hyperserial description are equal.*

The hyperserial presentation of numbers is no more definitive than the sign sequence presentation or the well-based series presentation. However, we expect that it is sufficient to serve as a basis to define derivations and compositions on $\mathbf{No}$.

Conventions

1 Prerequisites

This thesis is designed to be almost self-contained, and should be accessible to the patient average mathematician. This is with the exception of two areas of logic with which some familiarity will be useful.

First, although *model theory* does not play a major role in the thesis, it appears in decisive ways in certain sections, and it remained a guiding framework for my thought process. We will not systematically recall the definitions of basic model theoretic notions. Those can be found in most introductive model theory textbooks: see [27]. The reader can also find all necessary notions and more, including a treatment of case of many-sorted logic, in [4, Appendix B].

Secondly, it will be helpful for the reader to have a certain familiarity with *elementary set theory*, including set theoretic definitions of functions, unions and intersections, and the elementary theory of ordinal numbers and their arithmetic.

2 Axiomatic framework

2.1 NBG set theory

The underlying set theoretical framework of this paper is von Neumann, Bernays, and Gödel's set theory, henceforth referred to as *NBG set theory*, and more precisely Gödel's one-sorted version [54]. The language $\mathcal{L}_{\in,\mathrm{Set}}$ of NBG set theory is the first-order language having as primitives a binary symbol $\in$ interpreted as the membership relation, and a unary predicate Set which stands for the predicate "being a set", or "not being a proper class". NBG set theory and its language allow us to prove statements about classes. The reader will see that such powers are necessary when working with fields of transseries closed under exponentiation (see Section 3.2), hyperseries (see Section 4.1), or surreal numbers (see Part III).

We recall a few key features of the axiomatization. Sets are classes which can be members of classes. That is, the following statement holds

$$\forall x\,, (\mathrm{Set}(x) \Longleftrightarrow (\exists y(x \in y))). \tag{2.1}$$

Certain classes, called *proper classes*, are not sets. So proper classes are classes which lie in no class. This includes the class $\mathbf{V}$ of all sets, or the class of all sets that do not contain themselves (thus does NBG set theory avoid Russell's famous paradox).

If φ is an $\mathcal{L}_{\in}$-sentence, then we canonically define an $\mathcal{L}_{\in,\mathrm{Set}}$ sentence φ_{Set} in which each quantification is constrained to the predicate Set. For instance if φ is the sentence

$$\forall x \exists y(x \in y),$$

then φ_{Set} is the sentence

$$\forall x(\mathrm{Set}(x) \Longrightarrow (\exists y(\mathrm{Set}(y) \wedge x \in y))).$$

Crucially, NBG set theory is a conservative extension of ZFC (see [47]), which means that an $\mathcal{L}_{\in}$-sentence φ is a theorem of ZFC if and only if φ_{Set} is a theorem of NBG set theory. Thus, despite the fact that working with classes requires additional care, the reader who is familiar with elementary set theory should have no qualms with our use of classes.

An important feature of NBG set theory is the axiom *global choice* (**GC**). The axiom of global choice states that there is a function $\varsigma\colon \mathbf{V} \longrightarrow \mathbf{V}$ such that for any non-empty set x, we have $\varsigma(x) \in x$. A consequence of **GC** is the theorem of *limitation of size* (**LOS**). The theorem of limitation of size states that there is a well-ordering of $\mathbf{V}$. See [47] for a discussion of those results.

Remark 2.1. There exist several equivalent presentations of NBG set theory. Some use a two-sorted language with a sort for set and a sort for classes. Some don't use the Set predicate since it is defined in $\mathcal{L}_\in$ *via* (2.1). The translation between different presentations of NBG set theory is very straightforward. We introduced the predicate Set for clarity, but we will not rely on it in the body of the thesis. Thus our framework is similar to Mendelson's [80]. The reader can find more details and an axiomatization of NBG set theory in [80, Section 4.1].

2.2 Set-theoretic conventions and notations

If $\mathbf{X}$ and $\mathbf{Y}$ are classes, then a *function* $\mathbf{X} \longrightarrow \mathbf{Y}$ is a subclass of $\mathbf{X} \times \mathbf{Y}$ which has the usual functional property. If $\mathbf{X}$ is a proper class and $\mathbf{Y}$ is non-empty, then there is *no* class "$\mathbf{Y}^{\mathbf{X}}$" of functions $\mathbf{X} \longrightarrow \mathbf{Y}$. Indeed such functions are proper classes, and consequently cannot lie in classes. This means that we will have to use caution in the instances when we consider collections of functions $\mathbf{X} \longrightarrow \mathbf{Y}$. However, if X is a set, then each function $X \longrightarrow \mathbf{Y}$ is a set, and the class $\mathbf{Y}^X$ of functions $X \longrightarrow \mathbf{Y}$ is well-defined (and of course, it is a set if $\mathbf{Y}$ is a set). In most cases, it is enough, in order to prove that certain classes exists, to use the following consequence of the fact that NBG set theory is conservative over ZFC:

Class comprehension scheme: Given a formula $\varphi(x_1, \ldots, x_m, X_1, \ldots, X_n)$ in $\mathcal{L}_{\in,\mathrm{Set}}$, which quantifies only over sets, and classes $\mathbf{X}_1, \ldots, \mathbf{X}_n$, there is a class $\mathbf{X}_\varphi$ which contains all the tuples of sets satisfying φ, that is,

$$\forall x_1, \ldots, x_m, ((x_1, \ldots, x_m) \in \mathbf{X}_\varphi \Longleftrightarrow (\mathrm{Set}(x_1) \wedge \cdots \wedge \mathrm{Set}(x_m) \wedge \varphi(x_1, \ldots, x_m, \mathbf{X}_1, \ldots, \mathbf{X}_n))).$$

As is standard for sets, we write

$$\mathbf{X}_\varphi = \{(x_1, \ldots, x_m) : \varphi(x_1, \ldots, x_m, \mathbf{X}_1, \ldots, \mathbf{X}_n)\}.$$

Thus it is intended that the elements in the left-hand side of a bracket $\{x : [\ldots]\}$ are always sets, whereas the reader should not expect that the notation $\{x : [\ldots]\}$ itself always denote a set.

Given a class $\mathbf{I}$ and a formula $\varphi(x_0, x_1)$, possibly with parameters, we have a corresponding *family* $(\mathbf{X}_i)_{i \in \mathbf{I}}$ of classes indexed by $\mathbf{I}$, where for each $i \in \mathbf{I}$, we define

$$\mathbf{X}_i := \{y : \varphi(i, y)\}.$$

We will always understand families in this sense, the formula $\varphi(x_0, x_1)$ often being implicit. A family of functions is a family $(\mathbf{X}_i)_{i \in \mathbf{I}}$ where each class $\mathbf{X}_i$ is a function, a family of groups is a family $(\mathbf{X}_i)_{i \in \mathbf{I}}$ where each class $\mathbf{X}_i$ is a group, and so on…

When it is relevant, we will mention explicitly which classes are sets or proper classes. In general, we will use bold font letters $\mathbf{X}$, $\mathbf{G}$, $\mathbf{S}$, $\mathbf{N}$,… to denote classes which may be proper classes. We extend this to upper-case letters in blackboard bold font which we use for our fields $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$, $\mathbb{V}$,… of well-based series (the standard number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ being exceptions to this rule), or to upper-case fraktur font letters which we use for our ordered groups of monomials $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{U}$, $\mathfrak{W}$,… Indeed those will often turn out to be proper classes. Except in the cases of relations and functions between classes, we reserve regular fonts for sets.

3 Ordered and algebraic structures

3.1 Orderings

3.1.1 Orderings

An *ordering* on a class $\mathbf{X}$ is a binary relation $<$ on (i.e. a subclass of $\mathbf{X}^2$) with

O1. $x \nless x$ for all $x \in \mathbf{X}$.

O2. $(x < y \wedge y < z) \Longrightarrow (x < z)$ for all $x, y, z \in \mathbf{X}$.

We say that $(\mathbf{X},<)$ is an *ordered class*, or an *ordered set* if furthermore $\mathbf{X}$ is a set. An ordering $<$ on $\mathbf{X}$ is said *linear* if it satisfies

LO. $x=y\vee x<y\vee y<x$ for all $x,y\in\mathbf{X}$.

We then say that $(\mathbf{X},<)$ is a *linearly ordered class / set*. We will most of the time work with linear orderings. Consequently, we will sometimes say that $(\mathbf{X},<)$ is a *partially ordered class/ set* to specify that the ordering may not be linear.

If $(\mathbf{X},<)$ is an ordered class, then we write $>$ for the ordering on $\mathbf{X}$ where

$$\forall x,y\in\mathbf{X},(x>y\Longleftrightarrow y<x),$$

which is called the *reverse* of $<$. The ordering $<$ is linear if and only if $>$ is linear. We also write $\leqslant$ for the relation

$$\forall x,y\in\mathbf{X},(x\leqslant y)\Longleftrightarrow(x=y\vee x<y),$$

which we call the *large ordering* corresponding to $<$, and we write $\geqslant$ for the large ordering corresponding to $>$. Note that $\leqslant$ is *not* an ordering on $\mathbf{X}$.

Let $(\mathbf{X},<)$ be an ordered class, let $\mathbf{A},\mathbf{B}$ be subclasses of $\mathbf{X}$ and let $x,x_1,\ldots,x_n\in\mathbf{X}$. We write

$$\begin{aligned}
\mathbf{A}<\mathbf{B} &\iff \forall x,y\in\mathbf{X},(x\in\mathbf{A}\wedge y\in\mathbf{B}\Longrightarrow x<y)\\
\mathbf{A}\leqslant\mathbf{B} &\iff \forall x,y\in\mathbf{X},(x\in\mathbf{A}\wedge y\in\mathbf{B}\Longrightarrow x\leqslant y)\\
\mathbf{A}<x_1,\ldots,x_n &\iff \mathbf{A}<\{x_1,\ldots,x_n\}\\
x_1,\ldots,x_n\leqslant\mathbf{A} &\iff \{x_1,\ldots,x_n\}\leqslant\mathbf{A}\\
x_1,\ldots,x_n<\mathbf{A} &\iff \{x_1,\ldots,x_n\}<\mathbf{A}\\
\mathbf{A}<x_1,\ldots,x_n &\iff \mathbf{A}<\{x_1,\ldots,x_n\}.
\end{aligned}$$

For $\mathbf{A}\subseteq\mathbf{X}$, we will often write $(\mathbf{A},<)$ for the ordered class where $<$ is the intersection of $\mathbf{A}\times\mathbf{A}$ with $<$, called the induced ordering on $\mathbf{A}$. Indeed it is an ordering on $\mathbf{A}$, which is linearly ordered if $(\mathbf{X},<)$ is linearly ordered. Unless specified otherwise, we will always endow subclasses with the corresponding induced orderings.

Let $a\in\mathbf{A}$. We will say that $a\in\mathbf{A}$ is *minimal* in $\mathbf{A}$ if there is no $x\in\mathbf{A}$ with $x<a$. We will say that a is the minimum (resp. *maximum*) of $\mathbf{A}$ if we have $a\leqslant x$ (resp. $x\leqslant a$) for all $x\in\mathbf{A}$. In that case, we write $a=\min\mathbf{A}$ (resp. $a=\max\mathbf{A}$). The minimum (resp. maximum) of $\mathbf{A}$ when it exists is the unique minimal (resp. minimal) element of $\mathbf{A}$, and $\mathbf{A}$ has a minimum (resp. maximum) if and only if it has a unique minimal (resp. maximal) element.

3.1.2 Increasing functions

Let $(\mathbf{X},<_{\mathbf{X}})$ and $(\mathbf{Y},<_{\mathbf{Y}})$ be partially ordered classes and let $f\colon\mathbf{X}\longrightarrow\mathbf{Y}$ be a function. We say that f is *nondecreasing* if

$$\forall x,x'\in\mathbf{X},x<_{\mathbf{X}}x'\Longrightarrow f(x')\not<_{\mathbf{Y}}f(x),$$

that it is *increasing* if

$$\forall x,x'\in\mathbf{X},x<_{\mathbf{X}}x'\Longrightarrow f(x)\leqslant_{\mathbf{Y}}f(x'),$$

that is is *strictly increasing* if

$$\forall x,x'\in\mathbf{X},x<_{\mathbf{X}}x'\Longrightarrow f(x)<_{\mathbf{Y}}f(x'),$$

that it is an *(order) embedding* if

$$\forall x,x'\in\mathbf{X},x<_{\mathbf{X}}x'\Longleftrightarrow f(x)<_{\mathbf{Y}}f(x'),$$

(in particular, order embeddings are injective). We say that f is an *(order) isomorphism* if it is a bijective order embedding. Depending on whether $<_{\mathbf{X}}$ is partial or linear, we have the following logical implications among those properties:

Partial.	isomorphism	$\subseteq$	embedding	$\subseteq$	strictly increasing	$\subseteq$	increasing	$\subseteq$	nondecreasing
Linear.	isomorphism	$\subseteq$	embedding	$=$	strictly increasing	$\subseteq$	increasing	$=$	nondecreasing

Finally, we say that f is *nonincreasing* / *decreasing* / *strictly decreasing* if it is nondecreasing / increasing / strictly increasing for the reverse ordering $>_{\mathbf{Y}}$ on $\mathbf{Y}$.

3.1.3 Ordinals

We consider the class $\mathbf{On}$ of ordinals as a *generalized ordinal*. If $\boldsymbol{\nu}$ is a class, then $\boldsymbol{\nu} \leqslant \mathbf{On}$ means that $\boldsymbol{\nu} \in \mathbf{On}$ or $\boldsymbol{\nu} = \mathbf{On}$. For generalized ordinals, we use bold font notations $\boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\lambda}$ to suggest that $\boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\lambda}$ may be equal to $\mathbf{On}$, whereas the notations $\alpha, \gamma, \beta, \rho$ and so on are only used for true ordinals $\alpha, \gamma, \beta, \rho \in \mathbf{On}$. We also extend the relations $\leqslant$ and $<$ on $\mathbf{On}$ by making $\mathbf{On}$ maximal, with the convention that $\omega^{\mathbf{On}} := \mathbf{On}$. Given a linearly ordered *set* $(X, <)$, an ordinal $\alpha \in \mathbf{On}$, we say that α is the *order type* of X, and we write $\alpha = \operatorname{ord}(X, <)$, if there is an order isomorphism $(\alpha, \in) \longrightarrow (X, <)$, which is then unique.

We write $\mathbf{On}_{\mathrm{Lim}}$ for the class of limit ordinals. By convention, zero is a limit ordinal, and $\mathbf{On}$ is a limit generalized ordinal. A cardinal is an ordinal $\kappa \in \mathbf{On}$ such that there is no bijective map $\alpha \longrightarrow \kappa$ for any $\alpha \in \kappa$. Given a cardinal κ, we write κ^+ for its Hartog ordinal, or successor cardinal, i.e. for the smallest cardinal $> \kappa$.

3.2 Ordered algebraic structures

3.2.1 Model theoretic morphisms

Consider a first-order language $\mathcal{L}$ consisting of the function symbols $\underline{f_i}, i \in I$ with arities $\alpha_i, i \in I$ and the relation symbols $\underline{R_j}, j \in J$ with arities $\beta_j, j \in J$. We recall that given two first-order structures

$$\begin{aligned} \boldsymbol{M} &= (M, (f_i^{\boldsymbol{M}})_{i \in I}, (R_j^{\boldsymbol{M}})_{j \in J}) \quad \text{and} \\ \boldsymbol{N} &= (N, (f_i^{\boldsymbol{N}})_{i \in I}, (R_j^{\boldsymbol{N}})_{j \in J}), \end{aligned}$$

a *morphism* $\boldsymbol{M} \longrightarrow \boldsymbol{N}$ is a function $\Phi: M \longrightarrow N$ with

- $\Phi(f_i^{\boldsymbol{M}}(m_1, \ldots, m_{\alpha_i})) = f_i^{\boldsymbol{N}}(\Phi(m_1, \ldots, m_{\alpha_i}))$ for all $i \in I$ and $m_1, \ldots, m_{\alpha_i} \in M$, and
- $(\boldsymbol{M} \vDash R_j^{\boldsymbol{M}}(m_1, \ldots, m_{\beta_j})) \Longrightarrow (\boldsymbol{N} \vDash R_j^{\boldsymbol{N}}(\Phi(m_1, \ldots, m_{\beta_j})))$ for all $j \in J$ and $m_1, \ldots, m_{\beta_j} \in M$.

An *embedding* $\boldsymbol{M} \longrightarrow \boldsymbol{N}$ is a morphism $\Phi: \boldsymbol{M} \longrightarrow \boldsymbol{N}$ such that

$$\forall j \in J, \forall m_1, \ldots, m_{\beta_j} \in M (\boldsymbol{M} \vDash R_j^{\boldsymbol{M}}(m_1, \ldots, m_{\beta_j})) \Longleftrightarrow (\boldsymbol{N} \vDash R_j^{\boldsymbol{N}}(\Phi(m_1, \ldots, m_{\beta_j}))).$$

Finally, an *isomorphism* $\boldsymbol{M} \longrightarrow \boldsymbol{N}$ is a bijective embedding $\boldsymbol{M} \longrightarrow \boldsymbol{N}$.

Most of our structures have a strict ordering as single relation (besides the equality), so our morphism will be strictly increasing. In the cases where the structures are linearly ordered, the notions of morphism and embedding coincide.

3.2.2 Ordered algebraic structures

Here we state our conventions for ordered algebraic structures. Our ordered monoids $(\mathbf{M}, \cdot, 1 <)$ (in particular our ordered groups) satisfy

$$\forall x, y, z \in \mathbf{M}((x < y) \Longrightarrow (x z < y z \wedge z x < z y)).$$

If $(\mathbf{M}, \cdot, 1 <)$ is an ordered monoid, $\mathbf{X} \subseteq \mathbf{M}$ is a subclass and $x \in \mathbf{M}$, then we write

$$\begin{aligned} \mathbf{M}^{>\mathbf{X}} &:= \{a \in \mathbf{M} : a > \mathbf{X}\}, \\ \mathbf{M}^{\geqslant \mathbf{X}} &:= \{a \in \mathbf{M} : a \geqslant \mathbf{X}\}, \\ \mathbf{M}^{>x} &:= \mathbf{M}^{>\{x\}} = \{a \in \mathbf{M} : a > x\}, \\ \mathbf{M}^{\geqslant x} &:= \mathbf{M}^{\geqslant \{x\}} = \{a \in \mathbf{M} : a \geqslant x\}, \\ \mathbf{M}^{>} &:= \mathbf{M}^{>1}, \\ \mathbf{M}^{\geqslant} &:= \mathbf{M}^{\geqslant 1}, \quad \text{and} \\ \mathbf{M}^{\neq} &:= \mathbf{M} \setminus \{1\}. \end{aligned}$$

Note that the same applies to $(\mathbf{M}, \cdot, 1, >)$. For instance, we have $\mathbf{M}^{<} = \{a \in \mathbf{M} : a < 1\}$.

If $(\mathbf{G},+,0,<)$ is an additively denoted, Abelian ordered group, then for all $x\in\mathbf{G}$, we define the absolute value $|x|_{\mathbf{G}}$ of x as

$$|x|_{\mathbf{G}}:=\max(x,-x)\in\mathbf{G}.$$

We simply write $|x|:=|x|_{\mathbf{G}}$ if this does not lead to confusion.

Our rings are commutative, non-zero (i.e. $0\neq 1$). Our ordered domains $\mathbf{D}$ satisfy

$$\forall a,b\in\mathbf{D},((0<a\wedge 0<b)\Longrightarrow 0<a\,b).$$

Any ordered domain $\mathbf{D}$ contains a unique isomorphic copy of $(\mathbb{Z},+,\times)$ in the sense that there is a unique ordered ring embedding $\mathbb{Z}\longrightarrow\mathbf{D}$. We identify $\mathbb{Z}$ with the corresponding subring of $\mathbf{D}$. An element $x\in\mathbf{D}$ is said *infinitesimal* if $n\,|x|<1$ for all $n\in\mathbb{N}$, in which case we write $x\prec 1$. We write $\mathbf{D}^{\prec}$ for the class of infinitesimal elements of $\mathbf{D}$. Note that $\mathbb{R}^{\prec}=\mathbb{Q}^{\prec}=\mathbb{Z}^{\prec}=\{0\}$. If $\mathbf{F}$ is an ordered field, then the embedding $\mathbb{Z}\longrightarrow\mathbf{F}$ extends uniquely into an ordered ring embedding $\mathbb{Q}\longrightarrow\mathbf{F}$ and likewise, we identify $\mathbb{Q}$ with the corresponding subset of $\mathbf{F}$.

Consider a differential ring $(\mathbf{R},+,\cdot,0,1,\partial)$ where $\partial\colon\mathbf{R}\longrightarrow\mathbf{R}$ is a group morphism which satisfies the *Leibniz rule*

$$\forall x,y\in\mathbf{R},\partial(x\,y)=\partial(x)\,y+x\,\partial(y).$$

The function ∂ is called the *derivation* on $\mathbf{R}$. For $x\in\mathbf{R}$, we call $\partial(x)$ the *derivative* of x. For $k\in\mathbb{N}$, we write ∂^k for the k-fold iterate of ∂ (so $\partial^0=\mathrm{Id}_{\mathbf{R}}$) and we call $\partial^k(x)$ the k-th iterate derivative of x. We also sometimes write $x^{(k)}:=\partial^k(x)$ and $x'=\partial(x)$, $x''=\partial^2(x)$, and so on…

Part I

Well-based series

Formal series

Whatever

$$1+\varepsilon+\varepsilon^2+\varepsilon^3+\cdots \tag{1}$$

is, the reader expects that any field allowing its existence and construing ε as a small quantity should allow the relation

$$(1-\varepsilon)\,(1+\varepsilon+\varepsilon^2+\varepsilon^3+\cdots)=1. \tag{2}$$

Indeed (2) is a purely formal relation, assuming little more than distributivity of the product over transfinite sums. Whatever

$$f=\mathrm{e}^x\,x^{-1}+\mathrm{e}^x\,x^{-2}+\cdots+n!\,\mathrm{e}^x\,x^{-(n+1)}+\cdots \tag{3}$$

is, one expects that any differential field allowing its existence should allow the relation

$$\partial(f)=\mathrm{e}^x\,x^{-1}. \tag{4}$$

Whatever

$$g=x+\log x+\log\log x+\cdots \tag{5}$$

is, one expects that any group equipped with a composition law $\circ$ and that contains g should satisfy

$$g\circ\log x=g-x. \tag{6}$$

In general, formal series display the type of manipulations they may be subject to.

In fields of formal series, it is indeed possible to give precise meanings to (1), (3) and (5), and accordingly derive (2), (4) and (6). And this can be done without having to worry about refined analytic notions of convergence, nor about whatever

$$\xi+\log\xi+\log\log\xi+\cdots$$

could possibly signify for a number, series, or germ ξ. This comes at the high price of setting the formal realm of series apart from the analytic or geometric realm of (real, complex)-valued functions. The way back may be long, still this thesis is concerned almost exclusively with formal series in their abstract, model theoretic relation to numbers and functions.

Well-based series

In fact, giving a meaning to such infinite sums is a non-trivial task. If we are to take advantage of the formal setting, then the object ξ above should lie in a field of series equipped with a logarithm function log. In particular ξ should itself be a series. But then each term

$$\log_n\xi:=\underbrace{\log\cdots\log}_{n\text{ times}}\xi$$

is itself a series, and one has to make sense of the summability of the family $(\log_n\xi)_{n\in\mathbb{N}}$.

There is a simple and well-known case when such a notion of summability, and corresponding sums, exist. Here we are thinking of *formal Laurent series*

$$f=\sum_{k\in\mathbb{Z}}f_k\,x^k$$

over the real numbers, where $(f_k)_{k\in\mathbb{Z}}$ is an arbitrary family of real numbers which is zero for all k above a certain $n\in\mathbb{Z}$. That is, the *support*

$$\operatorname{supp}f:=\{k\in\mathbb{Z}:f_k\neq 0\}$$

of f is either empty, or has a maximum. Then it is known that for any formal Laurent series

$$s=s_{n_0}\,x^{n_0}+s_{n_0-1}\,x^{n_0-1}+\cdots,$$

with $n_0>0$ and $s_0\neq 0$, the sequence $(\sum_{k\geqslant -m}f_k\,s^k)_{m\in\mathbb{N}}$ converges to a series

$$\sum_{k\in\mathbb{Z}}f_k\,s^k$$

in the valuation topology. In fact, the set

$$S := \{p \in \mathbb{Z} : \exists k \in \mathbb{Z}, p \in \operatorname{supp} s^k\}$$

is either empty or has a maximum; for each $p \in S$, the set

$$I_p := \{k \in \mathbb{Z} : p \in \operatorname{supp} s^k\}$$

is finite, and we have

$$\sum_{k \in \mathbb{Z}} f_k s^k = \sum_{p \in S} \left(\sum_{k \in I_p} f_k \right) s^p. \tag{7}$$

For larger fields of series, such as Puiseux series, Levi-Civita series or transseries, the valuation topology is not suited to make sense of transfinite sums in sufficient generality. However, the order theoretic and finiteness conditions on S and $I_p, p \in S$ are retained in a large number of cases. This motivates studying a formal order theoretic setting in which families of series can be summed as in (7). Such is the purpose of *well-based series*.

The wide range of summable, so-called *well-based families*, and the rigid properties of summation are especially suited to the task of constructing very large fields of complicated series, and defining operations on them. In particular, we shall see that finiteness and well-orderedness conditions (generalizing those on S and I_p above) are fairly convenient to manipulate, while being very general.

In Chapter 1, we will define order theoretic notions and give many tools that will allow us to tackle the difficulties inherent to the manipulation of well-based series. The central object of (ordered) *fields of well-based series* (over $\mathbb{R}$) will be defined in Section 1.2 whereas the necessary order theoretic tools will be developed in Sections 1.1 and 1.3.

Calculus on well-based series

One of the main goals of this thesis is to establish a hyperserial calculus on the field **No** of surreal numbers. By calculus, we mean, for now, a way to let certain series act as partial differentiable functions (this is made precise in Section 2.1.2) on a class with appropriate structure.

An action of a field of well-based series $\mathbb{F}$ on a field of well-based series $\mathbb{S}$ can be conceived as a partially defined *composition law*

$$\circ_{\mathbb{S}} : \mathbb{F} \times \mathbb{S} \longrightarrow \mathbb{S} \tag{8}$$

such that each $f \in \mathbb{F}$ acts as a partial function $s \mapsto f \circ_{\mathbb{S}} s$ on $\mathbb{S}$. It is natural and useful to ask that elements in $\mathbb{F}$ act as series. That is, that given $s \in \mathbb{S}$, the function

$$\mathbb{F} \longrightarrow \mathbb{S}; f \mapsto f \circ_{\mathbb{S}} s,$$

should it be defined, must be a morphism of ordered fields which commutes with transfinite sums of well-based families. The field $\mathbb{F}$ should have enough structure that it be amenable to an internal law $\circ : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ whereby it act on itself.

Defining a composition law as in (8) can serve two purposes. Firstly, it induces a structure on $\mathbb{S}$ in terms of the action of $\mathbb{F}$, and allows us to study which equations

$$f \circ_{\mathbb{S}} s_1 = s_2, \quad f \in \mathbb{F}, s_1, s_2 \in \mathbb{S}$$

can be solved on $\mathbb{S}$. This is the case for fields of transseries and hyperseries, which are characterized by the action of a field $\mathbb{F}$ of so-called logarithmic transseries or hyperseries. Secondly, it provides a way to represent series in $\mathbb{F}$ as functions and ascribe them a subclass of $\mathbb{S}$ as a natural domain. If, for instance, given $f \in \mathbb{F}$, the partial function $\mathbb{S} \longrightarrow \mathbb{S}; s \mapsto f \circ_{\mathbb{S}} s$ is differentiable, and if its derivative has the form $s \mapsto \partial(f) \circ s$ for a unique series $\partial(f) \in \mathbb{F}$, then we also obtain a natural derivation ∂ on $\mathbb{F}$ which is compatible with the composition law.

There need not be a strong separation between the function field and the series field in the calculus. On the contrary, the main goal of this thesis is give a presentation of surreal numbers which should allow one to establish an equivalence between those notions for a very large class **No** of (surreal) numbers, that turn out to be (well-based) series, that should turn out to be (surreal-valued) functions…

Analytic functions

Analyticity is a condition on a function $\mathcal{A}: \mathbb{S} \longrightarrow \mathbb{S}$ in a field of well-based series, which states that $\mathcal{A}$ is locally determined by power series. That is, for each $s \in \mathbb{S}$, there is a fixed power series $P_s = \sum_{k \in \mathbb{N}} P_{s,k} z^k$ with coefficients in $\mathbb{S}$ such that for sufficiently small ε, we have a well-based expansion

$$\mathcal{A}(s+\varepsilon) = P_{s,0} + P_{s,1}\,\varepsilon + P_{s,2}\,\varepsilon^2 + \cdots. \tag{9}$$

This formal notion of analyticity is far from possessing the strength of analyticity in the case of real-valued functions. Nonetheless, we will see in Chapter 2 that it retains a few of its properties once properly stated in the formal setting. For instance, an analytic function $\mathcal{A}$ is infinitely differentiable, and (9) can always be rewritten as

$$\mathcal{A}(s+\varepsilon) = \mathcal{A}(s) + \mathcal{A}'(s)\,\varepsilon + \frac{\mathcal{A}''(s)}{2}\,\varepsilon^2 + \cdots. \tag{10}$$

Thus imposing analyticity for functions $\tilde{f}: s \mapsto f \circ_{\mathbb{S}} s$, $f \in \mathbb{F}$ in our calculi is a way to impose a compatibility between derivations and composition laws. Furthermore (10) provides a natural way to extend $\tilde{f}$ around a series s for which the properties of compositions on hyperseries already give us the expected values for $f \circ s$, $\partial(f) \circ s$, $\partial(\partial(f)) \circ s, \ldots$, but where it fails, perhaps for general model theoretic reasons, to provide defining equations for $f \circ (s+\varepsilon)$. In that case Taylor expansions can be construed as a natural way to define $f \circ (s+\varepsilon)$. In fact this type of expansion can turn out to be the *only* guide in establishing the value of $f \circ s$ for certain complicated hyperseries or surreal numbers f.

Transseries

Transseries are *prima facie* ideal candidates for the analytic calculi. Those generalized power series, by virtue of being well-based series involving exponentials and logarithms, can be endowed with transfinite sums and products, derivations, integration operators, composition laws, and have functional inversion. Their impressive closure properties and their formal nature make them amenable to algorithmic methods for solving equations and other mathematical problems, as van der Hoeven's thesis [60] illustrates. We will introduce transseries in Chapter 3 with the purposes of applying the content of Part I in simple contexts and of preparing the reader (and the proof writer) for the more demanding work on hyperseries.

Chapter 1
Strongly linear algebra

Strongly linear algebra, over $\mathbb{R}$ is the realm of vector spaces over $\mathbb{R}$ which are ordered fields, and which are equipped with a notion of summation with respect to which sums

$$\sum_{i \in I} f_i$$

of certain possibly infinite families $(f_i)_{i \in I}$ called well-based families can be defined. The criterion for summability is order-theoretic in nature, and requires non-trivial facts regarding ordered and partially ordered sets that are proved or stated throughout this chapter.

1.1 Well-based sets in ordered groups

In this section, we define ordered groups of well-based series, focusing on the notion of transfinite sums of well-based family irrespective of an additional structure of ordered field.

1.1.1 Well-ordered and well-based sets

We start with purely order theoretic simple statements.

Definition 1.1.1. *A* **well-founded** *ordering on a class* $\mathbf{X}$ *is a partial ordering on* $\mathbf{X}$ *for which each non-empty subclass of* $\mathbf{X}$ *has a minimal element. A* **well-ordered** *ordering on a class* $\mathbf{X}$ *is a linear and well-founded ordering on* $\mathbf{X}$*.*

So an ordered class $(\mathbf{X}, <)$ is well-ordered if and only if it is both well-founded and linearly ordered. The reader can check that for well-founded subclasses $\mathbf{Y}, \mathbf{Z} \subseteq \mathbf{X}$, the class $\mathbf{Y} \cup \mathbf{Z}$ is well-founded. We have a well-founded *induction principle*: if $(\mathbf{X}, <)$ is a well-founded ordered class, and $\mathbf{Y} \subseteq \mathbf{X}$ is a subclass with

$$\forall x \in \mathbf{Y}, ((\forall y \in \mathbf{X}, (y < x \Longrightarrow y \in \mathbf{Y})) \Longrightarrow x \in \mathbf{X}),$$

then $\mathbf{Y} = \mathbf{X}$.

If $\mathbf{X}$ is a class and $u: \mathbb{N} \longrightarrow \mathbf{X}$ is a sequence, then a *subsequence* of u is a sequence $u \circ \psi$ where $\psi: \mathbb{N} \longrightarrow \mathbb{N}$ is strictly increasing.

Proposition 1.1.2. (**Limitation Of Size**) *A partially ordered class* $(\mathbf{X}, <)$ *is well-founded if and only if every sequence* $u: \mathbb{N} \longrightarrow \mathbf{X}$ *in* $\mathbf{X}$ *has a nondecreasing subsequence.*

Proof. Assume that $(\mathbf{X}, <)$ is well-founded and consider a sequence $u: \mathbb{N} \longrightarrow \mathbf{X}$. By induction on $\mathbb{N}$, let us define a strictly increasing map $\psi: \mathbb{N} \longrightarrow \mathbb{N}$ such that $u \circ \psi$ is nondecreasing. For $m \in \mathbb{N}$, define X_m to be the set of positive integers $n > m$ such that u_n is minimal in $\{u_k : k > m\}$. Set $\psi(0) := 0$ and

$$\psi(m+1) := \min X_{\psi(m)}$$

for all $m \in \mathbb{N}$. Since $m < X_m$ for all $m \in \mathbb{N}$, the function ψ is strictly increasing. We claim that $u \circ \psi$ is nondecreasing. Indeed for $m, n \in \mathbb{N}$ with $m < n$, the element $u_{\psi(m)}$ is minimal in $\{u_k : k > \psi(m)\} \ni u_{\psi(n)}$, so we have $u_{\psi(n)} \not< u_{\psi(m)}$.

Now assume that $(\mathbf{X}, <)$ is not well-founded. We will define a sequence in $\mathbf{X}$ which has no nondecreasing subsequence. Consider a non-empty subclass $\mathbf{Y} \subseteq \mathbf{X}$ with no minimal element. By the axiom of limitation of size, we have a well-ordering $\blacktriangleleft$ of $\mathbf{Y}$. We define a strictly decreasing sequence $u: \mathbb{N} \longrightarrow \mathbf{Y}$ by induction. Define u_0 to be any element of $\mathbf{Y}$. If $n \in \mathbb{N}$ and $u_0 > \cdots > u_n$ are defined in $\mathbf{Y}$, then we note that u_n is not minimal in $\mathbf{Y}$, so there is a unique $\blacktriangleleft$-minimal element $u_{n+1} \in \mathbf{Y}$ with $u_{n+1} < u_n$, thus extending the sequence. For any strictly increasing map $\varphi: \mathbb{N} \longrightarrow \mathbb{N}$, the sequence $u \circ \varphi$ is strictly decreasing. In particular, the sequence u has no nondecreasing subsequence. □

Corollary 1.1.3. *A linearly ordered class* $(\mathbf{X}, <)$ *is well-ordered if and only if every sequence* u: $\mathbb{N} \longrightarrow \mathbf{X}$ *in* $\mathbf{X}$ *has an increasing subsequence.*

Corollary 1.1.4. *A partially ordered class* $(\mathbf{X}, <)$ *is well-founded if and only if each non-empty sub*set *of* $\mathbf{X}$ *has a minimal element. A linearly ordered class is well-ordered if and only if each non-empty sub*set *of* $\mathbf{X}$ *has a minimum.*

We say that an ordered class $(\mathbf{X}, <)$ is *well-based* if $(\mathbf{X}, >)$ is well-ordered, i.e. if any non-empty subset of $\mathbf{X}$ has a maximum. Well-based classes have opposite properties to well-ordered ones.

If $(\mathbf{X}, <_{\mathbf{X}})$ and $(\mathbf{Y}, <_{\mathbf{Y}})$ are partially-ordered classes, then we define their product $(\mathbf{X} \times \mathbf{Y}, <_{\mathbf{X}\times\mathbf{Y}})$. The underlying class of $\mathbf{X} \times \mathbf{Y}$ is the Cartesian product $\mathbf{X} \times \mathbf{Y}$, with the ordering

$$(x, y) <_{\mathbf{X}\times\mathbf{Y}} (x', y') \Longleftrightarrow ((x, y) \neq (x', y') \wedge x \leqslant_{\mathbf{X}} x' \wedge y \leqslant_{\mathbf{Y}} y').$$

It is well-known that $(\mathbf{X} \times \mathbf{Y}, <_{\mathbf{X}\times\mathbf{Y}})$ is a partially ordered class, and that it is linearly ordered if both $<_{\mathbf{X}}$ and $<_{\mathbf{Y}}$ are linear. We next justify that the product preserves well-foundedness and well-orderedness.

Proposition 1.1.5. *Let* $(\mathbf{X}, <_{\mathbf{X}})$ *and* $(\mathbf{Y}, <_{\mathbf{Y}})$ *be well-founded (resp. well-ordered) classes. Then* $(\mathbf{X} \times \mathbf{Y}, <_{\mathbf{X}\times\mathbf{Y}})$ *is a well-founded (resp. well-ordered) class.*

Proof. Let $\mathbf{A} \subseteq \mathbf{X} \times \mathbf{Y}$ be a non-empty subclass. Let x be a minimal element in the non-empty class $\{z \in \mathbf{X} : \exists t \in \mathbf{Y}, (z, t) \in \mathbf{A}\}$. Let y be a minimal element in the non-empty class $\{t \in \mathbf{Y} : (x, t) \in \mathbf{A}\}$. Then (x, y) is minimal in $\mathbf{A}$. Therefore $(\mathbf{X} \times \mathbf{Y}, <_{\mathbf{X}\times\mathbf{Y}})$ is well-founded. □

1.1.2 Multiplicative notations

We sometimes prefer to consider multiplicative notations for groups, even in the Abelian case. We then usually use fraktur letters to represents monomial groups and elements, writing $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{L}$, and so on for the group and $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{l}$ and so on for the elements. We also sometimes denote orderings of multiplicatively denoted groups with the symbol $\prec$ instead of $<$. Given such a group $\mathfrak{M}$, the neutral element is denoted 1. Given elements $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$, we write $\mathfrak{m}\,\mathfrak{n}$ for their product, and $\mathfrak{m}^{-1}$ for the inverse of $\mathfrak{m}$ in $\mathfrak{M}$. For $n \in \mathbb{N}$, we set

$$\begin{aligned}
\mathfrak{m}^0 &:= 1,\\
\mathfrak{m}^n &:= \mathfrak{m} \cdots \mathfrak{m} \quad (\mathfrak{m} \text{ multiplied with itself } n \text{ times), and}\\
\mathfrak{m}^{-n} &:= (\mathfrak{m}^n)^{-1} = (\mathfrak{m}^{-1})^n.
\end{aligned}$$

Given $\mathfrak{S}, \mathfrak{T} \subseteq \mathfrak{M}$ and $n \in \mathbb{N}$, we write

$$\begin{aligned}
\mathfrak{S} \cdot \mathfrak{T} &:= \{\mathfrak{s}\,\mathfrak{t} : \mathfrak{s} \in \mathfrak{S} \wedge t \in \mathfrak{T}\},\\
\mathfrak{S}^n &:= \underbrace{\mathfrak{S} \cdots \mathfrak{S}}_{n \text{ times}} = \{\mathfrak{s}_1 \cdots \mathfrak{s}_n : \mathfrak{s}_1, \ldots, \mathfrak{s}_n \in \mathfrak{S}\}, \quad \text{and}\\
\mathfrak{S}^\infty &:= \bigcup_{n \in \mathbb{N}} \mathfrak{S}^n = \{\mathfrak{s}_1 \cdots \mathfrak{s}_n : n \in \mathbb{N} \wedge \mathfrak{s}_1, \ldots, \mathfrak{s}_n \in \mathfrak{S}\}.
\end{aligned}$$

Warning 1.1.6. The notation $\mathfrak{S}^n$ conflicts with the standard Cartesian product abbreviation

$$\mathbf{X}^n = \mathbf{X} \times \cdots \times \mathbf{X},$$

which we nonetheless also adopt. We expect that this will not lead to confusions.

If $(\mathbf{G}, +, 0, <)$ is a linearly ordered Abelian group, then a *multiplicative copy* of $\mathbf{G}$ is simply $\mathbf{G}$ itself, re-branded in the multiplicative language. We often represent it as the set

$$\sigma^{\mathbf{G}} = \{\sigma^a : a \in \mathbf{G}\}$$

of formal terms σ^a, where $\sigma \in \{x, z, \mathrm{e}, \mathrm{e}_\beta, \ldots\}$ is a symbol that will vary depending on the contexts. For $\sigma^a, \sigma^b \in \sigma^{\mathbf{G}}$, we have a product

$$\sigma^a \sigma^b := \sigma^{a+b}$$

and an ordering

$$\sigma^a \prec \sigma^b \Longleftrightarrow a < b.$$

1.1.3 Hahn product groups

We next introduce a type of linearly ordered Abelian group, due to Hans Hahn [56], that will frequently appear in our work. See also [4, Section 2.4] and [67] for different generalizations.

Let $(\mathbf{G}, +, 0, <)$ denote a linearly ordered Abelian group, and let $\mathbf{L}$ be a linearly ordered non-empty class. We write $H[\mathbf{L}, \mathbf{G}]$ for the class of functions $f: X \longrightarrow \mathbf{G}^{\neq}$ where $X \subseteq \mathbf{L}$ is a well-based (possibly empty) sub*set*. We write

$$X = \operatorname{supp} f.$$

We define $f \in H[\mathbf{L}, \mathbf{G}]$ to be *strictly positive* if $\operatorname{supp} f \neq \varnothing$ and $f(\max \operatorname{supp} f) > 0$ in $\mathbf{G}$, where the maximum is taken in $(\mathbf{L}, <)$.

For in $f, g \in H[\mathbf{L}, \mathbf{G}]$, the set $\operatorname{supp} f \cup \operatorname{supp} g$ is well-based, so the set

$$X := \{l \in \operatorname{supp} f \cup \operatorname{supp} g : f(l) + g(l) \neq 0\}$$

is well-based. We define $f + g$ to be the element

$$(f + g): X \longrightarrow \mathbf{G}; l \mapsto f(l) + g(l)$$

of $H[\mathbf{L}, \mathbf{G}]$. We call the binary operation $+$ the *pointwise sum* on $H[\mathbf{L}, \mathbf{G}]$. Note that $H[\mathbf{L}, \mathbf{G}] = \mathbf{G}^L$ as a class whenever L is a well-based set. We will sometimes write $f(l) := 0$ for all $f \in H[\mathbf{L}, \mathbf{G}]$ and $l \in \mathbf{L} \setminus \operatorname{supp} f$. With this convention, we have

$$(f + g)(l) = f(l) + g(l)$$

for all $f, g \in H[\mathbf{L}, \mathbf{G}]$ and $l \in \mathbf{L}$. That is, the operation $+$ is indeed a pointwise sum. In particular, since $\mathbf{G}$ is an Abelian group, the structure $(H[\mathbf{L}, \mathbf{G}], +, 0)$ is an Abelian group. Its neutral element 0 is the element f with $f(l) = 0$ for all $l \in \mathbf{L}$, i.e. the empty function $\varnothing$. Moreover [51, Section 2.7], setting $f < g$ if and only if $g - f$ is strictly positive, we obtain a linear ordering on $H[\mathbf{L}, \mathbf{G}]$, such that $(H[\mathbf{L}, \mathbf{G}], +, 0, <)$ is a linearly ordered Abelian group.

We call $H[\mathbf{L}, \mathbf{G}]$ the *Hahn product* of $\mathbf{G}$ to the power $\mathbf{L}$. Note that for $f, g \in H[\mathbf{L}, \mathbf{G}]$, we have $f < g$ if and only if $f \neq g$, and for

$$l_0 = \max \{l \in \operatorname{supp} f \cup \operatorname{supp} g : f(l) \neq g(l)\},$$

we have $f(l_0) < g(l_0)$ in $\mathbf{G}$.

Example 1.1.7. Consider the linearly ordered set of positive integers $\mathbb{N}$ and the linearly ordered Abelian group of integers $(\mathbb{Z}, +, <)$. We write $\mathbb{N}^*$ for the ordered set $(\mathbb{N}, >)$ where $<$ is the reverse ordering on $\mathbb{N}$. We have a Hahn product group $H[\mathbb{N}^*, \mathbb{Z}]$. Since $(\mathbb{N}, >)$ is well-based, this simply consists in the set of anti-lexicographically ordered maps $\mathbb{N} \longrightarrow \mathbb{Z}$.

Remark 1.1.8. In the literature, it is often imposed that $\mathbf{L}$, and later monomials groups $\mathfrak{M}$ involved in fields of well-bsed series (see Section 1.2.1) be sets, and not possibly proper classes. Some authors would rather consider our types of Hahn products as "$\boldsymbol{\kappa}$-bounded Hahn products" for $\boldsymbol{\kappa} = \mathbf{On}$, meaning that sizes of supports of elements in the Hahn product groups are strictly bounded by the uncountable ordinal $\boldsymbol{\kappa}$.

Because our goal is to build large fields of hyperseries closed under exponentiation (by a result of Kuhlmann-Kuhlmann–Shelah [68], those must have a proper class as monomial group), and because surreal numbers themselves are an **On**-bounded Hahn series field, we think our choice is sound. Since Hahn product groups and fields of well-based series are recurring objects in the thesis, we allow ourselves to uniformly discard the expression **On**-bounded wherever it applies.

Lemma 1.1.9. *Let $\mathbf{G}$ be a linearly ordered Abelian group. Let $(\mathbf{L}_\mu)_{\mu \in \mathbf{On}}$ be a family of linearly ordered classes such that $\mathbf{L}_\mu \subseteq \mathbf{L}_\nu$ whenever $\mu < \nu$. Set $\mathbf{G}_\mu := H[\mathbf{L}_\mu, \mathbf{G}]$ for each μ, so $\mathbf{G}_\mu \subseteq \mathbf{G}_\nu$ for $\mu < \nu$. Set $\mathbf{L} := \bigcup_{\mu \in \mathbf{On}} \mathbf{L}_\nu$. Then*

$$\bigcup_{\mu \in \mathbf{On}} \mathbf{G}_\mu = H[\mathbf{L}, \mathbf{G}].$$

Proof. Set $\mathbf{G}_{\mathbf{On}} := \bigcup_{\mu \in \mathbf{On}} \mathbf{G}_\mu$. Clearly $\mathbf{G}_{\mathbf{On}} \subseteq H[\mathbf{L}, \mathbf{G}]$, so it remains to show the other inclusion. Let $f \in H[\mathbf{L}, \mathbf{G}]$. For each $l \in \operatorname{supp} f$, let μ_l be the least $\mu \in \mathbf{On}$ with $l \in \mathbf{L}_\mu$. Set

$$\mu_f := \sup \{\mu_l : l \in \operatorname{supp} f\}.$$

Then $f \in \mathbf{G}_{\mu_f} \subseteq \mathbf{G}_{\mathbf{On}}$. □

Remark 1.1.10. The previous result does not apply for set-sized unions. Consider for instance the subgroups $(2^{-n}\mathbb{Z}, +, <), n \in \mathbb{N}$ of the additive group $\mathbb{D}$ of dyadic rational numbers. We have $\mathbb{D} = \bigcup_{n \in \mathbb{N}} 2^{-n}\mathbb{Z}$. Now, the map

$$f: \mathbb{N} \longrightarrow \{2^{-n} : n \in \mathbb{N}\}; n \mapsto 2^{-n}$$

lies in $H[\mathbb{N}^*, \mathbb{D}]$, but not in $\bigcup_{n \in \mathbb{N}} H[\mathbb{N}^*, 2^{-n}\mathbb{Z}]$.

Hahn product groups will be ubiquitous in the sequel. Indeed, the monomials in our well-based series and the fields of well-based series themselves will be based on Hahn product groups.

1.1.4 Well-based families

Let $\mathbf{L}$, $\mathbf{G}$ and $H[\mathbf{L}, \mathbf{G}]$ be as in Section 1.1.3. We now introduce the central notion of (possibly transfinite) well-based families and their sums.

Definition 1.1.11. *Let $\mathbf{I}$ be a class. A family $F = (f_i)_{i \in \mathbf{I}}$ in $H[\mathbf{L}, \mathbf{G}]$ is said* **well-based** *if*

i. $\bigcup_{i \in \mathbf{I}} \operatorname{supp} f_i$ is a well-based set, and

ii. $\mathbf{I}_l := \{i \in \mathbf{I} : l \in \operatorname{supp} f_i\}$ is finite for all $l \in \mathbf{L}$.

Then we may define the sum $\sum_{i \in \mathbf{I}} f_i$ of $(f_i)_{i \in \mathbf{I}}$ as the series $\sum F$ with support

$$\operatorname{supp} \sum F := \left\{ l \in \mathbf{L} : \sum_{i \in \mathbf{I}_l} f_i(l) \neq 0 \right\}$$

and with $(\sum F)(l) := \sum_{i \in \mathbf{I}_l} f_i(l)$ for all $l \in \operatorname{supp} f$.

We will sometimes switch between the notations $\sum F$ and $\sum_{i \in \mathbf{I}} f_i$ for the sum of $F = (f_i)_{i \in \mathbf{I}}$.

Remark 1.1.12. This is nothing but a possibly infinite pointwise sum. Indeed the conditions above are designed to ensure that such a pointwise sum is defined, by imposing that

- the support of $\sum F$ is a well-based set (by i), and
- the sums $(\sum_{i \in \mathbf{I}} f_i)(l)$ have finite support (by ii),

so that $\sum F$ be a well-defined element of $H[\mathbf{L}, \mathbf{G}]$.

In fact, it would be possible in certain cases to relax the conditions. For instance, one could ask, instead of i, that the class $\{l\in\mathbf{L}:\sum_{i\in\mathbf{I}_l} f_i(l)\neq 0\}$ be a well-based set, or, instead of ii, that each family $(f_i(l))_{l\in\mathbf{L}}$ be summable for a certain notion of summability on $\mathbf{G}$. However, the sequel of this chapter will show that those strong conditions make the notions of well-based families and sums thereof very practical to manipulate. Furthermore, the aforementioned weakenings of i and ii can produce irregularities, such as the failure of Proposition 1.1.21 below.

Remark 1.1.13. The class $\mathbf{S}:=\bigcup_{i\in\mathbf{I}}\operatorname{supp} f_i$ is a set if and only if $s_i=0$ outside of a *subset* of $\mathbf{I}$. Indeed, assume that $\mathbf{S}$ is a set and consider a subclass $\mathbf{J}\subseteq\mathbf{I}$ with $s_j\neq 0$ for all $j\in\mathbf{J}$. For $l\in\mathbf{S}$, the class $\mathbf{J}_l:=\{j\in\mathbf{J}:l\in\operatorname{supp} f_j\}$ is finite, whence in particular a set. So $\mathbf{J}=\bigcup_{l\in\mathbf{S}}\mathbf{J}_l$ is a set.

Given $a\in\mathbf{G}$ and $l\in\mathbf{L}$, we write $a\,\chi_l$ for the unique function $\{l\}\longrightarrow\{a\}$. So $a\,\chi_l\in H[\mathbf{L},\mathbf{G}]$. If a family $(f_i)_{i\in\mathbf{I}}$ is well-based, then we have

$$\sum_{i\in\mathbf{I}} f_i=\sum_{l\in\bigcup_{i\in\mathbf{I}}\operatorname{supp} f_i}\left(\sum_{i\in\mathbf{I}_l} f_i(l)\right)\chi_l,$$

$$\sum_{i\in\mathbf{I}} f_i=\sum_{l\in\mathbf{L}}\left(\sum_{i\in\mathbf{I}} f_i(l)\right)\chi_l.$$

Conversely, given $f\in H[\mathbf{L},\mathbf{G}]$, the family $(f(l)\,\chi_l)_{l\in\operatorname{supp} f}$ is well-based, with

$$f=\sum_{l\in\mathbf{L}} f(l)\,\chi_l.$$

This yields a representation of elements in $H[\mathbf{L},\mathbf{G}]$ as well-based formal sums. A multiplicative copy of $H[\mathbf{L},\mathbf{G}]$ will in general be represented as a group of formal well-based products

$$f=\prod_{l\in\mathbf{L}}\chi_l^{f(l)}.$$

1.1.5 Properties of well-based families

Here, we derive elementary properties of well-based families. Those properties are well known, at least in the case of fields of well-based series and set-sized families, see [82, 60, 92, 62, 63].

We fix a Hahn product group $H[\mathbf{L},\mathbf{G}]$ and a class $\mathbf{I}$. We will use the two following elementary results, which follows easily from Definition 1.1.11, often without further mention:

Lemma 1.1.14. *Let $(f_i)_{i\in\mathbf{I}}$ be a well-based family in $H[\mathbf{L},\mathbf{G}]$ and let $(g_i)_{i\in\mathbf{I}}$ be a family in $H[\mathbf{L},\mathbf{G}]$ with*

$$\operatorname{supp} g_i\subseteq\operatorname{supp} f_i$$

for all $i\in\mathbf{I}$. Then $(g_i)_{i\in\mathbf{I}}$ is well-based.

Lemma 1.1.15. *Let $(f_i)_{i\in\mathbf{I}}$ be a well-based family in $H[\mathbf{L},\mathbf{G}]$, and let $\mathbf{J}\subseteq\mathbf{I}$ be a subclass. Then $(f_j)_{j\in\mathbf{J}}$ is well-based.*

We need *not* have $\operatorname{supp}\sum_{j\in\mathbf{J}} f_j\subseteq\operatorname{supp}\sum_{i\in\mathbf{I}} f_i$ in that case. For instance, in $H[\mathbb{N}^*,\mathbb{Z}]$, the family $(1\,\chi_n-1\,\chi_{n+1})_{n\in\mathbb{N}}$ is well-based, and we have $\sum_{n\in\mathbb{N}}(1\,\chi_n-1\,\chi_{n+1})=\chi_0$ whereas $\sum_{n>0}(1\,\chi_n-1\,\chi_{n+1})=\chi_1$.

Proposition 1.1.16. [62, Proposition 3.1(d)] *Let $F=(f_i)_{i\in\mathbf{I}}$ be a well-based family in $H[\mathbf{L},\mathbf{G}]$, let $\mathbf{J}$ be a class and let $\sigma\colon\mathbf{J}\longrightarrow\mathbf{I}$ be a bijection. The family $F\circ\sigma:=(f_{\sigma(j)})_{j\in\mathbf{J}}$ is well-based, with*

$$\sum F\circ\sigma=\sum F,$$

i.e.

$$\sum_{j\in\mathbf{J}} f_{\sigma(j)}=\sum_{i\in\mathbf{I}} f_i.$$

Proof. We have $\bigcup_{j\in\mathbf{J}}\operatorname{supp} f_{\sigma(j)}\subseteq\bigcup_{i\in\mathbf{I}}\operatorname{supp} f_i$, so $\bigcup_{j\in\mathbf{J}}\operatorname{supp} f_{\sigma(j)}$ is well-based. For $l\in\mathbf{L}$, we have

$$\mathbf{J}_l:=\{j\in\mathbf{J}:l\in\operatorname{supp} f_{\sigma(j)}\}\subseteq\sigma^{-1}(\mathbf{I}_l)$$

where $\mathbf{I}_l=\{i\in\mathbf{I}:l\in f_i\}$ is finite. Since σ is injective, we deduce that $\mathbf{J}_l$ is finite, so $(f_{\sigma(j)})_{j\in\mathbf{J}}$ is well-based. For $l\in\mathbf{L}$, we have

$$\begin{aligned}\sum_{j\in\mathbf{J}}(f_{\sigma(j)})(l) &= \sum_{j\in\mathbf{J}_l}(f_{\sigma(j)})(l)\\ &= \sum_{\sigma(j)\in\mathbf{I}_l}(f_{\sigma(j)})(l)\\ &= \sum_{i\in\mathbf{I}_l}(f_i)(l) && \text{(since } \sigma \text{ is bijective)}\\ &= \left(\sum_{i\in\mathbf{I}}f_i\right)(l).\end{aligned}$$

We deduce that $\sum_{j\in\mathbf{J}}f_{\sigma(j)}=\sum_{i\in\mathbf{I}}f_i$. □

Proposition 1.1.17. *Let $F=(f_i)_{i\in\mathbf{I}}$ and $G=(g_i)_{i\in\mathbf{I}}$ be well-based families in $H[\mathbf{L},\mathbf{G}]$. Then $(F+G)=(f_i+g_i)_{i\in\mathbf{I}}$ is well-based, with*

$$\sum(F+G)=\sum F+\sum G,$$

i.e.

$$\sum_{i\in\mathbf{I}}f_i+g_i=\sum_{i\in\mathbf{I}}f_i+\sum_{i\in\mathbf{I}}g_i.$$

Proof. For $i\in\mathbf{I}$, we have $\operatorname{supp}(f_i+g_i)\subseteq\operatorname{supp} f_i\cup\operatorname{supp} g_i$, so

$$\bigcup_{i\in\mathbf{I}}\operatorname{supp}(f_i+g_i)\subseteq\bigcup_{i\in\mathbf{I}}\operatorname{supp} f_i\cup\bigcup_{i\in\mathbf{I}}\operatorname{supp} g_i$$

is a well-based set. For $l\in\mathbf{L}$, the class

$$\mathbf{I}_l=\{i\in\mathbf{I}:l\in\operatorname{supp}(f_i+g_i)\}$$

is contained in the union of the two finite sets

$$\begin{aligned}\mathbf{I}_{F,l} &= \{i\in\mathbf{I}:l\in\operatorname{supp} f_i\}, \quad \text{and}\\ \mathbf{I}_{G,l} &= \{i\in\mathbf{I}:l\in\operatorname{supp} g_i\}.\end{aligned}$$

So $\mathbf{I}_l$ is a finite set. Therefore $(f_i+g_i)_{i\in\mathbf{I}}$ is well-based. For $l\in\mathbf{L}$, we have

$$\begin{aligned}\sum(F+G)(l) &= \sum_{i\in\mathbf{I}_l}(f_i+g_i)(l)\\ &= \sum_{i\in\mathbf{I}_l}f_i(l)+\sum_{i\in\mathbf{I}}g_i(l)\\ &= \sum_{i\in\mathbf{I}_{F,l}}f_i(l)+\sum_{i\in\mathbf{I}_{G,l}}f_i(l) && \text{(since } \mathbf{I}_{F,l} \text{ and } \mathbf{I}_{G,l} \text{ are finite)}\\ &= \left(\sum F\right)(l)+\left(\sum G\right)(l)\end{aligned}$$

So $\sum(F+G)=(\sum F)+(\sum G)$. □

Lemma 1.1.18. [62, Proposition 3.1(e)] *Let $\mathbf{I},\mathbf{J}$ be classes, and let $(\mathbf{I}_j)_{j\in\mathbf{J}}$ be a family of classes with $\mathbf{I}=\bigsqcup_{j\in\mathbf{J}}\mathbf{I}_j$. Let $(f_i)_{i\in\mathbf{I}}$ be a well-based family. Then for each $j\in\mathbf{J}$, the family $(f_i)_{i\in\mathbf{I}_j}$ is well-based. Setting $g_j:=\sum_{i\in\mathbf{I}_j}f_i$ for all $j\in\mathbf{J}$, the family $(g_j)_{j\in\mathbf{J}}$ is well-based, with*

$$\sum_{i\in\mathbf{I}}f_i=\sum_{j\in\mathbf{J}}g_j.$$

Proof. That each $(f_i)_{i\in \mathbf{I}_j}, j\in \mathbf{J}$ is well-based follows from Lemma 1.1.15. For $j\in \mathbf{J}$, we have

$$\operatorname{supp} g_j \subseteq \bigcup_{i\in \mathbf{I}} \operatorname{supp} f_i, \tag{1.1.1}$$

so $\bigcup_{j\in \mathbf{J}} \operatorname{supp} g_j \subseteq \bigcup_{i\in \mathbf{I}} \operatorname{supp} f_i$ is a well-based set. We also deduce from (1.1.1) that for $l\in \mathbf{L}$ and $j\in \mathbf{J}$ we have $(\mathbf{I}_j)_l \subseteq \mathbf{I}_l$ where we use the notations from Definition 1.1.11. So $(\mathbf{I}_j)_l$ is finite, and thus $(g_j)_{j\in \mathbf{J}}$ is well-based. For $l\in \mathbf{L}$, we have

$$\sum_{j\in \mathbf{J}} g_j(l) = \sum_{j\in \mathbf{J}} \sum_{i\in \mathbf{I}_j} f_i(l).$$

Since the sums involved have finite supports and $\mathbf{I} = \bigsqcup_{j\in \mathbf{J}} \mathbf{I}_j$, we obtain $\sum_{j\in \mathbf{J}} g_j(l) = \sum_{i\in \mathbf{I}} f_i(l)$, hence the result. □

Corollary 1.1.19. [62, Proposition 3.1(c)] *Let $F_1 = (f_i)_{i\in \mathbf{I}_1}$ and $F_2 = (f_i)_{i\in \mathbf{J}_1}$ be well-based families where $\mathbf{I} = \mathbf{I}_1 \sqcup \mathbf{I}_2$ is a disjoint union. Then $F_1 \amalg F_2 := (f_i)_{i\in \mathbf{I}}$ is well-based, with*

$$\sum F_1 \amalg F_2 = \sum F_1 + \sum F_2,$$

i.e.

$$\sum_{i\in \mathbf{I}} f_i = \sum_{i\in \mathbf{I}_1} f_i + \sum_{i\in \mathbf{i}_2} f_i.$$

We also have a formal Dirichlet rearrangement theorem:

Lemma 1.1.20. *Let $\mathbf{I}, \mathbf{J}$ be classes and let $(f_{i,j})_{(i,j)\in \mathbf{I}\times \mathbf{J}}$ be a well-based family in $H[\mathbf{L}, \mathbf{G}]$. For each $i_0 \in \mathbf{I}$ and for each $j_0 \in \mathbf{J}$, the families $(f_{i_0,j})_{j\in \mathbf{J}}$, $(f_{i,j_0})_{i\in \mathbf{I}}$ are well-based. Moreover families $(\sum_{j\in \mathbf{J}} f_{i,j})_{i\in \mathbf{I}}$ and $(\sum_{i\in \mathbf{I}} f_{i,j})_{j\in \mathbf{J}}$ are well-based, with*

$$\sum_{i\in \mathbf{I}} \left(\sum_{j\in \mathbf{J}} f_{i,j}\right) = \sum_{(i,j)\in \mathbf{I}\times \mathbf{J}} f_{i,j} = \sum_{j\in \mathbf{J}} \left(\sum_{i\in \mathbf{I}} f_{i,j}\right).$$

Proof. Apply Proposition 1.1.18 with $\mathbf{I}_j := \mathbf{I} \times \{j\}$ for all $j\in \mathbf{J}$ and $\mathbf{J}_i := \{i\} \times \mathbf{J}$ for all $i\in \mathbf{I}$. □

Proposition 1.1.21. *Let I be a set and let $(f_i)_{i\in I}$ be a well-based family with $f_i > 0$ for all $i\in I$. Then $\sum_{i\in I} f_i > 0$.*

Proof. Write $f = \sum_{i\in I} f_i$. Set $l := \max \bigcup_{i\in I} \operatorname{supp} f_i$, and let $j\in I$ with $\mathfrak{m} = \max \operatorname{supp} f_j$. For all $i\in I$, we have $l \geqslant \max \operatorname{supp} f_i$ so $f_i(l) \geqslant 0$. It follows that $f(l) = \sum_{i\in I} f_i(l) \geqslant f_j(l) > 0$. So $l = \max \operatorname{supp} f$ and $f > 0$. □

1.1.6 Well-based classes in Abelian linearly ordered groups

Let $(\mathfrak{M}, \cdot, 1, \prec)$ be a multiplicative, linearly ordered group. We now state Bernhard Neumann's important results on products $\mathfrak{S} \cdot \mathfrak{T}$, $\mathfrak{S}^n$ and $\mathfrak{S}^\infty$ for well-based subclasses $\mathfrak{S}, \mathfrak{T}$ of $\mathfrak{M}$ and $n\in \mathbb{N}$.

Lemma 1.1.22. [82, Lemma 3.2 and Corollary 3.21] *Let $\mathfrak{S}, \mathfrak{T} \subseteq \mathfrak{M}$ be well-based subclasses. Then the class $\mathfrak{S} \cdot \mathfrak{T}$ is well-based. Moreover, for all $\mathfrak{m} \in \mathfrak{S} \cdot \mathfrak{T}$, the class $\{(\mathfrak{u}, \mathfrak{v}) \in \mathfrak{S} \times \mathfrak{T} : \mathfrak{m} = \mathfrak{u}\,\mathfrak{v}\}$ is finite.*

Corollary 1.1.23. *Let $n\in \mathbb{N}$ and let $\mathfrak{S}_1, \ldots, \mathfrak{S}_n \subseteq \mathfrak{M}$ be well-based subclasses. The class*

$$\mathfrak{S} := \mathfrak{S}_1 \cdots \mathfrak{S}_n$$

is well-based. Moreover, for all $\mathfrak{m} \in \mathfrak{S}$, the class

$$\{(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \in \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_n : \mathfrak{m} = \mathfrak{u}_1 \cdots \mathfrak{u}_n\}$$

is finite.

Lemma 1.1.24. [82, Theorem 3.4] *Let $\mathfrak{S} \subseteq \mathfrak{M}$ be a well-based subclass with $\mathfrak{S} \preccurlyeq 1$. The class $\mathfrak{S}^\infty$ is well-based.*

Proof. Write $\mathfrak{T} := \mathfrak{S} \setminus \{1\}$. By [82, Theorem 3.4], the class $\mathfrak{T}^{\diamond} := \bigcup_{k>0} \mathfrak{T}^n$ is well-based. But then

$$\mathfrak{S}^{\infty} = \{1\} \cup \mathfrak{T}^{\diamond}$$

is well-based. □

Lemma 1.1.25. [82, Theorem 3.5] *Let $\mathfrak{S} \subseteq \mathfrak{M}$ be a well-based subclass with $\mathfrak{S} \prec 1$. For all $\mathfrak{m} \in \mathfrak{S}^{\infty}$, the class $\{n \in \mathbb{N} : \mathfrak{m} \in \mathfrak{S}^n\}$ is finite.*

1.2 Fields of well-based series

In this section, we define the ordered fields of well-based series that will be the underlying structures for hyperseries. We will only consider well-based series over $\mathbb{R}$, although most of our results apply in more general cases. The results in Sections 1.2.1, 1.2.2 and 1.2.3 are well-known: see [56, 82, 92].

1.2.1 Well-based series

Let $(\mathfrak{M}, \times, 1, \prec)$ be a *non-trivial*, linearly ordered, *Abelian* group. In particular $(\mathfrak{M}, \prec)$ is an infinite linearly ordered class. Considering the ordered additive group $(\mathbb{R}, +, 0, <)$ of real numbers, we have a well-defined linearly ordered Abelian Hahn product group $H[\mathfrak{M}, \mathbb{R}]$. We write

$$\mathbb{R}[[\mathfrak{M}]] := H[\mathfrak{M}, \mathbb{R}],$$

which for now is equipped with its additively denoted structure $(\mathbb{R}[[\mathfrak{M}]], +, 0, <)$ from Section 1.1.3. We call $(\mathfrak{M}, \times, 1, \prec)$ the *monomial group* of $\mathbb{R}[[\mathfrak{M}]]$. Recall that we have an inclusion $\mathbb{R} \times \mathfrak{M} \longrightarrow \mathbb{R}[[\mathfrak{M}]]$; $(r, \mathfrak{m}) \mapsto r\,\chi_{\mathfrak{m}}$, whose range is denoted $\mathbb{R}\,\mathfrak{M}$. For $(r, \mathfrak{m}) \in \mathbb{R} \times \mathfrak{M}$, we will simply write $r\,\mathfrak{m} := r\,\chi_{\mathfrak{m}}$. Elements of $\mathfrak{M} \subseteq \mathbb{R}[[\mathfrak{M}]]$ are called *monomials*, whereas those in $\mathbb{R}\,\mathfrak{M} \subseteq \mathbb{R}[[\mathfrak{M}]]$ are called *terms*. Finally, $\mathbb{R}^{\neq}\,\mathfrak{M}$ denotes the class of non-zero terms.

For the sequel, we write $\mathbb{S} := \mathbb{R}[[\mathfrak{M}]]$. We call elements s of $\mathbb{S}$ *well-based series*, and we write

$$s_{\mathfrak{m}} := s(\mathfrak{m}) \in \mathbb{R},$$

for the coefficients of series $s \in \mathbb{S}$, for all $\mathfrak{m} \in \mathfrak{M}$. Recall that each $s \in \mathbb{S}$ is the sum

$$s = \sum_{\mathfrak{m}} s_{\mathfrak{m}}\,\mathfrak{m}$$

of the well-based family $(s_{\mathfrak{m}}\,\mathfrak{m})_{\mathfrak{m} \in \mathfrak{M}}$. The *length* of s as a series is the order type $\operatorname{ot}(\operatorname{supp} s, \succ)$ for the reverse ordering $\succ$ on $\mathfrak{M}$. Terms are the well-based series of length $\leqslant 1$, and 0 is the unique series of length 0.

If s is a well-based series with $\operatorname{supp} s \neq \varnothing$, i.e. with $s \neq 0$, then we write

$$\begin{aligned} \mathfrak{d}_s &:= \max \operatorname{supp} s \in \mathfrak{M} \quad \text{and} \\ \tau_s &:= s_{\mathfrak{d}_s}\,\mathfrak{d}_s \in \mathbb{R}^{\neq}\,\mathfrak{M}. \end{aligned}$$

respectively for the *dominant monomial* and *dominant term* of s. For $\mathfrak{m} \in \mathfrak{M}$, we set

$$s_{\succ \mathfrak{m}} := \sum_{\mathfrak{n} \in \mathfrak{M}^{\succ \mathfrak{m}}} s_{\mathfrak{n}}\,\mathfrak{n},$$

and we write

$$\begin{aligned} s_{\succ} &= s_{\succ 1}, \\ s_{\asymp} &= s_1 \in \mathbb{R}, \\ s_{\prec} &= s - s_{\succ} - s_1, \end{aligned}$$

so $\operatorname{supp} s_{\succ} \succ 1$, $\operatorname{supp} s_{\asymp} \subseteq \{1\}$ and $\operatorname{supp} s_{\prec} \prec 1$.

For $s, t \in \mathbb{S}$, we say that t is a *strict truncation* of s and we write $t \lhd s$ if $t \neq s$ and $\operatorname{supp}(t - s) \prec \operatorname{supp} t$. The relation $\lhd$ is a well-founded partial ordering on $\mathbb{S}$ with minimum 0 and, we denote its corresponding non-strict ordering by $\trianglelefteq$. We also write $+\!\!\!\!+$ for the restriction of $+$ to the class $\{(t, s) \in \mathbb{S} \times \mathbb{S} : \operatorname{supp} s \succ \operatorname{supp} t\} = \{(s, t) \in \mathbb{S} \times \mathbb{S} : s \trianglelefteq s + t\}$. That is, the expression $s +\!\!\!\!+ t = u$ for s, $t, u \in \mathbb{S}$ means that $s + t = u$ and that $\operatorname{supp} s \succ \operatorname{supp} t$.

Let $s, t \in \mathbb{S}$. By Lemma 1.1.22, the set $(\operatorname{supp} s) \cdot (\operatorname{supp} t)$ is well-based and for each $\mathfrak{m} \in (\operatorname{supp} s) \cdot (\operatorname{supp} t)$, the set $\{(\mathfrak{u}, \mathfrak{v}) \in (\operatorname{supp} s) \times (\operatorname{supp} t) : \mathfrak{u}\,\mathfrak{v} = \mathfrak{m}\}$ is finite. Thus the family $(\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} s_{\mathfrak{u}}\, t_{\mathfrak{v}})_{\mathfrak{m} \in \mathfrak{M}}$ is well-based, and the Cauchy product

$$s\,t := \sum_{\mathfrak{m} \in \mathfrak{M}} \left(\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} s_{\mathfrak{u}}\, t_{\mathfrak{v}} \right) \mathfrak{m} \tag{1.2.1}$$

is well-defined. Note that

$$\operatorname{supp}(s\,t) \subseteq (\operatorname{supp} s) \cdot (\operatorname{supp} t). \tag{1.2.2}$$

Also note that the inclusion $\mathfrak{M} \subseteq \mathbb{S}^{>}$ preserves products. By [56], the class $(\mathbb{S}, +, \times, 0, 1, <)$ is an ordered field.

The ordering on $\mathfrak{M}$ extends into a partial ordering $\prec$ on $\mathbb{S}$ defined by $s \prec t$ if and only if $\mathbb{R}^{>}\,|s| < |t|$. We write $s \preccurlyeq t$ if $t \prec s$ is false, i.e. if there is $r \in \mathbb{R}^{>}$ with $|s| \leqslant r\,|t|$. We also write $s \asymp t$ if $s \preccurlyeq t$ and $t \preccurlyeq s$, i.e. if there is $r \in \mathbb{R}^{>}$ with $r\,|s| \geqslant |t|$ and $r\,|t| \geqslant s$. When s, t are non-zero, we have $s \prec t$ (resp. $s \preccurlyeq t$, resp. $s \asymp t$) if and only if $\mathfrak{d}_s \prec \mathfrak{d}_t$ (resp. $\mathfrak{d}_s \preccurlyeq \mathfrak{d}_t$, resp. $\mathfrak{d}_s = \mathfrak{d}_t$).

Then $\preccurlyeq$ is a dominance relation as per [4, Definition 3.1.1]. In other words, the relation $\preccurlyeq$ is a linear quasi-ordering on $\mathbb{S}$ with $1 \not\preccurlyeq 0$, and with

$$h \neq 0 \Longrightarrow f \preccurlyeq g \Longleftrightarrow f\,h \preccurlyeq g\,h \qquad \text{and} \qquad f \preccurlyeq h \wedge g \preccurlyeq h \Longrightarrow f + g \preccurlyeq h$$

for all $f, g, h \in \mathbb{S}$.

The relation $\preccurlyeq$ corresponds to the natural valuation on the ordered field $(\mathbb{S}, +, \times, <)$. In particular $(\mathbb{S}, +, \times, <, \preccurlyeq)$ is an ordered valued field with convex valuation ring

$$\mathbb{S}^{\preccurlyeq} := \{s \in \mathbb{S} : s \preccurlyeq 1\}.$$

More precisely, the dominant monomial function $\mathfrak{d} : \mathbb{S}^{\neq} \longrightarrow \mathfrak{M} ; s \mapsto \mathfrak{d}_s$ is a valuation on $\mathbb{S}$ with value group $(\mathfrak{M}, \times, \succ)$ (note the reverse ordering). In other words, it is a morphism $(\mathbb{S}^{\neq}, \times) \longrightarrow (\mathfrak{M}, \times)$ with

$$\mathfrak{d}_{s+t} \preccurlyeq \max(\mathfrak{d}_s, \mathfrak{d}_t)$$

whenever $s, t, s + t \in \mathbb{S}^{\neq}$.

Remark 1.2.1. The relation $\preccurlyeq$ on $\mathfrak{M}$ is the non-strict ordering corresponding to $\prec$, but the same is not true for $\prec$ and $\preccurlyeq$ on $\mathbb{S}$. On $\mathbb{S}$, the relation $\preccurlyeq$ is transitive and reflexive (such relations are sometimes called quasi-orders or pre-orders), and $\asymp$ is an equivalence relation.

We write

$$\begin{aligned} \mathbb{S}_{\succ} &:= \{s \in \mathbb{S} : \operatorname{supp} s \subseteq \mathfrak{M}^{\succ}\}, \\ \mathbb{S}^{\prec} &:= \{s \in \mathbb{S} : \operatorname{supp} s \subseteq \mathfrak{M}^{\prec}\} = \{s \in \mathbb{S} : s \prec 1\}, \quad \text{and} \\ \mathbb{S}^{>,\succ} &:= \{s \in \mathbb{S} : s > \mathbb{R}\} = \{s \in \mathbb{S} : s \geqslant 0 \wedge s \succ 1\}. \end{aligned}$$

Series in $\mathbb{S}_{\succ}$, $\mathbb{S}^{\prec}$ and $\mathbb{S}^{>,\succ}$ are respectively said *purely large*, *infinitesimal*, and *positive infinite*. We have an *additive decomposition*

$$\mathbb{S} = \mathbb{S}_{\succ} + \mathbb{R} + \mathbb{S}^{\prec}$$

where each $s \in \mathbb{S}$ decomposes uniquely as

$$s = s_{\succ} + s_{\asymp} + s_{\prec}. \tag{1.2.3}$$

We also have a *multiplicative decomposition*

$$\mathbb{S}^{\neq} = \mathbb{R}^{\neq} \cdot \mathfrak{M} \cdot (1 + \mathbb{S}^{\prec})$$

where each $s \neq 0$ decomposes uniquely as

$$s = r_s\, \mathfrak{d}_s\, (1 + \varepsilon_s), \tag{1.2.4}$$

with $r_s\, \mathfrak{d}_s = \tau_s \neq 0$ and $\varepsilon_s = \frac{s - \tau_s}{\tau_s}$ is infinitesimal.

Remark 1.2.2. Our notations for fields of well-based series follow the simple rule: symbols $\square$ appearing as exponents in a notation $\mathbb{S}^{\square}$ indicate that we consider series $s \in \mathbb{S}$ satisfying $\square$, e.g.

$$\mathbb{S}^{>} = \{s \in \mathbb{S} : s > 0\}, \qquad \mathbb{S}^{\prec} = \{s \in \mathbb{S} : s \prec 1\}, \quad \text{and} \quad \mathbb{S}^{>,\succ} = \{s \in \mathbb{T} : s > 0 \wedge s \succ 1\},$$

whereas symbols $\triangle$ appearing as indexes in $\mathbb{S}_\triangle$ pertain to conditions on supports of series, e.g.

$$\mathbb{S}_{\succ} = \{s \in \mathbb{S} : \operatorname{supp} s \succ 1\} \quad \text{and} \quad \mathbb{S}_{\succ,\alpha} = \left\{ s \in \mathbb{S}^{>,\succ} : \operatorname{supp} s \succ \frac{1}{L_{<\alpha}(E_\alpha(s))} \right\},$$

where the last notation appears when considering hyperserial fields.

Remark 1.2.3. The type of valued fields we are studying are very specific. Accordingly, we will mostly rely on valuation theory as a language and tool to state and prove results. Valued fields of well-based series, over other fields besides $\mathbb{R}$ and possibly with factor sets (see [82, 65]), among other generalizations, were studied in detail in particular by Irving Kaplansky. They naturally appear as so-called maximal valued fields, and are in particular Henselian.

In our case, fields of well-based series over $\mathbb{R}$ are even more specific. Indeed, by the Ax-Kochen-Erschov principle, the elementary theory of $(\mathbb{S}, +, \times, \prec)$ only depends on the theory of $(\mathfrak{M}, \times, \prec)$ as an ordered group. In particular, the elementary theory of $(\mathbb{S}, +, \times, \prec)$ is completely determined if $\mathfrak{M}$ is divisible. In that case $\mathbb{S}$ is in particular real-closed, as a consequence of [77, Theorem 1]. See for instance [4, Section 3.6] for more details.

Example 1.2.4. Consider a multiplicative copy $(x^{\mathbb{Z}}, \cdot, \prec)$ of $(\mathbb{Z}, +, <)$. Since $(\mathbb{Z}, +, <)$ is the smallest non-trivial linearly ordered Abelian group, the smallest field of well-based series is the field $\mathbb{R}[[x^{\mathbb{Z}}]]$ of so-called formal Laurent series over $\mathbb{R}$. Well-based subsets of $\mathbb{Z}$ are simply subsets of initial segments $(-\infty, n], n \in \mathbb{Z}$ of $\mathbb{Z}$. Thus any formal Laurent series s can be written as

$$s = \sum_{k=n}^{+\infty} s_k\, x^{-k}$$

where $n \in \mathbb{Z}$ and $(s_k)_{k \geqslant n}$ is a sequence of real numbers. So $\mathbb{R}[[x^{\mathbb{Z}}]]$ coincides with the usual field of formal Laurent series. For instance, the following are formal Laurent series:

$$1 + x^{-1} + \frac{1}{2} x^{-2} + \frac{1}{6} x^{-3} + \frac{1}{24} x^{-4} + \cdots$$

$$x^3 - \pi + 2\, x^{-2} - 16\, x^{-4} + 64\, x^{-6} - \cdots.$$

We will adopt the convention that x denotes an infinite variable whereas z stands for a infinitesimal one.

Example 1.2.5. Consider a multiplicative copy $(x^{\mathbb{R}}, \cdot, \prec)$ of the additive ordered group $(\mathbb{R}, +, <)$ of real numbers. We call $\mathbb{R}[[x^{\mathbb{R}}]]$ the field of *real-powered series*. It is well-known that for each countable ordinal $\alpha \in \omega_1$, there are well-ordered subsets of $\mathbb{Q}$, hence also of $\mathbb{R}$, that are order isomorphic to α. So elements of $\mathbb{R}[[x^{\mathbb{R}}]]$ may have arbitrary countable lengths as series. Similarly, we write $\mathbb{R}[[x^{\mathbb{Q}}]]$ for the field of well-based series whose monomial group is a multiplicative copy of $(\mathbb{Q}, +, <)$. We call those series *rational-powered series* and they properly contain Puiseux series.

Example 1.2.6. Let $\mathbf{L}$ be a linearly ordered class. Then the Hahn product group $H[\mathbf{L}, \mathbb{R}]$ is a linearly ordered Abelian group, so we can form the field of well-based series $\mathbb{R}[[H[\mathbf{L}, \mathbb{R}]]]$. A natural way to endow this field with additional structure is *via* certain partial functions $H[\mathbf{L}, \mathbb{R}] \longrightarrow \mathbb{R}[[H[\mathbf{L}, \mathbb{R}]]]$ or $\mathbb{R}[[H[\mathbf{L}, \mathbb{R}]]] \longrightarrow H[\mathbf{L}, \mathbb{R}]$. This is the spirit of [17], and a guiding principle in defining fields of transseries or hyperseries.

1.2.2 Products of well-based families

In this subsection, we gather results about well-based families in $\mathbb{S}$ pertaining to the Cauchy product. A subclass $\mathfrak{S}$ of $\mathfrak{M}$ is said *infinitesimal* if all its elements are infinitesimal. We say that $\mathfrak{S}$ is *small* if we have $\mathfrak{s} \preccurlyeq 1$ for all $\mathfrak{s} \in \mathfrak{S}$.

We will use the following elementary fact, often without mention:

Lemma 1.2.7. *Let $(s_i)_{i \in \mathbf{I}}$ be a well-based family and let $(r_i)_{i \in \mathbf{I}} \in \mathbb{R}^{\mathbf{I}}$. The family $(r_i\, s_i)_{i \in \mathbf{I}}$ is well-based.*

Proof. This follows from Lemma 1.1.14. □

Proposition 1.2.8. [92, Proposition 1.5.3(4)] *Let $S=(s_i)_{i\in\mathbf{I}}$ and $T=(t_j)_{j\in\mathbf{J}}$ be well-based families in $\mathbb{S}$ where $\mathbf{I}$ and $\mathbf{J}$ are classes. The family $S\cdot T:=(s_i\,t_j)_{(i,j)\in\mathbf{I}\times\mathbf{J}}$ is well-based, with*

$$\sum S\cdot T=\left(\sum S\right)\left(\sum T\right),$$

i.e. with

$$\sum_{(i,j)\in\mathbf{I}\times\mathbf{J}} s_i\,t_j=\left(\sum_{i\in\mathbf{I}} s_i\right)\left(\sum_{j\in\mathbf{J}} t_j\right).$$

Proof. Write

$$\begin{aligned}\mathfrak{S}_{\mathbf{I}} &:= \left(\bigcup_{i\in\mathbf{I}}\operatorname{supp} s_i\right),\quad\text{and}\\ \mathfrak{S}_{\mathbf{J}} &:= \left(\bigcup_{j\in\mathbf{J}}\operatorname{supp} t_j\right).\end{aligned}$$

Since S and T are well-based, those are well-based sets. For $(i,j)\in\mathbf{I}\times\mathbf{J}$, we have

$$\operatorname{supp} s_i\,t_j\subseteq(\operatorname{supp} s_i)\cdot(\operatorname{supp} t_j),$$

so $\bigcup_{(i,j)\in\mathbf{I}\times\mathbf{J}}\operatorname{supp} s_i\,t_j\subseteq\mathfrak{S}_{\mathbf{I}}\cdot\mathfrak{S}_{\mathbf{J}}$ is well-based by Lemma 1.1.22. For $\mathfrak{m}\in\mathfrak{M}$, Lemma 1.1.22 also implies that there are finitely many ordered pairs of monomials $(\mathfrak{u}_k,\mathfrak{v}_k)_{k\leqslant n}\in(\mathfrak{S}_{\mathbf{I}}\times\mathfrak{S}_{\mathbf{J}})^{n+1}, n\in\mathbb{N}$ with $\mathfrak{m}=\mathfrak{u}_k\,\mathfrak{v}_k$ for all $k\in\{0,\ldots,n\}$. The sets $\mathbf{I}_k=\{i\in\mathbf{I}:\mathfrak{u}_k\in\operatorname{supp} s_i\}$ and $\mathbf{J}_k=\{j\in\mathbf{J}:\mathfrak{v}_k\in\operatorname{supp} t_j\}$ being finite, the class

$$(\mathbf{I}\times\mathbf{J})_{\mathfrak{m}}=\{(i,j)\in\mathbf{I}\times\mathbf{J}:\mathfrak{m}\in\operatorname{supp} s_i\,t_j\}\subseteq\bigcup_{k=0}^{n}\mathbf{I}_k\times\mathbf{J}_k$$

is finite. So $S\cdot T$ is well-based. Since all sums involved have finite support, we have

$$\begin{aligned}\left(\sum_{(i,j)\in\mathbf{I}\times\mathbf{J}} s_i\,t_j\right)_{\mathfrak{m}} &= \left(\sum_{(i,j)\in(\mathbf{I}\times\mathbf{J})_{\mathfrak{m}}} s_i\,t_j\right)_{\mathfrak{m}}\\ &= \sum_{k=1}^{n}\sum_{(i,j)\in\mathbf{I}_k\times\mathbf{J}_k}(s_i)_{\mathfrak{u}_k}(t_j)_{\mathfrak{v}_k}\\ &= \sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}}\sum_{(i,j)\in\mathbf{I}\times\mathbf{J}}(s_i)_{\mathfrak{u}}(t_j)_{\mathfrak{v}}\\ &= \sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}}\left(\sum_{i\in\mathbf{I}}(s_i)_{\mathfrak{u}}\right)\left(\sum_{j\in\mathbf{J}}(t_j)_{\mathfrak{v}}\right)\\ &= \sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}}\left(\sum S\right)_{\mathfrak{u}}\left(\sum T\right)_{\mathfrak{v}}\\ &= \left(\left(\sum S\right)\cdot\left(\sum T\right)\right)_{\mathfrak{m}}.\end{aligned}$$

We deduce that $\sum S\cdot T=(\sum S)(\sum T)$. □

Notation 1.2.9. *Let $n\in\mathbb{N}$. If $\mathbf{X}$ is a class and $x=(x_1,\ldots,x_n)\in\mathbf{X}^n$, then for all $i\in\{1,\ldots,n\}$ we write $x_{[i]}:=x_i$. If $v\in\mathbb{N}^n$, then we also set $|v|:=\sum_{i=1}^{n}v_{[i]}$.*

Corollary 1.2.10. *Let $(t_j)_{j\in\mathbf{J}}$ be a well-based family and let $s\in\mathbb{S}$. The family $(s\,t_j)_{j\in\mathbf{J}}$ is well-based, with*

$$\sum_{j\in\mathbf{J}} s\,t_j=s\sum_{j\in\mathbf{J}} t_j.$$

Lemma 1.2.11. [92, Corollary 1.5.6] *Let $n\in\mathbb{N}^>$. For all infinitesimal series $\varepsilon_1,\ldots,\varepsilon_n\in\mathbb{S}^{\prec}$ and for all $(r_v)_{v\in\mathbb{N}^n}\in\mathbb{R}^{\mathbb{N}^n}$ the family $(r_v\,\varepsilon_1^{v_{[1]}}\cdots\varepsilon_n^{v_{[n]}})_{v\in\mathbb{N}^n}$ is well-based.*

Proof. By Lemma 1.2.7, we may assume that $r_v = 1$ for all $v \in \mathbb{N}^n$. By Proposition 1.2.8 and Lemma 1.1.15 we may assume that $n = 1$. We write $\varepsilon := \varepsilon_1$.

The set $\mathfrak{S} := \operatorname{supp} \varepsilon$ is infinitesimal, so $\mathfrak{S}^\infty$ is well-based by Lemma 1.1.24. For all $k \in \mathbb{N}$ we have

$$\operatorname{supp} \varepsilon^k \subseteq \mathfrak{S}^\infty,$$

so $\bigcup_{k \in \mathbb{N}} \operatorname{supp} \varepsilon^k$ is a well-based set. Consider an $\mathfrak{m} \in \mathfrak{M}$. The set $I_\mathfrak{m} = \{k \in \mathbb{N} : \mathfrak{m} \in \mathfrak{S}^k\}$ is finite by Lemma 1.1.25, so $\{k \in \mathbb{N} : \mathfrak{m} \in \operatorname{supp} \varepsilon^k\}$ is finite, whence $(\varepsilon^k)_{k \in \mathbb{N}}$ is well-based. □

Corollary 1.2.12. [82] *For $\varepsilon \in \mathbb{S}^{\prec}$, we have*

$$\sum_{k \in \mathbb{N}} \varepsilon^k = (1 - \varepsilon)^{-1}.$$

Proof. Note that $\operatorname{supp} \varepsilon \prec 1$ so by Lemma 1.2.11, the family $(\varepsilon^k)_{k \in \mathbb{N}}$ is well-based. Set $t := \sum_{k \in \mathbb{N}} \varepsilon^k$. We have

$$\begin{aligned}
(1 - \varepsilon)\, t &= \sum_{k \in \mathbb{N}} \varepsilon^k + \sum_{k \in \mathbb{N}} (-\varepsilon^{k+1}) && \text{(by Corollary 1.2.10)}\\
&= \sum_{k \in \mathbb{N}} \varepsilon^k - \sum_{k \in \mathbb{N}} \varepsilon^{k+1} && \text{(by Corollary 1.2.10)}\\
&= \left(1 + \sum_{k > 0} \varepsilon^k\right) - \sum_{k > 0} \varepsilon^k && \text{(by Proposition 1.1.16 and Corollary 1.1.19)}\\
&= 1.
\end{aligned}$$

Therefore $t = (1 - \varepsilon)^{-1}$. □

Lemma 1.2.13. *Let $(s_i)_{i \in \mathbf{I}}$ be a family in $\mathbb{S}$ where $\mathbf{I}$ is a class. Assume that there is a well-based and infinitesimal set $\mathfrak{T} \subseteq \mathfrak{M}$, a well-based set $\mathfrak{S} \subseteq \mathfrak{M}$ and a function $\mathrm{N} : \mathbf{I} \longrightarrow \mathbb{N}$ such that we have*

$$\operatorname{supp} s_i \subseteq \mathfrak{T}^{\mathrm{N}(i)} \cdot \mathfrak{S} \quad \textit{for all } i \in \mathbf{I}.$$

Assume that $(s_j)_{j \in \mathbf{J}}$ is well-based whenever $\mathbf{J} \subseteq \mathbf{I}$ and $\mathrm{N}(\mathbf{J})$ is finite. Then $(s_i)_{i \in \mathbf{I}}$ is well-based.

Proof. Assume for contradiction that $(s_i)_{i \in \mathbf{I}}$ is not well-based. So there is an injective sequence $(i_k)_{k \in \mathbb{N}} \in \mathbf{I}^\mathbb{N}$ and a sequence $(\mathfrak{m}_k)_{k \in \mathbb{N}} \in \mathfrak{M}^\mathbb{N}$ with $\mathfrak{m}_0 \preccurlyeq \mathfrak{m}_1 \preccurlyeq \cdots$ and $\mathfrak{m}_k \in \operatorname{supp} s_{i_k}$ for all $k \in \mathbb{N}$. We have $\{\mathfrak{m}_k : k \in \mathbb{N}\} \subseteq \mathfrak{T}^\infty \cdot \mathfrak{S}$ where $\mathfrak{T}^\infty \cdot \mathfrak{S}$ is well-based by Lemmas 1.1.22 and 1.1.24. So $\{\mathfrak{m}_k : k \in \mathbb{N}\}$ is well-based and we may assume that $(\mathfrak{m}_k)_{k \in \mathbb{N}}$ is constant. By Lemma 1.1.22 and 1.1.25, the set $\{n \in \mathbb{N} : \mathfrak{m}_0 \in \mathfrak{T}^n \cdot \mathfrak{S}\}$ is finite. In particular $\{N(i_k) : k \in \mathbb{N}\}$ is finite, so $(s_{i_k})_{k \in \mathbb{N}}$ is well-based: a contradiction. □

Corollary 1.2.14. *Let $(s_{n,m})_{(n,m) \in \mathbb{N}^2}$ be a family in $\mathbb{S}$ such that each $(s_{n,m})_{m \in \mathbb{N}}$ for $n \in \mathbb{N}$ is well-based. Assume that there is a well-based and infinitesimal set $\mathfrak{T} \subseteq \mathfrak{M}$ and a well-based set $\mathfrak{S} \subseteq \mathfrak{M}$ with*

$$\forall n, m \in \mathbb{N},\ \operatorname{supp} s_{n,m} \subseteq \mathfrak{T}^n \cdot \mathfrak{S}.$$

Then $(s_{n,k})_{(n,k) \in \mathbb{N}^2}$ is well-based.

Proposition 1.2.15. *Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series. Let I be a set and let f: $I \longrightarrow \mathbb{N}$ be an arbitrary function. Let $(s_i)_{i \in I}$ be a well-based family in $\mathbb{S}$ and let $\delta \preccurlyeq 1$. The family $(s_i\, \delta^{f(i)})_{i \in I}$ is well-based.*

Proof. Write $\delta = r + \varepsilon$ where $r \in \mathbb{R}$ and $\varepsilon \prec 1$, and set $\mathfrak{S} := \bigcup_{i \in I} \operatorname{supp} s_i$. So $\mathfrak{S}$ is well-based. For $(i, k) \in I \times \mathbb{N}$, write $s_{i,k} := s_i \binom{f(i)}{k} r^{f(i) - k} \varepsilon^k$, so

$$\operatorname{supp} s_{i,k} \subseteq \mathfrak{S} \cdot (\operatorname{supp} \varepsilon)^k.$$

If $J \subseteq \mathbb{N}$ is finite, then $(s_{i,k})_{i \in I, k \in J}$ is well-based as a finite union of well-based families. We deduce with Lemma 1.2.13 that $(s_{i,k})_{i \in I, k \in \mathbb{N}}$ is well-based. In particular $\left(\sum_{k=0}^{f(i)} s_{i,k}\right)_{i \in I} = (s_i\, \delta^{f(i)})_{i \in I}$ is well-based by Lemma 1.1.18. □

1.2.3 Flatness

Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series. For $s \in \mathbb{S}^{>}$, we write $s^{+} := \max(s, s^{-1})$ and $s^{-} := (s^{+})^{-1} s^{+} := \max(s, s^{-1})$. So $s^{+} = s^{-1}$ if $s < 1$ and $s^{+} = s$ otherwise. As in [92, 64], it is useful to consider the following orderings on $\mathbb{S}^{>}$:

Definition 1.2.16. *Let $s, t \in \mathbb{S}^{>}$. We say that s is* **flatter** *than t and we write*

$$s \prec\!\!\prec t \quad \text{if} \quad (s^{+})^{n} < t^{+} \quad \text{for all } n \in \mathbb{N}^{>}, \text{ and}$$

$$s \asymp\!\!\asymp t \quad \text{if there are } m, n \in \mathbb{N}^{>} \text{ with} \quad t^{+} < (s^{+})^{m} < (t^{+})^{n}.$$

We also write $s \preceq\!\!\preceq t$ if $s \prec\!\!\prec t$ or $s \asymp\!\!\asymp t$.

The relation $\prec\!\!\prec$ is a partial ordering on $\mathbb{S}^{>}$. We sometimes extend it to $\mathbb{S}^{\neq}$ by writing $s \prec\!\!\prec t$ whenever $|s| \prec\!\!\prec |t|$. Note that $s \prec\!\!\prec t$ if and only if $v\mathfrak{d}_s > v\mathfrak{d}_t$ where v is the natural (or standard, or Archimedean) valuation on the ordered group $\mathfrak{M}$. See [4, p 83–84], for more details.

Example 1.2.17. In the field $\mathbb{R}[[x^{\mathbb{Z}}]]$ of formal Laurent series, we have $f \asymp\!\!\asymp g$ for all $f, g \in \mathbb{R}[[x^{\mathbb{Z}}]]$ with $f, g \not\asymp 1$. The existence of strictly flatter elements in fields of well-based series typically involves the existence of a logarithm or an exponential. For instance, in the field **No** of surreal numbers equipped with Gonshor's exponential and logarithm functions exp and log [55, Chapter 10] (see also Chapter 11), we have

$$1 \prec\!\!\prec \log(\log(a)) \prec\!\!\prec \log a \prec\!\!\prec \exp\left(\sqrt{\log a}\right) \prec\!\!\prec a \prec\!\!\prec \exp(a) \prec\!\!\prec \exp(a^{2}) \prec\!\!\prec \exp(\exp(a))$$

for all $a \in \mathbf{No}^{>,\succ}$. Note that this reflects the asymptotics of the corresponding real-valued functions at $+\infty$.

Lemma 1.2.18. *Let $L: (\mathbb{S}^{>}, \times) \longrightarrow (\mathbb{S}, +)$ be a strictly increasing morphism. Then for all $s, t \in \mathbb{S}^{\neq}$, we have*

$$\begin{aligned} s \prec\!\!\prec t &\iff L(s) \prec L(t), \\ s \preceq\!\!\preceq t &\iff L(s) \preccurlyeq L(t), \\ s \asymp\!\!\asymp t &\iff L(s) \asymp L(t). \end{aligned}$$

Proof. This follows from the relation $L(s^{n}) = n\, s$ for all $s \in \mathbb{S}^{>}$ and $n \in \mathbb{N}$ and the fact that L is strictly increasing. □

We will frequently use the following consequences of Lemma 1.2.18, sometimes without mention:

Corollary 1.2.19. *Assume that there is a strictly increasing morphism $L: (\mathbb{S}^{>}, \times) \longrightarrow (\mathbb{S}, +)$. Then for $s, t, u \in \mathbb{S}^{>}$, we have*

a) $s\, t \preceq\!\!\preceq \max(s^{+}, t^{+})$.

b) $s \asymp t \Longrightarrow s \asymp\!\!\asymp t$.

c) $s \prec\!\!\prec t \Longrightarrow s\, t \asymp\!\!\asymp t$.

Proof. The assertions *a*), *c*) follow from the classical valuation theoretic properties of $\preccurlyeq$. The assertion *b*) is an immediate consequence of Lemma 1.2.18. □

1.3 Strong linearity

The notion of well-based families allows us to consider strongly linear operators. Those are operators between fields of well-based series which commute with transfinite sums of well-based families. This will be the case for derivations and right compositions that we will define in later sections. In the present section, we define this notion and introduce ways to define strongly linear operators.

1.3.1 Strongly linear functions

Let $\mathbb{S}=\mathbb{R}[[\mathfrak{M}]]$ and $\mathbb{T}=\mathbb{R}[[\mathfrak{N}]]$ be fields of well-based series. Consider a function $\Phi\colon\mathbb{S}\longrightarrow\mathbb{T}$ which is $\mathbb{R}$-linear. Then Φ is said *strongly linear* if for every well-based family $(s_i)_{i\in\mathbf{I}}$ in $\mathbb{S}$, the family $(\Phi(s_i))_{i\in\mathbf{I}}$ in $\mathbb{T}$ is well-based, with

$$\Phi\left(\sum_{i\in\mathbf{I}} s_i\right)=\sum_{i\in\mathbf{I}}\Phi(s_i).$$

If $\Phi\colon\mathfrak{M}\longrightarrow\mathbb{T}$ is a function, then we say that it is *well-based* if for any well-based family $(\mathfrak{m}_i)_{i\in I}$ in $\mathfrak{M}$, the family $(\Phi(\mathfrak{m}_i))_{i\in I}$ in $\mathbb{T}$ is well-based. Then Φ extends uniquely into a strongly linear map $\hat{\Phi}\colon\mathbb{S}\longrightarrow\mathbb{T}$ [62, Proposition 3.5]. As a consequence, we have the following characterization of strong linearity.

Lemma 1.3.1. *An $\mathbb{R}$-linear function $\Phi\colon\mathbb{S}\longrightarrow\mathbb{T}$ is strongly linear if and only if for all $s\in\mathbb{S}$, the family $(s_{\mathfrak{m}}\Phi(\mathfrak{m}))_{\mathfrak{m}\in\mathfrak{M}}$ is well-based, with*

$$\Phi(s)=\sum_{\mathfrak{m}\in\mathfrak{M}} s_{\mathfrak{m}}\Phi(\mathfrak{m}).$$

If $\Phi\colon\mathfrak{M}\longrightarrow\mathbb{T}$ is well based, then $\hat{\Phi}$ is strictly increasing whenever Φ is strictly increasing and it is a ring morphism whenever $\Phi(\mathfrak{m}\,\mathfrak{n})=\Phi(\mathfrak{m})\,\Phi(\mathfrak{n})$ for all $\mathfrak{m},\mathfrak{n}\in\mathfrak{M}$ [62, Corollary 3.8]. In particular, if $\Phi(\mathfrak{m})\in\mathfrak{N}$ for all $\mathfrak{m}\in\mathfrak{M}$ and Φ is strictly increasing, then Φ is well-based. Hence:

Proposition 1.3.2. *Let $\mathfrak{S}\subseteq\mathfrak{M}$ and $\mathfrak{T}\subseteq\mathfrak{N}$ and consider an order-preserving map $\Psi\colon\mathfrak{S}\longrightarrow\mathfrak{T}$. Then there is a unique strongly linear function $\hat{\Psi}\colon\mathbb{R}[[\mathfrak{S}]]\longrightarrow\mathbb{R}[[\mathfrak{T}]]$ that extends Ψ. If moreover $\mathfrak{S},\mathfrak{T}$ are subgroups and Ψ is a group morphism, then $\hat{\Psi}$ is an embedding of ordered fields.* □

1.3.2 Operator supports

We fix two fields of well-based series $\mathbb{S}=\mathbb{R}[[\mathfrak{M}]]$ and $\mathbb{T}=\mathbb{R}[[\mathfrak{N}]]$. A very convenient way to prove that certain families related to certain operators are well-based is to rely on the notions of operator support [33, p. 10] and relative operator support [14, Definition 2.4]. We recall the definitions here.

Definition 1.3.3. *Let $\Phi\colon\mathfrak{M}\longrightarrow\mathbb{T}$ be a function. If $\mathfrak{M}\subseteq\mathfrak{N}$ (as ordered groups), then the* **support** $\operatorname{supp}\Phi$ *of Φ is the class*

$$\operatorname{supp}\Phi:=\bigcup_{\mathfrak{m}\in\mathfrak{M}}\frac{\operatorname{supp}\Phi(\mathfrak{m})}{\mathfrak{m}}.$$

The **relative support** $\operatorname{supp}_{\odot}\Phi$ *of Φ is the class*

$$\operatorname{supp}_{\odot}\Phi:=\bigcup_{\mathfrak{m}\in\mathfrak{M}}\frac{\operatorname{supp}\Phi(\mathfrak{m})}{\mathfrak{d}_{\Phi(\mathfrak{m})}}.$$

If $\Psi\colon\mathbb{S}\longrightarrow\mathbb{T}$ is a linear function, then we define its support and relative support as

$$\begin{aligned}\operatorname{supp}\Psi &:= \operatorname{supp}(\Psi\upharpoonright\mathfrak{M}) \quad \textit{and}\\ \operatorname{supp}_{\odot}\Psi &:= \operatorname{supp}_{\odot}(\Psi\upharpoonright\mathfrak{M}) \quad \textit{respectively.}\end{aligned}$$

Example 1.3.4. Consider the ordered field $\mathbb{R}[[x^{\mathbb{Z}}]]$ of formal Laurent series, where $x^{\mathbb{Z}}$ is a multiplicative copy of $(\mathbb{Z},+,<)$. We have a derivation given by

$$\left(\sum_{k=n}^{+\infty} a_k\,x^{-k}\right)':=\sum_{k=n}^{+\infty} k\,a_k\,x^{-(k+1)}.$$

In that case, for any monomial $x^n\in x^{\mathbb{Z}}$, we have $(x^n)'=n\,x^{n-1}$ so

$$\frac{\operatorname{supp}(x^n)'}{x^n}\subseteq\{x^{-1}\}.$$

This shows that the set $\{x^{-1}\}$ is a (well-based) support for the derivation.

Example 1.3.5. For $f \in \mathbb{R}[[x^{\mathbb{Z}}]]$, we have a well-defined sum

$$f \circ (x+1) := \sum_{k \in \mathbb{N}} \frac{f^{(k)}}{k!} \tag{1.3.1}$$

which corresponds to the composition of f with $x+1$ on the right, as formal series. We claim that the function $x^{\mathbb{Z}} \longrightarrow \mathbb{R}[[x^{\mathbb{Z}}]]; x^n \mapsto x^n \circ (x+1)$ has well-based relative support $\{x^{-k} : k \in \mathbb{N}\}$. Indeed, for $n \in \mathbb{Z}$, we have

$$x^n \circ (x+1) = x^n + n\, x^{n-1} + \frac{n\,(n-1)}{2}\, x^{n-2} + \cdots.$$

So $\mathfrak{d}_{x^n \circ (x+1)} = x^n$ and

$$\frac{\operatorname{supp} x^n \circ (x+1)}{\mathfrak{d}_{x^n \circ (x+1)}} \subseteq \frac{\bigcup_{k \in \mathbb{N}} \operatorname{supp} (x^n)^{(k)}}{x^n} \subseteq \frac{\{x^{n-k} : k \in \mathbb{N}\}}{x^n} = \{x^{-k} : k \in \mathbb{N}\}.$$

That (1.3.1) is well-defined then follows from Proposition 1.3.7 below.

We next include two useful results.

Proposition 1.3.6. [33, Lemma 2.9] *Let $\Phi \colon \mathfrak{M} \longrightarrow \mathbb{T}$ have well-based support. Then Φ is well-based.*

Proof. By Lemma 1.3.1, we have to show that given a well-based subset $\mathfrak{S} \subseteq \mathfrak{M}$, the family $(\Phi(\mathfrak{m}))_{\mathfrak{m} \in \mathfrak{S}}$ is well-based. For $\mathfrak{m} \in \mathfrak{S}$, we have $\operatorname{supp} \Phi(\mathfrak{m}) \subseteq \mathfrak{S} \cdot (\operatorname{supp} \Phi)$, so $\bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m})$ is well-based by Lemma 1.1.22. Moreover Lemma 1.1.22 implies that for all $\mathfrak{n} \in \mathfrak{N}$, the set

$$\{\mathfrak{m} \in \mathfrak{S} : \mathfrak{n} \in \mathfrak{S} \cdot (\operatorname{supp} \Phi(\mathfrak{m}))\} \subseteq \{\mathfrak{m} \in \mathfrak{S} : \mathfrak{n} \in \mathfrak{S} \cdot (\operatorname{supp} \Phi)\}$$

is finite, so $(\Phi(\mathfrak{m}))_{\mathfrak{m} \in \mathfrak{S}}$ is well-based. □

Proposition 1.3.7. [14, Proposition 2.5] *Let $\Phi \colon \mathfrak{M} \longrightarrow \mathbb{T}$ be relatively well-based. Assume that $0 \notin \Phi(\mathfrak{M})$ and that $\mathfrak{d} \circ \Phi \colon \mathfrak{M} \longrightarrow \mathfrak{N}$ is strictly increasing. Then Φ is well-based and its strongly linear extension $\hat{\Phi}$ is injective.*

Proof. Consider a well-based subset $\mathfrak{S} \subseteq \mathfrak{M}$. We have

$$\bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m}) \subseteq \{\mathfrak{d}_{\Phi(\mathfrak{m})} : \mathfrak{m} \in \mathfrak{S}\} \cdot (\operatorname{supp}_{\odot} \Phi),$$

so $\bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m})$ is a well-based subset of $\mathfrak{N}$. For any $\mathfrak{n} \in \bigcup_{\mathfrak{m} \in \mathfrak{S}} \operatorname{supp} \Phi(\mathfrak{m})$, the set of pairs $(\mathfrak{m}, \mathfrak{u}) \in \mathfrak{S} \times \operatorname{supp}_{\odot} \Phi$ with $\mathfrak{d}_{\Phi(\mathfrak{m})}\, \mathfrak{u} = \mathfrak{n}$ forms a finite antichain. Since any $\mathfrak{m} \in \mathfrak{S}$ with $\mathfrak{n} \in \operatorname{supp} \Phi(\mathfrak{m})$ induces such a pair $(\mathfrak{m}, \mathfrak{n} / \mathfrak{d}_{\Phi(\mathfrak{m})})$, it follows that the set of all such $\mathfrak{m}$ is also finite. This completes the proof that Φ is well-based. To see that $\hat{\Phi}$ is injective, let $s \in \mathbb{S}^{\neq}$ and take $r \in \mathbb{R}^{\neq}$ with $s \sim r\, \mathfrak{d}_s$. The assumption that $\mathfrak{d} \circ \Phi$ is strictly increasing gives $\hat{\Phi}(s - r\, \mathfrak{d}_s) \prec \hat{\Phi}(r\, \mathfrak{d}_s) = r\, \Phi(\mathfrak{d}_s) \neq 0$, so $\hat{\Phi}(s) \neq 0$. □

Chapter 2
Analytic functions on well-based series

In this chapter, we introduce the notion of analytic function on a field $\mathbb{S}$ of well-based series, which will play an important role in Part II. The general idea is that a function f is is analytic if

$$f(s+\varepsilon)=\sum_{k\in\mathbb{N}}\frac{f^{(k)}(s)}{k!}\varepsilon^k \tag{2.0.1}$$

for all s in its domain, for sufficiently small ε in absolute value. This leads us in particular to study the behavior of formal power series

$$P=\sum_{k\in\mathbb{N}} P_k\, z^k\in\mathbb{S}[[z]],$$

and the conditions under which such power series converge at points $\varepsilon\in\mathbb{S}$, i.e. yield well-based families $(P_k\,\varepsilon^k)_{k\in\mathbb{N}}$.

Such work was done in some detail by Norman Alling in his book *Foundations of analysis over surreal number fields* [1] (see in particular [1, Chapters 8 and 9]). Alling showed that formal power series had similar behavior to real convergent power series on their domain of convergence. Unfortunately, Alling's results are restricted to the specific context of surreal numbers. Even more unfortunate for us is his focus on a domain of convergence for power series which is rather too small for our purposes in later sections of the thesis. We will have to show that certain result of Alling extend to larger domains. The notion of analytic functions on surreal numbers also appears in [19, Section 7.3].

Taylor series such as (2.0.1), are also a way to define composition laws on differential fields of well-based series *via* Taylor expansions. They were used in that manner recently [33] in order to define the composition law on logarithmic hyperseries. On a simpler level, the extension of the so-called restricted real-analytic functions to fields of well-based series [34] also belongs to this type of development. In particular, the definition of logarithms and exponentials on fields of transseries crucially depends on those functions being analytic. There is one surprising (and ultimately deceiving) exception: even though Gonshor's definition [55, Chapter 10] of the surreal exponential function does not require analyticity, we will see that it turns out to be an analytic function [55, Theorem 10.3]. This suggests to us that analyticity and Taylor expansions are not only sound and practical, but also natural, in developing formal calculus on fields of well-based series.

2.1 Elementary analysis on ordered fields

We first introduce generalization of classical notions in real calculus to general ordered fields. The content of this section (in the set sized context) is often considered common folklore.

2.1.1 The order topology

Let $\mathbf{F}$ be an ordered field, possibly class-sized. There is a natural topology on $\mathbf{F}$, called the *order topology*, which turns it into a topological field. The order topology has open intervals as a basis, so a non-empty subclass $\mathbf{O}\subseteq\mathbf{F}$ is *open* if for all $x\in\mathbf{O}$, there is a $\delta\in\mathbf{F}^{>}$ such that the interval $(x-\delta,x+\delta)$ is contained in $\mathbf{O}$. A *neighborhood* of $x\in\mathbf{F}$ is a subclass of $\mathbf{F}$ containing an open subclass of $\mathbf{F}$ which itself contains x.

Remark 2.1.1. If $\mathbf{F}$ is a proper class, then the intended topology, which should be the class of open classes, does not exists in general as per NBG set theory. This is because no class may contain a proper class whereas any infinite interval in a class-sized ordered field is a proper class. Therefore we will not be using topology here and will instead only rely of the well-defined notions of open subclasses, continuity, differentiability, and so on.

One can define the continuity of a function $g: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ between subclasses of $\mathbf{F}$ in the classical way, i.e. g is *continuous* at $x \in \mathbf{A}_1$ if for any neighborhood $\mathbf{O}$ of $g(x)$, the class

$$g^{-1}(\mathbf{A}_2 \cap \mathbf{O}) = \{y \in \mathbf{A}_1 : g(y) \in \mathbf{O}\}$$

is the intersection of $\mathbf{A}_1$ with a neighborhood of x. This translates into the famous "ε-δ definition": the function g is continuous at x if and only if

$$\forall \varepsilon \in \mathbf{F}^{>}, \exists \delta \in \mathbf{F}^{>}, \forall y \in \mathbf{A}_1, |y - x| < \delta \Longrightarrow |g(y) - g(x)| < \varepsilon.$$

2.1.2 Differentiable functions

Our main interest will be that of differentiable functions:

Definition 2.1.2. *Let $g: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ be a function between subclasses of $\mathbf{F}$, and let $x \in \mathbf{A}_1$ be such that $\mathbf{A}_1$ is a neighborhood of x. We say that g is* **differentiable** *at x if there is an $l \in \mathbf{F}$, such that for all $\varepsilon \in \mathbf{F}^{>}$, there is a $\delta \in \mathbf{F}^{>}$, such that for all $h \in \mathbf{F}$ with $|h| < \delta$, we have*

$$|g(x+h) - g(x) - h\,l| < \varepsilon\,|h|.$$

The element l is unique when it exists. It is called the *derivative* of g at x, and written $l =: g'(x)$. If $\mathbf{A}_1$ is open and g is differentiable at each $x \in \mathbf{A}_1$, then we say that g is *differentiable*, and we write g' for the function $\mathbf{A}_1 \longrightarrow \mathbf{A}_2; x \mapsto g'(x)$, which we call the *derivative* of g. A classical Calculus 101 proof of the chain rule for differentiable real-valued functions applies in our setting, and yields:

Proposition 2.1.3. *Let $f: \mathbf{A}_2 \longrightarrow \mathbf{A}_3$ and $g: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ be functions between subclasses of $\mathbf{F}$. Let $x \in \mathbf{F}$ such that $\mathbf{A}_1$ is a neighborhood of x and $\mathbf{A}_2$ is a neighborhood of $g(x)$. If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x with*

$$(f \circ g)'(x) = g'(x) \times f'(g(x)).$$

Similarly, it can be shown that many "elementary" results in real analysis can be recovered in this setting: differentiable functions are continuous, sums and products of continuous (resp. differentiable) functions are continuous (resp. differentiable)... Unfortunately, the list does not include any interesting theorem in real analysis. Recall that $\mathbb{R}$ is unique up to unique isomorphism as an ordered field with the least upper bound property. For *any* ordered field besides the real numbers, *none* of the following properties are true:

1. Every non-empty bounded subclass has a least upper bound.
2. Every interval is connected (i.e. cannot be written as the union of two open subclasses).
3. There is a connected infinite interval.
4. Every locally constant function is constant.
5. Every continuous function on a closed interval is bounded.
6. The direct image of an interval by a continuous function is an interval.
7. Every continuous and injective function between intervals is strictly monotone.
8. Every differentiable function on an interval with zero derivative is constant.
9. Rolle's theorem for differentiable functions.

10. The mean value theorem for differentiable functions.

11. The L'Hospital rule.

See [84] for a discussion and proofs of most of these results. We'll see that we can retain some of the above properties which pertain to functions when working with certain classes of objects in our calculi. For instance, the statements 5, 7 and 8 will hold in all our calculi.

2.2 Power series

Let us now study elementary properties of power series and show how they can act as functions on power series fields. Throughout the section, we fix power series fields $\mathbb{S}$, $\mathbb{T}$ and $\mathbb{U}$ over $\mathbb{R}$, and we write $\mathfrak{M}$ for the monomial group of $\mathbb{S}$, i.e. $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$.

2.2.1 Power series

Let $\mathbf{D}$ be a domain. We write $\mathbf{D}[[z_1, \ldots, z_n]]$ for the ring of power series

$$P := \sum_{v \in \mathbb{N}^n} P_v \, z_1^{v[1]} \cdots z_n^{v[n]} \quad \text{where } (P_v)_{v \in \mathbb{N}^n} \in \mathbf{D}^{\mathbb{N}^n},$$

which is also a domain. We write $\mathbf{D}[[z]]$ for the ring of power series $P = \sum_{k \in \mathbb{N}} P_k \, z^k$ in one variable.

For $P = \sum_{k \in \mathbb{N}} P_k \, z^k$, the *derivative* P' of P is the power series

$$P' := \sum_{k \in \mathbb{N}} (k+1) \, P_{k+1} \, z^k \in \mathbf{D}[|z|].$$

We write $P^{(n)}$ for the n-th iterated derivative of P, i.e. $P^{(0)} = P$ and $P^{(n+1)} = (P^{(n)})'$ for all $n \in \mathbb{N}$.

Consider the subdomain $z \, \mathbf{D}[[z]]$ of $\mathbf{D}[[z]]$ of power series $P = \sum_{k \in \mathbb{N}} P_k \, z^k$ with $P_0 = 0$. We have a composition law $\circ \colon \mathbf{D}[[z]] \times z \, \mathbf{D}[[z]] \longrightarrow \mathbf{D}[[z]]$. Indeed for $P = \sum_{k \in \mathbb{N}} P_k \, z^k$, $Q = \sum_{k \in \mathbb{N}} Q_k \, z^k \in \mathbb{S}[[z]]$ with $Q_0 = 0$, we have a *composite power series*

$$P \circ Q := P_0 + \sum_{k \in \mathbb{N}} \left(\sum_{m_1 + \cdots + m_n = k} P_n \, Q_{m_1} \cdots Q_{m_n} \right) z^k \in \mathbb{S}[[z]].$$

For $P \in \mathbf{D}[[z]]$ and $Q, R \in z \, \mathbf{D}[[z]]$, we have $Q \circ R \in z \, \mathbf{D}[[z]]$ and

$$P \circ (Q \circ R) = (P \circ Q) \circ R.$$

2.2.2 Convergence of power series

Definition 2.2.1. *Given a power series*

$$P = \sum_{v \in \mathbb{N}^n} P_v \, z_1^{v[1]} \cdots z_n^{v[n]} \in \mathbb{S}[[z_1, \ldots, z_n]],$$

and $s_1, \ldots, s_n \in \mathbb{S}$, *we say that* P **converges at** $(s_1, \ldots, s_n)$ *if the family* $(P_v \, s_1^{v[1]} \cdots s_n^{v[n]})_{v \in \mathbb{N}^n}$ *is well-based. We then set*

$$\tilde{P}(s_1, \ldots, s_n) := \sum_{v \in \mathbb{N}^n} P_v \, s_1^{v[1]} \cdots s_n^{v[n]}.$$

We write $\operatorname{Conv}(P)$ *for the class of tuples* $(s_1, \ldots, s_n) \in \mathbb{S}^n$ *at which* P *converges.*

Remark 2.2.2. This notion of convergence, like the notion of well-based family, does not correspond to the convergence of sequences for the valuation topology on $\mathbb{S}$.

Example 2.2.3. Any real power series $P = \sum_{k \in \mathbb{N}} r_k z^k \in \mathbb{R}[[z]]$ converges on $\mathbb{S}^{\prec}$ by Lemma 1.2.11. In fact, since the sequence $(s^k)_{k \in \mathbb{N}}$ is $\prec$-nondecreasing whenever $s \succcurlyeq 1$, we have $\mathrm{Conv}(P) = \mathbb{S}^{\prec}$ unless P is a polynomial.

The following shows that for $P \in \mathbb{S}[[z]]$, the class $\mathrm{Conv}(P)$ is an ultrametic ball.

Proposition 2.2.4. [92, Corollary 1.5.8] *For all $P \in \mathbb{S}[[z]]$, and $\varepsilon, \delta \in \mathbb{S}$ with $\delta \in \mathrm{Conv}(P)$, we have $\varepsilon \preccurlyeq \delta \Longrightarrow \varepsilon \in \mathrm{Conv}(P)$.*

Proof. Write $P = \sum_{k \in \mathbb{N}} P_k z^k$ and $u := \varepsilon/\delta \preccurlyeq 1$. By Proposition 1.2.15 for $I = \mathbb{N}$ and $f = \mathrm{Id}_{\mathbb{N}}$, the family $(P_k \delta^k u^k)_{k \in \mathbb{N}} = (P_k \varepsilon^k)_{k \in \mathbb{N}}$ is well-based. □

Lemma 2.2.5. *Let $P \in \mathbb{S}[[z]]$ with $\mathrm{Conv}(P) \neq \{0\}$. Then $\mathrm{Conv}(P)$ is open.*

Proof. Consider a $\varepsilon \in \mathrm{Conv}(P)$. Given a positive $\delta \in \mathrm{Conv}(P) \setminus \{0\}$, we have $\varepsilon + \delta \preccurlyeq \delta$ or $\varepsilon + \delta \preccurlyeq \varepsilon$, and likewise $\varepsilon - \delta \preccurlyeq \delta$ or $\varepsilon - \delta \preccurlyeq \varepsilon$. In any case, we obtain $\varepsilon + \delta, \varepsilon - \delta \in \mathrm{Conv}(P)$ by Proposition 2.2.4. Therefore $\mathrm{Conv}(P)$ is open. □

Example 2.2.6. The case $\mathrm{Conv}(P) = \{0\}$ can occur. For instance, on the field $\mathbb{R}[[x^{\mathbb{Z}}]]$ of formal Laurent series, the power series

$$P_0 := \sum_{k \in \mathbb{N}} x^k z^k$$

satisfies $\mathrm{Conv}(P_0) = \{0\}$. This does not mean that there cannot be a larger field $\mathbb{V} \supseteq \mathbb{R}[[x^{\mathbb{Z}}]]$ on which P_0 has a non trivial domain of convergence. For instance, identifying x with the surreal number ω, the power series P_0 converges at $1/\omega^{\omega}$ in the field **No** of surreal numbers.

Lemma 2.2.7. *Let $P = \sum_{k \in \mathbb{N}} P_k z^k \in \mathbb{S}[[z]]$ be a power series. For all $n \in \mathbb{N}$, we have $\mathrm{Conv}(P) = \mathrm{Conv}(P^{(n)})$.*

Proof. It suffices to prove the result for $n = 1$. We have $0 \in \mathrm{Conv}(P) \cap \mathrm{Conv}(P')$. Recall that

$$P' = \sum_{k \in \mathbb{N}} (k+1) P_{k+1} z^k.$$

For $\varepsilon \in \mathbb{S}^{\neq}$, we have the following equivalences:

$(P_k \varepsilon^k)_{k \in \mathbb{N}}$ is well-based

$\Longleftrightarrow (P_{k+1} \varepsilon^{k+1})_{k \in \mathbb{N}}$ is well-based (by Proposition 1.1.16 and Corollary 1.1.19)

$\Longleftrightarrow ((k+1) P_{k+1} \varepsilon^k)_{k \in \mathbb{N}}$ is well-based. (by Corollary 1.2.10)

We deduce that $\mathrm{Conv}(P) = \mathrm{Conv}(P')$. □

Proposition 2.2.8. *Let $P = \sum_{k \in \mathbb{N}} P_k z^k \in \mathbb{S}[[z]]$ be a power series and let $\varepsilon, \delta \in \mathrm{Conv}(P)$. Write $P_{+\varepsilon}$ for the power series $P_{+\varepsilon} := \sum_{k \in \mathbb{N}} \frac{\widetilde{P^{(k)}}(\varepsilon)}{k!} z^k$. We have $\delta \in \mathrm{Conv}(P_{+\varepsilon})$ and*

$$\widetilde{P_{+\varepsilon}}(\delta) = \tilde{P}(\varepsilon + \delta).$$

Proof. Note that $P_{+0} = P$ and that $P_{+\varepsilon}(0) = P(\varepsilon)$, so we may assume that ε and δ are non-zero. The power series $P_{+\varepsilon}$ is well-defined by Lemma 2.2.7. We have

$$\bigcup_{i,k \in \mathbb{N}} \mathrm{supp}\,(P_{k+i} \varepsilon^{k+i}) = \bigcup_{j \in \mathbb{N}} \mathrm{supp}\,(P_j \varepsilon^j),$$

where the right hand set is well-based since $(P_j \varepsilon^j)_{j \in \mathbb{N}}$ is well-based. For each monomial $\mathfrak{m} \in \mathfrak{M}$, the set $I_{\mathfrak{m}} := \{(i,k) \in \mathbb{N}^2 : \mathfrak{m} \in \mathrm{supp}(P_{i+k} \delta^{k+i})\}$ is contained in $\{(i,k) \in \mathbb{N}^2 : i + k \in J_{\mathfrak{m}}\}$ where

$$J_{\mathfrak{m}} := \{j \in \mathbb{N} : \mathfrak{m} \in \mathrm{supp}(P_j \delta^j)\}.$$

Since $(P_j \varepsilon^j)_{j \in \mathbb{N}}$ is well-based, we deduce that $J_{\mathfrak{m}}$, and hence $I_{\mathfrak{m}}$ are finite. This shows that $(P_{k+i} \varepsilon^{k+i})_{i,k \in \mathbb{N}}$ is well-based. Likewise $(P_{k+i} \delta^{k+i})_{i,k \in \mathbb{N}}$ is well-based.

For $k\in\mathbb{N}$, we have

$$\frac{\widetilde{P^{(k)}}(\varepsilon)}{k!}\,\delta^k=\sum_{i\in\mathbb{N}}\binom{k+i}{k}P_{k+i}\,\varepsilon^i\,\delta^k. \tag{2.2.1}$$

Therefore it suffices to show that the family $(P_{k+i}\,\varepsilon^i\,\delta^k)_{i,k\in\mathbb{N}}$ is well-based in order to prove that $\delta\in\mathrm{Conv}(P_{+\varepsilon})$. For $i,k\in\mathbb{N}$, write

$$\varepsilon^i\,\delta^k=u^{i+k}\,v^k$$

where $(u,v)=(\varepsilon,{}^{\delta}\!/_{\varepsilon})$ if $\delta\preccurlyeq\varepsilon$ and $(u,v)=(\delta,{}^{\varepsilon}\!/_{\delta})$ if $\varepsilon\prec\delta$. In any case, we have $v\preccurlyeq 1$ and the family $(P_{i+k}\,u^{i+k})_{i,k\in\mathbb{N}}$ is well-based. Applying Proposition 1.2.15 for $I=\mathbb{N}\times\mathbb{N}$ and $f=(a,b)\mapsto a+b$, we see that the family $(P_{i+k}\,u^{i+k}\,v^k)_{i,k\in\mathbb{N}}=(P_{k+i}\,\varepsilon^i\,\delta^k)_{i,k\in\mathbb{N}}$ is well-based.

On the other hand we have $\delta+\varepsilon\preccurlyeq\varepsilon$ or $\delta+\varepsilon\preccurlyeq\delta$, so $\delta+\varepsilon\in\mathrm{Conv}(P)$ and $(P_k\,(\delta+\varepsilon)^k)_{k\in\mathbb{N}}$ is well-based. By Lemma 1.1.20, we have

$$\begin{aligned}
\sum_{k\in\mathbb{N}}\frac{\widetilde{P^{(k)}}(\varepsilon)}{k!}\,\delta^k &= \sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{N}}\binom{k+i}{k}P_{k+i}\,\varepsilon^i\,\delta^k\\
&= \sum_{i,k\in\mathbb{N}}\binom{k+i}{k}P_{k+i}\,\varepsilon^i\,\delta^k\\
&= \sum_{j\in\mathbb{N}}\sum_{l\leqslant j}\binom{j}{l}P_j\,\varepsilon^{j-l}\,\delta^l\\
&= \sum_{j\in\mathbb{N}}P_j\,(\varepsilon+\delta)^j\\
&= \tilde{P}(\varepsilon+\delta),
\end{aligned}$$

where we use Proposition 1.1.16 for the bijection

$$\psi\colon\mathbb{N}^2\longrightarrow\{(j,k):j\in\mathbb{N}\wedge k\leqslant j\};(i,k)\mapsto(i+k,k)$$

to obtain the third equality above. □

Lemma 2.2.9. *Let $P=\sum_{k\in\mathbb{N}}P_k\,z^k\in\mathbb{S}[[z]]$ be a power series with* $\mathrm{Conv}(P)\neq\{0\}$*. The function $\tilde{P}$ is infinitely differentiable on* $\mathrm{Conv}(P)$ *with $\tilde{P}^{(n)}=\widetilde{P^{(n)}}$ on* $\mathrm{Conv}(P)$ *for all $n\in\mathbb{N}$.*

Proof. We first prove that $\tilde{P}$ is differentiable on $\mathrm{Conv}(P)$ with $\tilde{P}'=\widetilde{P'}$. Let $\varepsilon>0$ and let $s\in\mathrm{Conv}(P)$. For all $h\in\mathbb{S}$ with $|h|<|s|$, we have $h\preccurlyeq s$, so Proposition 2.2.8 yields

$$\begin{aligned}
\tilde{P}(s+h)-\tilde{P}(s) &= \sum_{k>0}\frac{\widetilde{P^{(k)}}(s)}{k!}\,h^k\\
&= \widetilde{P'}(s)\,h+h^2\,u,
\end{aligned}$$

where $u:=\sum_{k\in\mathbb{N}}\frac{\widetilde{P^{(k+2)}}(s)}{(k+2)!}\,h^k$. If $u=0$, then we set $\delta:=|s|$. If $u\neq 0$, then we set $\delta:={}^{\varepsilon}\!/_{|u|}$. In both cases, we obtain $|\tilde{P}(s+h)-\tilde{P}(s)-\widetilde{P'}(s)\,h|<\varepsilon\,|h|$ whenever $|h|<\delta$. So $\tilde{P}$ is differentiable at s with $\tilde{P}'(s)=\widetilde{P'}(s)$. The result for all n follows by induction. □

2.2.3 Zeros of power series

We next consider zeros of power series functions. A *zero* of a power series $P\in\mathbb{S}[[z_1,\ldots,z_n]]$ is an element $(s_1,\ldots,s_n)\in\mathrm{Conv}(P)$ with $\tilde{P}(s_1,\ldots,s_n)=0$.

Example 2.2.10. Non-zero power series in one variable may have infinitely many zeros. Here we give an example due to van der Hoeven. Set $\mathbb{S}=\mathbb{R}[[x^{\mathbb{Z}}]]$ and write $\pi(n,k)$ for the number of partitions of n into k parts, for all $k\leqslant n<\omega$. For $k\in\mathbb{N}$, set

$$s_k:=(-1)^k\sum_{n\in\mathbb{N}}\pi(n,k)\,x^{-n}\in\mathbb{S}.$$

Then the power series $P=\sum_{k\in\mathbb{N}} s_k z^k$ has x^n as a zero for each $n\in\mathbb{N}$. This power series can be obtained by formally expanding the product $\prod_{n\in\mathbb{N}}(1-x^{-n} z)$.

Lemma 2.2.11. *Suppose that $\mathfrak{M}$ is uncountable and let $P\in\mathbb{S}[[z_1,\ldots,z_n]]$ be a power series with $(\mathbb{S}^{\prec})^n\subseteq\operatorname{Conv}(P)$. If $P\neq 0$, then $\tilde{P}(s_1,\ldots,s_n)\neq 0$ for some $s_1,\ldots,s_n\in\mathbb{S}^{\prec}$.*

Proof. We prove this by induction on n. If $n=1$, then write $P=\sum_{k\in\mathbb{N}} P_k z^k$. Suppose that $P\neq 0$ and let $\mathcal{R}\subseteq\mathbb{S}^{\prec}$ be the set of non-zero infinitesimal zeros of P. Fix $s\in\mathcal{R}$ and let m be such that $0\neq P_m s^m\succcurlyeq P_k s^k$ for all k. Since $P(s)=0$, there exists an index $k\neq m$ with $P_m s^m\asymp P_k s^k$. Then $0\neq s\asymp(P_m^{-1} P_k)^{1/(m-k)}$, whence

$$\{\mathfrak{d}_s: s\in\mathcal{R}\}\subseteq\left\{\left(\frac{\mathfrak{d}_{P_k}}{\mathfrak{d}_{P_m}}\right)^q: k,m\in\mathbb{N}, q\in\mathbb{Q}, P_k, P_m\neq 0\right\}.$$

In particular, $\{\mathfrak{d}_s: s\in\mathcal{R}\}$ is countable, whereas $\{\mathfrak{d}_s: s\in\mathbb{T}^{\prec}, s\neq 0\}=\mathfrak{M}^{\prec}$ is uncountable, so $\mathcal{R}\neq\mathbb{T}^{\prec}$.

Now suppose that $n>1$ and write $P=\sum_{k\in\mathbb{N}} P_k X_n^k$ where each P_k lies in $\mathbb{S}[[z_1,\ldots,z_{n-1}]]$. Assume that $P\neq 0$. By the induction hypothesis, we can find $s_1,\ldots,s_{n-1}\in\mathbb{S}^{\prec}$ and $k\in\mathbb{N}$ such that $\widetilde{P_k}(s_1,\ldots,s_{n-1})\neq 0$. Fix such elements $s_1,\ldots,s_{n-1}$ and let $\mathcal{R}\subseteq\mathbb{S}^{\prec}$ be the set of $s\in\mathbb{T}^{\prec}$ such that $\tilde{P}(s_1,\ldots,s_{n-1},s)=0$. By the special case when $n=1$, we see that $\mathcal{R}\neq\mathbb{S}^{\prec}$. Thus $\tilde{P}(s_1,\ldots,s_{n-1},s)\neq 0$ for some $s\in\mathbb{T}$. □

We will also need similar results in the univariable case.

Lemma 2.2.12. *Let $P=\sum_{n\in\mathbb{N}} P_n z^n\in\mathbb{S}[[z]]$ be a power series and let $\mathcal{R}\subseteq\operatorname{Conv}(P)$ be an uncountable set of zeros of P with pairwise distinct dominant terms. We have $P=0$.*

Proof. Assume for contradiction that there is a non-zero term P_n in the sequence and consider $s\in\mathcal{R}$. Since the sum of $(P_n s^n)_{n\in\mathbb{N}}$ is zero, for each number m with $P_m\neq 0$, there must exist at least one number $n\neq m$ with $\tau_{P_m}\tau_s^m=\tau_{P_n}\tau_s^n$. Then $\tau_s=\left(\frac{\tau_{s_m}}{\tau_{s_n}}\right)^{1/(m-n)}$, so we deduce that

$$\mathcal{R}\subseteq\left\{\left(\frac{\tau_{s_m}}{\tau_{s_n}}\right)^q: n,m\in\mathbb{N}, q\in\mathbb{Q}, P_m, P_n\neq 0\right\}.$$

Therefore $\mathcal{R}$ is countable: a contradiction. □

Lemma 2.2.13. *Let $P=\sum_{k\in\mathbb{N}} P_k z^k\in\mathbb{S}[[z]]$ be a power series, and let κ be an infinite cardinal. Let $\mathcal{R}\subseteq\mathbb{S}$ be a set of zeros of P with cardinality $\geqslant\kappa^+$ such that for each $s\in\mathcal{R}$, the order type of $(\operatorname{supp} s,\succ)$ is $\leqslant\kappa$. Then $P=0$.*

Proof. Assume for contradiction that $P\neq 0$. We will call *large* the subsets X of $\mathcal{R}$ with $|\mathcal{R}\setminus X|\leqslant\kappa$. For $\alpha\leqslant\kappa$ and $s\in\mathbb{S}$, we let $s_{|\alpha}$ denote the $\trianglelefteq$-maximal truncation of s such that the order type of $(\operatorname{supp} s_{|\alpha},\succ)$ is $\leqslant\alpha$, and we write $s_{\alpha|}:=s-s_{|\alpha}$. Let $\mathcal{I}$ denote the set of ordinals $\alpha\leqslant\kappa$ such that there is a large subset $X_\alpha\subseteq\mathcal{R}$ with $t_{|\beta}=u_{|\beta}$ for all $\beta<\alpha$ and $t,u\in X_\alpha$. Note that $\mathcal{I}$ contains 0 trivially and 1 by Lemma 2.2.12. Let us show that $\kappa\in\mathcal{I}$. Let $\alpha\leqslant\kappa$ with $\beta\in\mathcal{I}$ for all $\beta<\alpha$.

If α is limit, then for each $\beta<\alpha$, pick a large subset $X_\beta\subseteq\mathcal{R}$ satisfying the condition and consider the set $X_\alpha:=\bigcap_{\beta<\alpha} X_\beta$. This set is large since $\alpha<\kappa^+$ and κ^+ is regular. Moreover it satisfies the condition for α by definition. So $\alpha\in\mathcal{I}$.

Assume now that $\alpha=\beta+1$ where $\beta\in\mathcal{I}$ and $\beta>0$. We fix a set X_β satisfying the condition for β. For $t\in X_\beta$, since $\beta>0$, we have $t_{|\beta}\preccurlyeq t$, so $t_{|\beta}\in\operatorname{Conv}(P)$ is defined. We deduce with Lemma 2.2.9 that $t_{|\beta}\in\operatorname{Conv}(P^{(k)})$ for all $k\in\mathbb{N}$. By Proposition 2.2.8 we have $\tilde{P}(t)=\tilde{P}(t_{|\beta}+t_{\beta|})=\widetilde{P_{+t_{|\beta}}}(t_{\beta|})$. Assume for contradiction that $P_{+t_{|\beta}}=0$. Then $\widetilde{P^{(i)}}(t_{|\beta})=0$ for all $i\in\mathbb{N}$, so $\tilde{P}(t_{|\beta}+\varepsilon)=0$ for all $\varepsilon\in\mathbb{S}$ with $\varepsilon\preccurlyeq t_{|\beta}$. In particular, given $\gamma<\beta$, we have $\tilde{P}(t_{|\gamma}+r\,\mathfrak{d}_{t_{|\gamma}})=0$ for all $r\in\mathbb{R}$, which contradicts Lemma 2.2.12. We deduce that $P_{+t_{|\beta}}$ is non-zero. By Lemma 2.2.12, there is a co-countable subset of X_β, hence large subset X_α of $\mathcal{R}$ with $(t_{\beta|})_{|1}=(u_{\beta|})_{|1}$, hence $u_{|\alpha}=v_{|\alpha}$ for all $u,v\in X_\alpha$. This proves that $\alpha\in\mathcal{I}$. By induction, we deduce that $\kappa\in\mathcal{I}$. For $u,v\in X_\kappa$, we have $u=u_{|\kappa}=v_{|\kappa}=v$, which contradicts the fact that X_κ is large. □

We note two corollaries to this result.

Corollary 2.2.14. *Let $P \in \mathbb{S}[[z]]$ and let $\varepsilon \in \operatorname{Conv}(P)$ with $\varepsilon \neq 0$. If $\tilde{P}(\delta) = 0$ for all $\delta \preccurlyeq \varepsilon$ then $P = 0$.*

Proof. Consider the set $\mathcal{S}$ of series $s \in \mathbb{S}$ with $s \preccurlyeq \varepsilon$ and such that the order type of $(\operatorname{supp} s, \succ)$ is at most ω. Fix an $\mathfrak{m} \in \mathfrak{M}^{\prec}$ with $\mathfrak{m} \preccurlyeq \varepsilon$. For each binary sequence $u \in 2^{\mathbb{N}}$, we have a single element $\sum_{n \in \mathbb{N}} u(n) \, \mathfrak{m}^n \in \mathcal{S}$, so $\mathcal{S}$ is uncountably infinite. It follows by Lemma 2.2.13 for $\kappa = \omega$ that $P = 0$. □

Corollary 2.2.15. *Let $P \in \mathbb{S}[[z]]$ with $\operatorname{Conv}(P) \neq \{0\}$ and let $\delta \in \operatorname{Conv}(P)$. We have $\operatorname{Conv}(P_{+\delta}) = \operatorname{Conv}(P)$ and $P = (P_{+\delta})_{+(-\delta)}$.*

Proof. We may assume that $\delta \neq 0$. Proposition 2.2.8 shows that $\operatorname{Conv}(P_{+\delta}) \supseteq \operatorname{Conv}(P)$. By $\prec$-initiality of $\operatorname{Conv}(P)$, we have $-\delta \in \operatorname{Conv}(P)$. So $-\delta \in \operatorname{Conv}(P_{+\delta})$, which means that the power series $(P_{+\delta})_{+(-\delta)}$ is well-defined. Since $\operatorname{Conv}(P_{+\delta})$ is $\prec$-initial and contains δ, Proposition 2.2.8 yields

$$\widetilde{(P_{+\delta})_{+(-\delta)}}(\varepsilon) = \widetilde{P_{+\delta}}(\varepsilon - \delta) = \tilde{P}(\varepsilon)$$

for all $\varepsilon \preccurlyeq \delta$. We deduce by Corollary 2.2.14 that $P = (P_{+\delta})_{+(-\delta)}$. Applying Proposition 2.2.8, this time to $(P_{+\delta}, -\delta)$, we get $\operatorname{Conv}(P_{+\delta}) \subseteq \operatorname{Conv}(P)$, hence the equality. □

2.3 Analytic functions

Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$, $\mathbb{T} = \mathbb{R}[[\mathfrak{N}]]$ and $\mathbb{U} = \mathbb{R}[[\mathfrak{O}]]$ be fixed fields of well-based series over $\mathbb{R}$ with $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{O}$ as ordered groups, whence $\mathbb{S} \subseteq \mathbb{T} \subseteq \mathbb{U}$ as ordered valued fields. We also fix a non-empty open subclass $\mathbf{O}$ of $\mathbb{S}$.

2.3.1 Analyticity

Definition 2.3.1. *Let $f: \mathbf{O} \longrightarrow \mathbb{T}$ be a function and let $s \in \mathbf{O}$. We say that f is* **analytic at** *s if there is a power series $f_s \in \mathbb{T}[[z]]$ with $\operatorname{Conv}(f_s) \neq \{0\}$ and a $\delta \in \operatorname{Conv}(f_s) \setminus \{0\}$ such that for all $\varepsilon \preccurlyeq \delta$, we have*

$$(s + \varepsilon \in \mathbf{O}) \Longrightarrow f(s + \varepsilon) = \tilde{f}_s(\varepsilon).$$

We say that f_s is a **Taylor series** *of f at s. We say that f is* **analytic** *if it is analytic at each $s \in \mathbf{O}$.*

Lemma 2.3.2. *Let $f: \mathbf{O} \longrightarrow \mathbb{S}$ be analytic at $s \in \mathbf{O}$. Then f_s is the unique Taylor series of f at s.*

Proof. Let $P \in \mathbb{S}[[z]]$ and $\delta \in \operatorname{Conv}(P) \setminus \{0\}$ with $s + \varepsilon \in \mathbf{O}$ and $f(s + \varepsilon) = \tilde{P}(\varepsilon)$ for all $\varepsilon \preccurlyeq \delta$. Then the function $\widetilde{f_s - P}$ is zero on the class of series $s \preccurlyeq \delta$, so we have $f_s = P$ by Corollary 2.2.14. □

If $f: \mathbf{O} \longrightarrow \mathbb{S}$ is analytic at $s \in \mathbf{O}$ where $\mathbf{O}$ is open, then we can define

$$\operatorname{Conv}(f)_s := \{t \in \mathbf{O} : t - s \in \operatorname{Conv}(f_s) \wedge f(t) = \tilde{f}_s(t - s)\}.$$

Proposition 2.3.3. *Let $P \in \mathbb{T}[[z]]$ with $\operatorname{Conv}(P) \neq \{0\}$. Then $\tilde{P}$ is analytic on $\operatorname{Conv}(P)$ with $\tilde{P}_\delta = P_{+\delta}$ and $\operatorname{Conv}(\tilde{P})_\delta = \operatorname{Conv}(P)$ for all $\delta \in \operatorname{Conv}(P)$.*

Proof. Let $\delta \in \operatorname{Conv}(P)$. The class $\operatorname{Conv}(P)$ is open by Lemma 2.2.5, with $\operatorname{Conv}(P_{+\delta}) = \operatorname{Conv}(P)$. By Proposition 2.2.8, we have $\tilde{P}(\delta + \varepsilon) = \widetilde{P_{+\delta}(\varepsilon)}$ for all $\varepsilon \in \operatorname{Conv}(P)$, so $\tilde{P}$ is indeed analytic on $\operatorname{Conv}(P)$ with $\operatorname{Conv}(\tilde{P})_\delta \supseteq \operatorname{Conv}(P_{+\delta}) = \operatorname{Conv}(P)$. But we also have $\operatorname{Conv}(\tilde{P})_\delta \subseteq \operatorname{Conv}(P_{+\delta}) = \operatorname{Conv}(P)$ by definition, hence the result. □

Corollary 2.3.4. *Let $f: \mathbf{O} \longrightarrow \mathbb{T}$ be analytic at $s \in \mathbf{O}$. Then there is an open neighborhood $\mathbf{O}_s$ of s such that the restriction $f \upharpoonright \mathbf{O}_s$ of f to $\mathbf{O}_s$ is analytic.*

Proof. Define $\mathbf{O}_s = \{s + \mathbb{S}^{\prec \delta}\}$ where δ is any element of $\mathrm{Conv}(f_s) \setminus \{0\}$. Then Proposition 2.3.3 yields the result. □

Proposition 2.3.5. *Let $f: \mathbf{O} \longrightarrow \mathbb{S}$ be analytic at $s \in \mathbf{O}$ and let $\mathbf{U} \subseteq \mathrm{Conv}(f)_s$ be a non-empty open subclass containing 0. Then f is analytic on $s + \mathbf{U}$, with $f_{s+\delta} = (f_s)_{+\delta}$ for all $\delta \in \mathbf{U}$.*

Proof. Let $\delta \in \mathbf{U}$ and set $t := s + \delta$. Since $\mathbf{U} \ni 0$ is open and non-empty, we find a $\rho \neq 0$ with $\delta + \varepsilon \in \mathbf{U}$ for all $\varepsilon \preccurlyeq \rho$. Thus $f(t + \varepsilon) = \tilde{f}_s(\delta + \varepsilon)$ whenever $\varepsilon \preccurlyeq \rho$. But given such ε, we have $\tilde{f}_s(\delta + \varepsilon) = \widetilde{(f_s)_{+\delta}}(\varepsilon)$ by Proposition 2.2.8, whence

$$f(t + \varepsilon) = \tilde{f}_s(\delta + \varepsilon) = \widetilde{(f_s)_{+\delta}}(\varepsilon).$$

So f is analytic at t with $f_t = (f_s)_{+(t-s)}$. □

Proposition 2.3.6. *Let $f: \mathbf{O} \longrightarrow \mathbb{S}$ be analytic at $s \in \mathbf{O}$. Then f is infinitely differentiable at s, and each $f^{(n)}$ for $n \in \mathbb{N}$ is analytic at s with $\mathrm{Conv}(f^{(n)})_s \supseteq \mathrm{Conv}(f)_s$. Moreover, we have*

$$f_s = \sum_{k \in \mathbb{N}} \frac{f^{(k)}(s)}{k!} z^k.$$

Proof. Recall that $\tilde{f}_s$ is infinitely differentiable on $\mathrm{Conv}(f_s)$. By Lemma 2.2.9, each derivative $\tilde{f}_s^{(n)}$ for $n \in \mathbb{N}$ is a power series function on $\mathrm{Conv}(f_s)$, and is thus analytic on $\mathrm{Conv}(f_s)$ by Proposition 2.3.3. It follows since $\mathrm{Conv}(f)_s$ is a neighborhood of s that f is infinitely differentiable at s. By Lemma 2.2.9, given $\delta \in \mathrm{Conv}(f)_s$, we have $f^{(n)}(s + \delta) = \tilde{f}_s^{(n)}(\delta) = \widetilde{(f_s)^{(n)}}(\delta)$. Therefore $f^{(n)}$ is analytic at s with $f_s^{(n)} = (f_s)^{(n)}$ and $\mathrm{Conv}(f^{(n)})_s \supseteq \mathrm{Conv}(f)_s$. Write $f_s = \sum_{k \in \mathbb{N}} s_k z^k$. We have $f^{(k)}(s) = \widetilde{(f_s)}^{(k)}(0) = \widetilde{(f_s)^{(k)}}(0) = k!\, s_k$. We deduce that $f_s = \sum_{k \in \mathbb{N}} \frac{f^{(k)}(s)}{k!} z^k$. □

Corollary 2.3.7. *Let $\mathbf{O} \subseteq \mathbb{S}$ be open and non-empty and assume that $\mathbf{O} = \bigsqcup_{i \in \mathbf{I}} \mathbf{O}_i$ where each $\mathbf{O}_i$ is open and non-empty. Let $(s_i)_{i \in \mathbf{I}}$ be a family where $s_i \in \mathbf{O}_i$ for all $i \in \mathbf{I}$. Let $(P_i)_{i \in \mathbf{I}}$ be a family of power series in $\mathbb{S}[[z]]$ with $(s_i + \mathrm{Conv}(P_i)) \supseteq \mathbf{O}_i$. The function $f: \mathbf{O} \longrightarrow \mathbb{S}$ such that for all $i \in \mathbf{I}$ and $s \in \mathbf{O}_i$, we have $f(s) = P_i(s - s_i)$ is well-defined and analytic.*

Proof. Let $s \in \mathbf{O}$ and let $i \in \mathbf{I}$ with $s \in \mathbf{O}_i$. We have $s - s_i \in \mathbf{O}_i - s_i \subseteq \mathrm{Conv}(P_i)$ so $\tilde{P}_i(s - s_i)$ is defined. In particular f is well-defined. The class $\mathbf{O}_i - s_i$ is a neighborhood of 0, so there is a $\delta \in \mathrm{Conv}(P_i) \setminus \{0\}$ such that $s_i + \varepsilon \in \mathbf{O}_i$ whenever $\varepsilon \preccurlyeq \delta$. Given $\varepsilon \preccurlyeq \delta$, we have

$$f(s + \varepsilon) = P_i(s + \varepsilon - s_i) = (P_i)_{+(s - s_i)}(\varepsilon)$$

by Proposition 2.2.8. Therefore f is analytic at s with $f_s = (P_i)_{+(s - s_i)}$. □

We leave it to the reader to check that analyticity, at a point or on an open class, is preserved by sums and products. The following result will be used extensively in the thesis to show that a composition of analytic functions is analytic.

Proposition 2.3.8. *Let $\mathbf{U} \subseteq \mathbb{T}$ be open. Let $f: \mathbf{U} \longrightarrow \mathbb{U}$, $g: \mathbf{O} \longrightarrow \mathbf{U}$ and let $s \in \mathbf{O}$ such that g is analytic at s and f is analytic at $g(s)$. Write*

$$f_{g(s)} = \sum_{n \in \mathbb{N}} a_n z^n \qquad \text{and} \qquad g_s = \sum_{n \in \mathbb{N}} b_n z^n.$$

Let $\varepsilon_f \in \mathrm{Conv}(f)_{g(s)}$ and $\varepsilon \in \mathrm{Conv}(g)_s$ with

$$\forall m \in \mathbb{N}^{>}, b_m \varepsilon^m \prec \varepsilon_f. \tag{2.3.1}$$

The function $f \circ g$ is analytic at s with $\varepsilon \in \mathrm{Conv}(f \circ g)_s$, and $(f \circ g)_s = f_{g(s)} \circ (g_s - g(s))$.

Proof. For $n \in \mathbb{N}$ and $k \in \mathbb{N}^{>}$, set $X_{n,k} := \{v \in (\mathbb{N}^{>})^n : |v| = k\}$.

$$c_{n,k} := \sum_{v \in X_{n,k}} a_n b_{v_{[1]}} \cdots b_{v_{[n]}},$$

so $f_{g(s)} \circ (g_s - g(s)) = f(g(s)) + \sum_{k \in \mathbb{N}^>} (\sum_{n \in \mathbb{N}} c_{n,k}) z^k$. Note that since $\varepsilon \in \mathrm{Conv}(g)_s \subseteq \mathrm{Conv}(g_s)$, the set

$$\mathfrak{S}_g := \bigcup_{m \in \mathbb{N}} \mathrm{supp}\,(b_m \varepsilon^m)$$

is well-based. We have $\mathfrak{S}_g \prec \varepsilon_f$ by (2.3.1). Set $\mathfrak{m} := \mathfrak{d}_{\varepsilon_f}$, so $\mathfrak{S}_g \prec \mathfrak{m} \preccurlyeq \varepsilon_f$. The set $\mathfrak{S}_f := \bigcup_{n \in \mathbb{N}} \mathrm{supp}\,(a_n \mathfrak{m}^n)$ is well-based. For $n \in \mathbb{N}$ and $k \in \mathbb{N}^>$, we have

$$\mathrm{supp}\, c_{n,k} \varepsilon^k \subseteq (\mathfrak{S}_g \cdot \mathfrak{m}^{-1})^n \cdot \mathfrak{S}_f,$$

where $\mathfrak{S}_g \cdot \mathfrak{m}^{-1}$ is well-based and infinitesimal, and $\mathfrak{S}_f$ is well-based. Since each family $(c_{n,k} \varepsilon^k)_{k>0}$ for $n \in \mathbb{N}$ is well-based with sum $(g(s+\varepsilon) - g(s))^n$, we conclude with Corollary 1.2.14 that $(c_{n,k} \varepsilon^k)_{n \in \mathbb{N}, k>0}$ is well-based. We deduce by Lemma 1.1.20 that

$$\begin{aligned} f(g(s+\varepsilon)) &= \sum_{n \in \mathbb{N}} a_n (g(s+\varepsilon) - g(s))^n \\ &= \sum_{n \in \mathbb{N}} a_n \left(\sum_{k \in \mathbb{N}^>} b_k \varepsilon^k \right)^n \\ &= f(g(s)) + \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}^>} c_{n,k} \varepsilon^k \\ &= f(g(s)) + \sum_{k \in \mathbb{N}^>} \left(\sum_{n \in \mathbb{N}} c_{n,k} \right) \varepsilon^k \\ &= (\widetilde{f_{g(s)} \circ (g_s - g(s))})(\varepsilon). \end{aligned}$$

By Proposition 2.2.4, we deduce that $f \circ g$ is analytic at s, hence the result. □

2.3.2 Basic examples of analytic functions

We give a few examples of analytic functions. Perhaps the most *un*interesting examples of analytic functions are locally constant functions, such as the dominant monomial map

$$\mathbb{S}^{\neq} \twoheadrightarrow \mathfrak{M}; s \mapsto \mathfrak{d}_s,$$

or the purely large part function

$$\mathbb{S} \twoheadrightarrow \mathbb{S}_{\succ}; s \mapsto s_{\succ}.$$

Note that these functions are monotone on any interval on which they are defined.

This illustrates the fact that analytic functions need not behave in a similar way as real-analytic or holomorphic functions. In fact, the disconnectedness of any non-trivial field of well-based series implies that any purely local notion of regularity for functions is subject to this type of phenomenon.

Example 2.3.9. Recall that for $\varepsilon \in \mathbb{T}^{\prec}$, we have

$$\frac{1}{1+\varepsilon} = \sum_{k \in \mathbb{N}} (-1)^k \varepsilon^k.$$

Thus for $s \in \mathbb{T}^{\neq}$ and $\delta \prec s$, we have

$$\begin{aligned} \frac{1}{s+\delta} &= s^{-1} \times \frac{1}{1 + (\delta s^{-1})} \\ &= s^{-1} \sum_{k \in \mathbb{N}} (-1)^k s^{-k} \delta^k. \end{aligned}$$

Thus the reciprocal function $\mathcal{R}: \mathbb{T}^{\neq} \longrightarrow \mathbb{T}^{\neq}; s \mapsto \frac{1}{s}$ is analytic with

$$\begin{aligned} \mathrm{Conv}(\mathcal{R}_s) &= \{\delta \in \mathbb{T} : \delta \prec s\} \quad \text{and} \\ \mathcal{R}_s &= \sum_{k \in \mathbb{N}} ((-1)^k s^{-(k+1)}) z^k \end{aligned}$$

for all $s \neq 0$.

2.4 Real-analytic functions on well-based series

A well-known type of analytic function is that of restricted real-analytic function of [31, 34]. We recall some of the definitions.

2.4.1 Real-analytic functions

For $n \in \mathbb{N}$ and $v \in \mathbb{N}^n$, write $\frac{\partial^{|v|}}{\partial x^v}$ for the partial derivative operator

$$\frac{\partial^{|v|}}{\partial x^v} \leftrightharpoons \frac{\partial^{|v|}}{\partial^{v_{[1]}} x_1 \cdots \partial^{v_{[n]}} x_n}.$$

We also write $v! := v_{[1]}! \cdots v_{[n]}!$.

If U is a non-empty open subset of $\mathbb{R}^n$ and $f: U \longrightarrow \mathbb{R}$ is an analytic function, then by Lemma 1.2.11 it extends into a function $\overline{f}: U + (\mathbb{S}^{\prec})^n \longrightarrow \mathbb{R} + \mathbb{S}^{\prec}$ given by

$$\forall r \in \mathbb{R}^n, \forall \varepsilon_1, \ldots, \varepsilon_n \prec 1, \overline{f}(r + (\varepsilon_1, \ldots, \varepsilon_n)) := \sum_{v \in \mathbb{N}^n} \frac{1}{v!} \frac{\partial^{|v|} f}{\partial x^v}(r)\, \varepsilon_1^{v_{[1]}} \cdots \varepsilon_n^{v_{[n]}}.$$

We say that $\overline{f}$ is a *restricted real-analytic function* on $\mathbb{S}$.

Assume in particular that $n = 1$ and $U = I$ is a non-empty open interval of $\mathbb{R}$. Then we have

$$\begin{aligned} \overline{f}: I + \mathbb{S}^{\prec} &\longrightarrow \mathbb{R} + \mathbb{S}^{\prec} \\ (r + \varepsilon) &\longmapsto \sum_{k \in \mathbb{N}} \frac{f^{(k)}(r)}{k!} \varepsilon^k. \end{aligned}$$

For each $r \in I$, the function $\overline{f}$ is analytic on $r + \mathbb{S}^{\prec}$ by Proposition 2.3.6. Since $I + \mathbb{S}^{\prec}$ is the disjoint union $I + \mathbb{S}^{\prec} = \bigsqcup_{r \in I} r + \mathbb{S}^{\prec}$, it follows that $\overline{f}$ is analytic.

In particular, we have a real-analytic calculus through which each element f of the ring $\mathbb{A}\mathrm{n}$ of real analytic functions $\mathbb{R} \longrightarrow \mathbb{R}$ acts as an analytic function on $\mathbb{S}^{\preccurlyeq} = \mathbb{R} + \mathbb{S}^{\prec}$.

Lemma 2.4.1. *Let $\mathbb{T}$ be a field of well-based series over $\mathbb{R}$. Let $\Psi: \mathbb{S} \longrightarrow \mathbb{T}$ be a strongly linear morphism of ordered rings. Let I be a non-empty interval of $\mathbb{R}$ and let $f: I \longrightarrow \mathbb{R}$ be real-analytic. For all $s \in I + \mathbb{S}^{\prec}$ we have $\Psi(\overline{f}(s)) = \overline{f}(\Psi(s))$.*

Proof. We have $s = r + \varepsilon$ for a unique $(r, \varepsilon) \in I \times \mathbb{S}^{\prec}$. Since Ψ is a strictly increasing morphism of rings, we have $\Psi(s) = r + \Psi(\varepsilon)$ where $\Psi(\varepsilon) \prec 1$. Therefore,

$$\overline{f}(\Psi(s)) = \sum_{k \in \mathbb{N}} \frac{f^{(k)}(r)}{k!} \Psi(\varepsilon)^k = \Psi\left(\sum_{k \in \mathbb{N}} \frac{f^{(k)}(r)}{k!} \varepsilon^k\right) = \Psi(\overline{f}(s)). \qquad \square$$

2.4.2 Model theory of restricted real-analytic functions

Fix $n \in \mathbb{N}$. Let $\mathbb{R}\{z_1, \ldots, z_n\}$ denote the set of power series

$$P = \sum_{v \in \mathbb{N}^n} P_v\, z_1^{v_{[1]}} \cdots z_n^{v_{[n]}} \in \mathbb{R}[[z_1, \ldots, z_n]],$$

where $n \in \mathbb{N}$, such that there is an open neighborhood U of $[-1, 1]^n$ in $\mathbb{R}$ where P converges for the euclidean topology. The corresponding function

$$f_P: U \longrightarrow \mathbb{R}; r \mapsto \sum_{v \in \mathbb{N}^n} P_v\, r_{[1]}^{v_{[1]}} \cdots r_{[n]}^{v_{[n]}}$$

is thus analytic on U. Note that $P_v = \frac{\partial^{|v|} f}{\partial x^v}(0)$ for all $v \in \mathbb{N}^n$, so $\overline{f_P}$ coincides with the function $\tilde{P}$ on $U + (\mathbb{S}^{\prec})^n$. We write $\mathbb{R}\{z_1, z_2, \ldots\} := \bigcup_{n \in \mathbb{N}} \mathbb{R}\{z_1, \ldots, z_n\}$.

Consider the first-order language $\mathcal{L}_{\mathrm{an}}$ expanding the language $\{+,-,\times,{}^{-1},<\}$ of ordered rings with a n-ary function symbol P for each $P\in\mathbb{R}\{z_1,\ldots,z_n\}$. We have two $\mathcal{L}_{\mathrm{an}}$-structures

- Define $\mathbb{R}_{\mathrm{an}}$ to be the real ordered field where each $P\in\mathbb{R}\{z_1,\ldots,z_n\}$ is interpreted as the map

$$\begin{array}{rcl}\mathbb{R}^n & \longrightarrow & \mathbb{R}\\ r & \longmapsto & \begin{cases} P(r) & \text{if } r\in[-1,1]^n\\ 0 & \text{otherwise.}\end{cases}\end{array}.$$

- Define $\mathbb{S}_{\mathrm{an}}$ to be the ordered field $\mathbb{S}$ where each $P\in\mathbb{R}\{z_1,\ldots,z_n\}$ is interpreted as the map

$$\begin{array}{rcl}\mathbb{R}^n+(\mathbb{S}^{\prec})^n & \longrightarrow & \mathbb{R}\\ s & \longmapsto & \begin{cases} f_P(s) & \text{if } s\in[-1,1]^n\\ 0 & \text{otherwise.}\end{cases}\end{array}.$$

Then the elementary theory T_{an} of $\mathbb{R}_{\mathrm{an}}$ is model complete and o-minimal [32]. It also has quantifier elimination in an extended language (see [31, Theorem 4.6] and [34, Proposition 2.9]). Moreover [34, Corollary 2.11], if $\mathbb{S}$ is real-closed (i.e. if $\mathfrak{M}$ is divisible), then $\mathbb{S}_{\mathrm{an}}$ is a model of T_{an}.

2.4.3 Exponential and logarithm

An important example of unary restricted analytic function is the exponential function

$$\begin{array}{rcl}\overline{\exp}:\mathbb{T}^{\preccurlyeq} & \longrightarrow & \mathbb{R}^{>}+\mathbb{T}^{\prec}\\ r+\varepsilon & \longmapsto & \mathrm{e}^r\sum_{k\in\mathbb{N}}\dfrac{\varepsilon^k}{k!}.\end{array}$$

We also have a logarithm

$$\begin{array}{rcl}\overline{\log}:\mathbb{R}^{>}+\mathbb{T}^{\prec} & \longrightarrow & \mathbb{T}^{\preccurlyeq}\\ r+\varepsilon & \longmapsto & \log r+\sum_{k\in\mathbb{N}}\dfrac{(-1)^k r^k}{k+1}\varepsilon^{k+1}.\end{array}$$

Proposition 2.4.2. *The function $\overline{\log}$ and $\overline{\exp}$ are functional inverses of each other. Moreover, for each $r\in\mathbb{R}$, the function $\overline{\log}\restriction(\mathrm{e}^r+\mathbb{S}^{\prec})$ is the functional inverse of $\overline{\exp}\restriction(r+\mathbb{S}^{\prec})$.*

Proof. Since the real-valued log and exp are mutually inverse analytic functions, the composite of the Taylor series of log and exp at e^r and r respectively is the identity series $z\in\mathbb{R}[[z]]$. Thus the composite of the Taylor series of $\overline{\log}$ and $\overline{\exp}$ at e^r and r respectively is z. We conclude by Proposition 2.3.8 that for all $\varepsilon\in\mathbb{T}^{\prec}$, we have

$$\overline{\log}(\overline{\exp}(r+\varepsilon))=\log(\mathrm{e}^r)+\tilde{z}(\varepsilon)=r+\varepsilon.$$

Symmetric arguments prove that $\overline{\exp}(\overline{\log}(r+\varepsilon))=r+\varepsilon$ for all $r\in\mathbb{R}^{>}$ and $\varepsilon\in\mathbb{S}^{\prec}$, hence the result. □

Consider the first-order language $\mathcal{L}_{\mathrm{an,exp}}$ extending $\mathcal{L}_{\mathrm{an}}$ with two unary function symbols exp and log. We expand $\mathbb{R}_{\mathrm{an}}$ into an $\mathcal{L}_{\mathrm{an,exp}}$ structure $\mathbb{R}_{\mathrm{an,exp}}$ by interpreting exp as the real exponential function and log as the natural logarithm, extended to $\mathbb{R}$ with

$$\log(r):=0$$

whenever $r\leqslant 0$.

Extending Wilkie's theorem [97], van den Dries and Miller [37] showed that the first-order theory $T_{\mathrm{an,exp}}$ of $\mathbb{R}_{\mathrm{an,exp}}$ is model-complete. Furthermore, van den Dries, Macintyre and Marker [34] later showed that it has quantifier elimination in $\mathcal{L}_{\mathrm{an,exp}}$, and that it is o-minimal,. As is well known [30, 39, 60, 68, 92, 72], it is possible to extend the restricted analytic exponential and logarithm to $\mathbb{S}$ and $\mathbb{S}^{>}$ respectively, provided the field $\mathbb{S}$ has additional structure. This leads us into the realm of *transseries*. In that case $\mathbb{S}_{\mathrm{an}}$ may also be expanded to an $\mathcal{L}_{\mathrm{an,exp}}$-structure. We shall elaborate on this in Section 3.1.

2.5 Real powers

In this section, we fix a field $\mathbb{S}=\mathbb{R}[[\mathfrak{M}]]$ of well-based series. We will show that under a condition on $\mathfrak{M}$, we can define a composition law $\circ: \mathbb{R}[[x^{\mathbb{R}}]] \times \mathbb{S}^{>,\succ} \longrightarrow \mathbb{S}$ where $\mathbb{R}[[x^{\mathbb{R}}]]$ is the field of real-powered series of Example 1.2.5. This can be seen as a toy example of the type of arguments we will use in later chapters of the thesis. It is, in most part, a consequence of the work in [33].

2.5.1 Real powering operation

Given $r \in \mathbb{R}$, the real power function $\mathbb{R}^{>} \longrightarrow \mathbb{R}^{>}; y \mapsto y^r$ is analytic. Therefore it induces a restricted analytic function on $\mathbb{R}^{>}+\mathbb{S}^{\prec}$. Here we study a case when this real power function extends to the whole class $\mathbb{S}^{>}$.

We say that $\mathfrak{M}$ *has real powers*, if it comes with a real power operation

$$\mathbb{R} \times \mathfrak{M} \rightarrow \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$$

for which $\mathfrak{M}$ is a multiplicative ordered $\mathbb{R}$-vector space, i.e. an ordered $\mathbb{R}$-vector space with multiplication and real powering in the roles of addition and scalar multiplication. Note in particular that $\mathfrak{M}=\mathfrak{M}^{\mathbb{Q}}$ is then divisible, so $\mathbb{S}$ is real-closed. In the sequel of the section, we assume that $\mathfrak{M}$ has real powers. For $k \in \mathbb{N}$ and $r \in \mathbb{R}$, the generalized binomial coefficient $\binom{r}{k} \in \mathbb{R}$ is defined by

$$\binom{r}{k} := \frac{r\,(r-1)\cdots(r-k)}{k!}.$$

The real powering operation on $\mathfrak{M}$ extends to $\mathbb{S}^{>}$ as follows: for $\varepsilon \in \mathbb{S}^{\prec}$, we set

$$(1+\varepsilon)^r := \sum_{k \in \mathbb{N}} \binom{r}{k} \varepsilon^k \tag{2.5.1}$$

For a multiplicatively decomposed series $s=c\,\mathfrak{m}\,(1+\varepsilon) \in \mathbb{S}^{>}$ where $c \in \mathbb{R}^{>}$, $\mathfrak{m} \in \mathfrak{M}$, and $\varepsilon \in \mathbb{S}^{\prec}$ (see (1.2.4)), we set

$$s^r := c^r\,\mathfrak{m}^r\,(1+\varepsilon)^r. \tag{2.5.2}$$

We first note elementary properties of this definition.

Proposition 2.5.1. *For $r, r' \in \mathbb{R}$ and $s, t \in \mathbb{S}^{>}$ we have*

$$\begin{aligned} (s^r)^{r'} &= s^{r r'} \quad \textit{and} \\ (s\,t)^r &= s^r\,t^r. \end{aligned}$$

Proof. For $s, t \sim 1$, the first two relations follow from basic power series manipulations; see [62, Corollary 16]. The extension to the general case when $s, t \in \mathbb{S}^{>}$ is straightforward and left to the reader. □

Proposition 2.5.2. *For $r \in \mathbb{R}^{>}$ and $s, t \in \mathbb{S}^{>}$ with $s<t$, we have $s^r<t^r$.*

Proof. Since $(s/t)^r=s^r/t^r$, it suffices to show that $(s/t)^r<1$. Write $s/t=c\,\mathfrak{m}\,(1+\varepsilon)$ where $c \in \mathbb{R}^{>}$, $\mathfrak{m} \in \mathfrak{M}$, and $\varepsilon \in \mathbb{S}^{\prec}$. Since $0<s<t$, we have $s/t<1$, so either $\mathfrak{m} \prec 1$, or $\mathfrak{m}=1$ and $c<1$, or $\mathfrak{m}=c=1$ and $\varepsilon<0$. If $\mathfrak{m} \prec 1$, then $\mathfrak{m}^r \prec 1$, so $(s/t)^r \prec 1$. If $\mathfrak{m}=1$ and $c<1$, then $c^r<1$ and $(s/t)^r=c^r\,(1+\varepsilon)^r \in c^r+\mathbb{S}^{\prec}<1$. If $\mathfrak{m}=c=1$ and $\varepsilon<0$, then $(s/t)^r-1=(1+\varepsilon)^r-1 \sim r\,\varepsilon<0$, so $(s/t)^r<1$. □

Thus the extended real power operation $\mathbb{R} \times \mathbb{S}^{>} \rightarrow \mathbb{S}^{>}; (r, s) \mapsto s^r$ gives the multiplicative group $\mathbb{S}^{>}$ a structure of ordered $\mathbb{R}$-vector space. Accordingly, we will say that $\mathbb{S}$ *has real powers*.

Proposition 2.5.3. *The field $\mathbb{S}$ is real-closed. Any ring morphism $\mathbb{S} \longrightarrow R$, where R is an ordered domain, is strictly increasing.*

Proof. Recall that $\mathbb{S}$ is real-closed, hence Euclidean. Given an ordered domain R, a morphism of rings $\Psi: \mathbb{S} \longrightarrow R$ and a series $s \in \mathbb{S}^{>}$, we have $s=(s^{1/2})^2$, whence $\Psi(s)=\Psi(s^{1/2})^2$ is strictly positive. Therefore Ψ is strictly increasing. □

Example 2.5.4. Consider the field $\mathbb{R}[[x^{\mathbb{R}}]]$ of real-powered series of Example 1.2.5. Recall that the ordered group $x^{\mathbb{R}}$ is a multiplicative copy of the additive ordered group $(\mathbb{R}, +, <)$ of real numbers. As such, the natural structure of ordered $\mathbb{R}$-vector space on $(\mathbb{R}, +, <)$ yields a real powering operation on $x^{\mathbb{R}}$, hence on $\mathbb{R}[[x^{\mathbb{R}}]]$.

Example 2.5.5. Let $e: (\mathbb{S}_{\succ}, +, <) \longrightarrow (\mathfrak{M}, \times, \prec)$ be an isomorphism of ordered groups and let l denote its functional inverse. Then we can lift the law of ordered vector space over $\mathbb{R}$ on $(\mathbb{S}_{\succ}, +, <)$ into a real powering operation on $\mathfrak{M}$ by setting

$$\mathfrak{m}^r := e(r\, l(\mathfrak{m}))$$

for all $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}$. In particular, some of our fields of transseries and hyperseries (see Chapter 3) will be equipped with such operations.

2.5.2 Real-powered calculus

Now let us show how to extend the real powering operation $\mathbb{R} \times \mathbb{S}^{>} \longrightarrow \mathbb{S}^{>}$ into a calculus of real-powered series on $\mathbb{S}$, i.e. we will define a composition law $\circ: \mathbb{R}[[x^{\mathbb{R}}]] \times \mathbb{S}^{>,\succ} \longrightarrow \mathbb{S}$.

Lemma 2.5.6. *If $I \subseteq \mathbb{R}$ is well-based, then $(s^r)_{r \in I}$ is well-based for all $s \in \mathbb{S}^{>,\succ}$.*

Proof. Let $s = c\, \mathfrak{m}\, (1 + \varepsilon)$ with $c \in \mathbb{R}^{>}$, $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and $\varepsilon \prec 1$. Note in view of Definition 2.5.1 that there is a sequence of real constants $(c_r)_{r \in I}$ such that $(s^r)_{r \in I}$ is a subfamily of the product family $(c_r\, \mathfrak{m}^r\, \varepsilon^k)_{r \in \mathrm{I} \wedge k \in \mathbb{N}}$. Since $(\mathfrak{m}^r)_{r \in I}$ and $(\varepsilon^k)_{k \in \mathbb{N}}$ are both well-based, we conclude with Lemmas 1.1.15, 1.2.7 and Proposition 1.2.8. □

Given $p = \sum_{r \in \mathbb{R}} p_r\, x^r \in \mathbb{R}[[x^{\mathbb{R}}]]$ and $s \in \mathbb{S}^{>,\succ}$, the family $(p_r\, s^r)_{r \in \mathbb{R}}$ is well-based by the above lemma, so we may define

$$p \circ s := \sum_{r \in \mathbb{R}} p_r\, s^r. \tag{2.5.3}$$

So $x^r \circ s = s^r$ for all $r \in \mathbb{R}$ and $s \in \mathbb{S}^{>}$.

Fix $s \in \mathbb{S}^{>}$. We have $(x^r\, x^{r'}) \circ s = (x^r \circ s)\, (x^{r'} \circ s)$ for all $r, r' \in \mathbb{R}$ by Proposition 2.5.1. By Proposition 1.3.2, the function $\mathbb{R}[[x^{\mathbb{R}}]] \longrightarrow \mathbb{S}; p \mapsto p \circ s$ is a strongly linear morphism of rings, which is strictly increasing by Proposition 2.5.3.

Recall that $\mathbb{R}[[x^{\mathbb{R}}]]$ itself has real powers, so similarly, for $p, q \in \mathbb{R}[[x^{\mathbb{R}}]]$ with $q > 0$, we have a well-defined series $p \circ q \in \mathbb{R}[[x^{\mathbb{R}}]]$.

Proposition 2.5.7. *Let $r \in \mathbb{R}$, $p \in \mathbb{R}[[x^{\mathbb{R}}]]$ with $p > 0$ and $s \in \mathbb{S}^{>,\succ}$. We have $p^r \circ s = (p \circ s)^r$.*

Proof. Write $p = c\, \mathfrak{m}\, (1 + \varepsilon)$ where $c \in \mathbb{R}^{>}$, $\mathfrak{m} := \mathfrak{d}_p$, and $\varepsilon \prec 1$. We have $g^r = c^r\, \mathfrak{m}^r \sum_{k \in \mathbb{N}} \binom{r}{k}\, \varepsilon^k$, so $g^r \circ s = c^r\, (\mathfrak{m}^r \circ s) \sum_{k \in \mathbb{N}} \binom{r}{k}\, \varepsilon^k \circ s$. We also have

$$(g \circ s)^r = c^r\, (\mathfrak{m} \circ s)^r\, (1 + \varepsilon \circ s)^r = c^r\, (\mathfrak{m} \circ s)^r \sum_{k \in \mathbb{N}} \binom{r}{k}\, (\varepsilon \circ s)^k.$$

Since $\varepsilon^k \circ s = (\varepsilon \circ s)^k$, we only need to show that $(\mathfrak{m} \circ s)^r = \mathfrak{m}^r \circ s$. Now $\mathfrak{m} = x^{r'}$ for some $r' \in \mathbb{R}$, so

$$(\mathfrak{m} \circ s)^r = (x^{r'} \circ s)^r = s^{r' r} = x^{r' r} \circ s = \mathfrak{m}^r \circ s$$

by Proposition 2.5.1. □

Corollary 2.5.8. *Let $p = \mathbb{R}[[x^{\mathbb{R}}]]$, $q \in \mathbb{R}[[x^{\mathbb{R}}]]^{>}$, and $s \in \mathbb{S}^{>,\succ}$. We have $p \circ (q \circ s) = (p \circ q) \circ s$.*

Proof. We have

$$p \circ (q \circ s) = \sum_{r \in \mathbb{R}} p_r\, (q \circ s)^r = \sum_{r \in \mathbb{R}} p_r\, (q^r \circ s) = \left(\sum_{r \in \mathbb{R}} p_r\, q^r \right) \circ s = (p \circ q) \circ s,$$

where the second equality follows from Proposition 2.5.7 and the third one follows from the strong linearity of the composition with s. □

Given $p\in\mathbb{R}[[x^{\mathbb{R}}]]$, we set $\partial(p):=\sum_{r\in\mathbb{R}} r\,p_r\,x^{r-1}$. Note that $\operatorname{supp}\partial=\{x^{-1}\}$ is well-based, and that ∂ is a strongly linear derivation $\mathbb{R}[[x^{\mathbb{R}}]]\longrightarrow\mathbb{R}[[x^{\mathbb{R}}]]$. We write $p^{(0)}=p$ and $p^{(k+1)}:=\partial(p^{(k)})$ for all $k\in\mathbb{N}$. So $p^{(k)}$ is the k-th derivative of p.

Proposition 2.5.9. *For $p\in\mathbb{R}[[x^{\mathbb{R}}]]$, $t\in\mathbb{S}^{>,\succ}$ and $\delta\in\mathbb{S}$ with $\delta\prec t$, the family $((p^{(k)}\circ t)\,\delta^k)_{k\in\mathbb{N}}$ is well-based, and we have*

$$p\circ(t+\delta)=\sum_{k\in\mathbb{N}}\frac{p^{(k)}\circ t}{k!}\,\delta^k.$$

Proof. Set $\varepsilon:={}^{\delta}/_{t}$. We first handle the case when $f=x^r$, for a fixed $r\in\mathbb{R}$. We have $\varepsilon\prec 1$, so

$$(1+\varepsilon)^r=\sum_{k\in\mathbb{N}}\binom{r}{k}\varepsilon^k.$$

For $k\in\mathbb{N}$, we also have $(x^r)^{(k)}=k!\binom{r}{k}x^{r-k}$, so $(x^r)^{(k)}\circ t=k!\binom{r}{k}t^{r-k}$. Therefore,

$$\begin{aligned}(t+\delta)^r &= t^r(1+\varepsilon)^r=t^r\sum_{k\in\mathbb{N}}\binom{r}{k}\varepsilon^k=t^r\sum_{k\in\mathbb{N}}\frac{(x^r)^{(k)}\circ t}{k!\,t^{r-k}}\,\varepsilon^k\,x\\ &= \sum_{k\in\mathbb{N}}\frac{(x^r)^{(k)}\circ t}{k!}\,\delta^k.\end{aligned}$$

Now consider a general series $p=\sum_{r\in\mathbb{R}}p_r\,x^r\in\mathbb{R}[[x^{\mathbb{R}}]]$. For $x^r\in\operatorname{supp}p$ and $k\in\mathbb{N}$, we have

$$\operatorname{supp}((x^r)^{(k)}\circ t)\,\delta^k\subseteq x^r\cdot(\operatorname{supp}\partial)^k\cdot(\operatorname{supp}\varepsilon)^{\infty}\subseteq\{x^{-1}\}^k\cdot(\operatorname{supp}p)\cdot(\operatorname{supp}\varepsilon)^{\infty}.$$

The set $\{x^{-1}\}$ is infinitesimal and well-based, whereas $(\operatorname{supp}p)\cdot(\operatorname{supp}\varepsilon)^{\infty}$ is well-based. Applying Lemma 1.2.13 to $\mathbf{I}:=(\operatorname{supp}p)\times\mathbb{N}$, $s_{x^r,k}=(x^r)^{(k)}\circ t$ and $f(x^r,k)=k$ for all $(x^r,k)\in\mathbf{I}$, we see that $((x^r)^{(k)}\circ t)_{x^r\in\operatorname{supp}p,k\in\mathbb{N}}$ is well-based. We deduce by Lemma 1.1.20 that

$$\begin{aligned}p\circ(t+\delta) &= \sum_{r\in\mathbb{R}}p_r\,(t+\delta)^r\\ &= \sum_{r\in\mathbb{R}}\sum_{k\in\mathbb{N}}p_r\frac{(x^r)^{(k)}\circ t}{k!}\,\delta^k\\ &= \sum_{k\in\mathbb{N}}\sum_{r\in\mathbb{R}}p_r\frac{(x^r)^{(k)}\circ t}{k!}\,\delta^k\\ &= \sum_{k\in\mathbb{N}}\frac{p^{(k)}\circ t}{k!}\,\delta^k,\end{aligned}$$

where we used the strong linearity of ∂ to obtain the last identity. □

We see with Proposition 2.3.6 that each $p\in\mathbb{R}[[x^{\mathbb{R}}]]$ induces an analytic function $\tilde{p}\colon\mathbb{S}^{>,\succ}\longrightarrow\mathbb{S}$; $s\mapsto p\circ s$ with with $\tilde{p}'=\widetilde{\partial(p)}$.

Lemma 2.5.10. *For all $p\in\mathbb{R}[[x^{\mathbb{R}}]]^{\neq}$ and $s\in\mathbb{S}^{>,\succ}$, we have $p\circ s\sim\tau_p\circ\tau_s$.*

Proof. Write $\tau_p=c\,x^r$ for $c\in\mathbb{R}^{\neq}$ and $r\in\mathbb{R}$. We have $p\sim\tau_p$. Since $\mathbb{R}[[x^{\mathbb{R}}]]\longrightarrow\mathbb{S}$; $q\mapsto q\circ s$ is an $\mathbb{R}$-linear embedding of ordered fields, we have $p\circ s\sim\tau_p\circ s=c\,s^r$. Now by (2.5.2) we have $s^r\sim\tau_s^r$, hence the result. □

Proposition 2.5.11. *Let $p\in\mathbb{R}[[x^{\mathbb{R}}]]^{>,\succ}$. For $s,t\in\mathbb{S}^{>,\succ}$ with $s<t$, we have $p\circ s<p\circ t$.*

Proof. Suppose first that $s\sim t$ and write $\delta:=t-s$, so $\delta\prec t$. We have

$$p\circ s=p\circ(t+\delta)=p\circ t+\sum_{k>0}\frac{p^{(k)}\circ t}{k!}\,\delta^k.$$

Let $k>0$. Since $\operatorname{supp} \partial^k = \{x^{-k}\}$, we have $(p^{(k+1)} \circ t)\, \delta^{k+1} \prec t^{-k}\, \delta^k\, (\partial(p) \circ t)\, \delta$ where $t^{-k}\, \delta^k = ({}^{\delta}/_{t})^k$ is infinitesimal. So $(p^{(k+1)} \circ t)\, \delta^{k+1} \prec (\partial(p) \circ t)\, \delta$. In particular we have

$$p \circ s - p \circ t \sim (\partial(p) \circ t)\, \delta.$$

We have $p > \mathbb{R}$ so $\partial(p) \sim p\, x^{-1} > 0$, whence $\partial(p) \circ t > 0$. The series $\delta = s - t$ is negative, so $p \circ s < p \circ t$.

Suppose now that $s \not\asymp t$. Then Lemma 2.5.10 yields $p \circ s \sim \tau_p \circ \tau_s$ and $\tau_p \circ \tau_t \sim p \circ t$ where $\tau_p \circ \tau_s < \tau_p \circ \tau_t$, hence the result. □

We thus have a calculus of analytic real-powered series/functions on $\mathbb{S}$, whose properties are summed up in the following theorem.

Theorem 2.5.12. *The function* $\circ\colon \mathbb{R}[[x^{\mathbb{R}}]] \times \mathbb{S}^{>,\succ} \longrightarrow \mathbb{S}$ *defined above has the following properties:*

a) *For all* $s \in \mathbb{S}^{>}$*, the function* $\mathbb{R}[[x^{\mathbb{R}}]] \longrightarrow \mathbb{S}; p \mapsto p \circ s$ *is a strongly linear embedding of ordered fields.*

b) *For all* $p, q \in \mathbb{R}[[x^{\mathbb{R}}]]$ *with* $q \in \mathbb{R}[[x^{\mathbb{R}}]]^{>,\succ}$ *and all* $s \in \mathbb{S}^{>,\succ}$*, we have* $p \circ (q \circ s) = (p \circ q) \circ s$*.*

c) *For all* $p \in \mathbb{R}[[x^{\mathbb{R}}]]^{>,\succ}$ *and* $s, t \in \mathbb{S}^{>,\succ}$ *with* $s < t$*, we have* $p \circ s < p \circ t$*.*

d) *For all* $p \in \mathbb{R}[[x^{\mathbb{R}}]]^{>,\succ}$ *and* $t, \delta \in \mathbb{S}$ *with* $t > \mathbb{R}$ *and* $\delta \prec t$*, the family* $((p^{(k)} \circ t)\, \delta^k)_{k \in \mathbb{N}}$ *is well-based with*

$$p \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{p^{(k)} \circ t}{k!}\, \delta^k.$$

Similar properties will hold in the case of the hyperserial calculus on hyperseries (see Section 7.1.1), and ultimately on surreal numbers (see Result 1.2 in the Conclusion).

Chapter 3

Transseries

We now introduce certain well-based series, expanding power series with formal exponential and logarithm symbols e^x and $\log x$, called *transseries*. As an introduction to hyperseries and as ground work to study surreal numbers, we will recall part of Schmeling's work [92, Chapters 2, 4 and 5]. We introduce a looser notion of transseries field than Schmeling's in Section 3.1. In Section 3.2, we study exponentiation on transserial fields. In Section 3.3, we define a transserial calculus $(\mathbb{T}_{\log}, \partial, \circ)$ on transserial fields.

3.1 Transserial fields

In this section, we fix a non-trivial ordered Abelian group $\mathfrak{M}$, and we set $\mathbb{T} := \mathbb{R}[[\mathfrak{M}]]$. We will see how to define a logarithm function on $\mathbb{T}^{>}$ by relying on its formal Taylor series. Consider the following power series in $\mathbb{R}[[z]]$:

$$L := \sum_{k \in \mathbb{N}^{>}} \frac{(-1)^{k-1}}{k} z^k \qquad \text{and} \qquad E := \sum_{k \in \mathbb{N}} \frac{1}{k!} z^k$$

Note that $\tilde{L}(\varepsilon) \in \mathbb{T}^{\prec}$ and $\tilde{E}(\varepsilon) \in 1 + \mathbb{T}^{\prec}$ for all $\varepsilon \prec 1$. The functions $\tilde{L}$ and $\tilde{E}$ are respectively the restricted analytic functions $\varepsilon \mapsto \overline{\log}(1+\varepsilon)$ and $\varepsilon \mapsto \overline{\exp}(\varepsilon)$ on $\mathbb{T}^{\prec}$ (see Section 2.4). As a corollary of Proposition 2.4.2, we have

$$\tilde{E}(\tilde{L}(\varepsilon)) = 1 + \varepsilon \quad \text{and} \qquad \tilde{L}(\tilde{E}(\varepsilon) - 1) = \varepsilon \tag{3.1.1}$$

for all $\varepsilon \prec 1$.

Lemma 3.1.1. [92, Example 2.1.3] *The function $\tilde{E}$ is strictly increasing on $\mathbb{T}^{\prec}$.*

Proof. Let $\varepsilon, \delta \prec 1$ with $\varepsilon < \delta$. We have

$$\tilde{E}(\varepsilon) - \tilde{E}(\delta) = \varepsilon - \delta + \frac{1}{2}(\varepsilon^2 - \delta^2) + \cdots.$$

For all $k > 1$, we have $\varepsilon^k - \delta^k = (\varepsilon - \delta)\, u$ where $u := \varepsilon^{k-1} + \varepsilon^{k-2}\delta + \cdots + \varepsilon\,\delta^{k-2} + \delta^{k-1} \prec 1$. Thus $\varepsilon^k - \delta^k \prec \varepsilon - \delta$. We deduce that $\tilde{E}(\varepsilon) - \tilde{E}(\delta) \sim \varepsilon - \delta > 0$. □

Lemma 3.1.2. [92, Lemma 2.1.4] *For all $\varepsilon, \delta \prec 1$, we have*

$$\tilde{E}(\varepsilon + \delta) = \tilde{E}(\varepsilon)\, \tilde{E}(\delta) \qquad \textit{and} \qquad \tilde{L}(\varepsilon + \delta + \varepsilon\,\delta) = \tilde{L}(\varepsilon) + \tilde{L}(\delta).$$

Proof. Note that $E^{(k)} = E$ for all $k \in \mathbb{N}$. By Proposition 2.2.8, we have

$$\begin{aligned} \tilde{E}(\varepsilon + \delta) &= \widetilde{E_{+\varepsilon}}(\delta) \\ &= \sum_{k \in \mathbb{N}} \frac{\widetilde{E^{(k)}}(\varepsilon)}{k!} \delta^k \\ &= \tilde{E}(\varepsilon) \sum_{k \in \mathbb{N}} \frac{1}{k!} \delta^k \\ &= \tilde{E}(\varepsilon)\, \tilde{E}(\delta). \end{aligned}$$

By (3.1.1), we have

$$\begin{aligned}\tilde{E}(\tilde{L}(\varepsilon+\delta+\varepsilon\,\delta)) &= 1+\varepsilon+\delta+\varepsilon\,\delta\\ &= (1+\varepsilon)\,(1+\delta)\\ &= \tilde{E}(\tilde{L}(\varepsilon))\,\tilde{E}(\tilde{L}(\delta))\\ &= \tilde{E}(\tilde{L}(\varepsilon)+\tilde{L}(\delta)).\end{aligned}$$

By Lemma 3.1.1, we deduce that $\tilde{L}(\varepsilon+\delta+\varepsilon\,\delta)=\tilde{L}(\varepsilon)+\tilde{L}(\delta)$. □

Lemma 3.1.3. *For all $\varepsilon\prec 1$, we have $\tilde{L}(\varepsilon)\leqslant\varepsilon$.*

Proof. Indeed, we have $\tilde{L}(\varepsilon)=\varepsilon-\varepsilon^2\left(\frac{1}{2}+\delta\right)$ where $\delta:=\sum_{k>0}\frac{(-1)^{k-1}}{k+2}\varepsilon^k$ is infinitesimal. We deduce that $\varepsilon^2\left(\frac{1}{2}+\delta\right)\geqslant 0$, so $\tilde{L}(\varepsilon)\leqslant\varepsilon$. □

3.1.1 Well-based series with a logarithm

Definition 3.1.4. *Let $\log\colon(\mathbb{T}^{>},\times)\longrightarrow(\mathbb{T},+)$ be a strictly increasing group morphism with*

- **TF1.** $\log\mathfrak{m}\in\mathbb{T}_{\succ}$ *for all* $\mathfrak{m}\in\mathfrak{M}$.
- **TF2.** $\log s\leqslant s-1$ *for all* $s\in\mathbb{T}^{>}$.
- **TF3.** $\log r\,\mathfrak{m}\,(1+\varepsilon)=\log\mathfrak{m}+\log r+\tilde{L}(\varepsilon)$.

Then we say that $(\mathbb{T},\log)$ *is a* **transserial field**, *and that* $\log$ *is the* **logarithm** *for* $(\mathbb{T},\log)$.

Remark 3.1.5. Since log is a group morphism, an equivalent version of **TF3** is

$$\forall u\in\mathbb{T}^{>}, u\preccurlyeq 1\Longrightarrow\log u=\overline{\log u}$$

where $\overline{\log}$ is the restricted analytic function of Section 2.4.

Remark 3.1.6. This definition is similar to Schmeling's definition of transseries fields [92, Definition 1.1.2], except for the fact that Schmeling imposes a fourth axiom **T4** which we do not impose. Our definition is also a slightly stronger version of Ehrlich and Kaplan's notion of logarithmic Hahn field [46, Definition 6.1].

We will write $\log_n$ instead of $\log^{[n]}$ for the partially defined n-fold iterate of log, so $\log_0=\mathrm{Id}_{\mathbb{T}^{>}}$ and $\log_{n+1}=\log_n\circ\log$ for all $n\in\mathbb{N}$.

Lemma 3.1.7. [92, Proposition 2.2.4(1)] *Let $(\mathbb{T},\log)$ be a transserial field. For $s\in\mathbb{T}^{>,\succ}$, we have $\log s\ll s$.*

Proof. Let $n\in\mathbb{N}$. We have $s>\mathbb{R}$ so $\log s>\mathbb{R}$. By **TF2**, we have

$$\frac{1}{2}\log s\leqslant\log s-n=\log\mathrm{e}^{-n}\,s\leqslant\mathrm{e}^{-n}\,s-1<\mathrm{e}^{-n}\,s.$$

So $\log\mathfrak{m}<\mathbb{R}^{>}\,s$, that is, $\log s\prec s$. Applying this to $\log s\in\mathbb{T}^{>,\succ}$, we obtain $\log_2 s\prec\log s$, whence $\log\,(\log s)^n=n\log_2 s<\log s$ for all $n\in\mathbb{N}$. But then $(\log s)^{\mathbb{N}}<s$, whence $\log s\ll s$. □

Proposition 3.1.8. *The function $\log\colon\mathbb{T}^{>}\longrightarrow\mathbb{T}$ is analytic with*

$$\mathrm{Conv}(\log)_s=\mathbb{T}^{\prec s}=\{\delta\in\mathbb{T}:\delta\prec s\}$$

and

$$\log^{(k)}(s)=(-1)^{k-1}\,(k-1)!\,s^{-k}$$

for all $s>0$.

Proof. Let $s\in\mathbb{T}^{>}$. For $k\in\mathbb{N}$, set $a_{k,s}:=(-1)^{k-1}\,(k-1)!\,s^{-k}$. For $\delta\prec s$ and $k>0$, we have

$$\frac{a_{k,s}\,\delta^k}{k!}=\frac{(-1)^{k-1}}{k}\left(\frac{\delta}{s}\right)^k.$$

Since ${}^{\delta}/_{s} \prec 1$, the family $(a_{k,s}\,\delta^k)_{k\in\mathbb{N}^>}$ is well-based with $\sum_{k\in\mathbb{N}^>}\frac{a_{k,s}\,\delta^k}{k!} = \tilde{L}\left(\frac{\delta}{s}\right)$. By **TF3**, we have

$$\log(s+\delta) = \log\left(s\left(1+\frac{\delta}{s}\right)\right) = \log s + \tilde{L}\left(\frac{\delta}{s}\right).$$

That is, the function log is analytic at s with $\log_s = \log(s) + L\circ(s^{-1}\,z)$ and $\operatorname{Conv}(\log)_s \supseteq \{\delta\in\mathbb{T}: \delta\prec s\}$. Note that $1\notin\operatorname{Conv}(L)$ so $s\notin\operatorname{Conv}(\log_s)$. It follows since $\operatorname{Conv}(\log)_s$ is $\prec$-initial that $\operatorname{Conv}(\log)_s = \{\delta\in\mathbb{T}:\delta\prec s\}$. By Proposition 2.3.6, for each $k\in\mathbb{N}$, the series $\frac{\log^{(k)}(s)}{k!}$ is the $(k+1)$-th coefficient $\frac{a_{k,s}}{k!}$ of $\log_s$. So $\log^{(k)}(s) = a_{k,s} = (-1)^{k-1}\,(k-1)!\,s^{-k}$. □

Proposition 3.1.9. *Let* $(\mathbb{U},\log)$ *be a transserial field and let* $\Psi:\mathbb{T}\longrightarrow\mathbb{U}$ *be a strongly linear and strictly increasing morphism of rings with* $\Psi(\log\mathfrak{m}) = \log\Psi(\mathfrak{m})$ *for all* $\mathfrak{m}\in\mathfrak{M}$*. Then we have*

$$\Psi(\log s) = \log\Psi(s)$$

for all $s\in\mathbb{T}^>$*.*

Proof. Let $s\in\mathbb{T}^>$ and write $s = r\,\mathfrak{d}_s\,(1+\varepsilon)$ where $r\in\mathbb{R}^>$ and $\varepsilon\prec 1$. Since Ψ is $\mathbb{R}$-linear and strictly increasing, we have $\Psi(r\,(1+\varepsilon))\preccurlyeq 1$. Lemma 2.4.1 yields

$$\Psi(\log(r\,(1+\varepsilon))) = \Psi(\overline{\log}(r\,(1+\varepsilon))) = \overline{\log}\,\Psi(r\,(1+\varepsilon)) = \log\Psi(r\,(1+\varepsilon)).$$

It follows that

$$\Psi(\log s) = \Psi(\log\mathfrak{d}_s) + \Psi(\log(r\,(1+\varepsilon))) = \log\Psi(\mathfrak{d}_s) + \log\Psi(r\,(1+\varepsilon)) = \log\Psi(s).$$ □

3.1.2 Extending partial logarithms

Defining a logarithm on $\mathbb{R}[[\mathfrak{M}]]$ reduces to defining its restriction to $\mathfrak{M}$, as we next show. All our results here can be found in [14].

Proposition 3.1.10. *Let* $L_1:\mathfrak{M}\longrightarrow\mathbb{T}_\succ$ *be a strictly increasing group morphism. There is a unique extension of* L_1 *into a strictly increasing group morphism*

$$\log:(\mathbb{T}^>,\times)\longrightarrow(\mathbb{T},+)$$

which extends the natural logarithm on $\mathbb{R}^>$ *and with*

$$\log(1+\varepsilon) = \tilde{L}(\varepsilon)\quad\textit{for all } \varepsilon\in\mathbb{T}^\prec.$$

For $s\in\mathbb{T}^>$*, writing* $s = r\,\mathfrak{d}_s\,(1+\varepsilon)$ *for* $r\in\mathbb{R}^>$ *and* $\varepsilon\in\mathbb{T}^\prec$*, we have*

$$\log s = L_1(\mathfrak{d}_s) + \log r + \tilde{L}(\varepsilon). \tag{3.1.2}$$

Proof. Note that each $s\in\mathbb{T}^>$ can be written as $s = r\,\mathfrak{d}_s\,(1+\varepsilon)$ in a unique way. Thus (3.1.2) yields a well-defined function $\log:\mathbb{T}^>\longrightarrow\mathbb{T}$. We also deduce that any extension of L_1 satisfying the conditions must be given by (3.1.2). So it remains only to prove that log satisfies the conditions. Let $s = r\,\mathfrak{d}_s\,(1+\varepsilon)$ and $t = d\,\mathfrak{d}_t\,(1+\delta)$ be as above. We have $s\,t = (r\,d)\,\mathfrak{d}_{st}\,(1+u)$ where $r\,d>0$ and $u := \varepsilon+\delta+\varepsilon\,\delta\prec 1$. Recall that $\tilde{L}(u) = \tilde{L}(\varepsilon)+\tilde{L}(\delta)$ by Lemma 3.1.2. Since $\mathfrak{d}_{st} = \mathfrak{d}_s\,\mathfrak{d}_t$, we have

$$\begin{aligned}\log(s\,t) &= L_1(\mathfrak{d}_s\,\mathfrak{d}_t) + \log(r\,d) + \tilde{L}(u)\\ &= L_1(\mathfrak{d}_s) + L_1(\mathfrak{d}_t) + \log(r) + \log(d) + \tilde{L}(\varepsilon) + \tilde{L}(\delta)\\ &= \log s + \log t.\end{aligned}$$

To see that log is order-preserving, we need only check that $\log s>0$ for all $s>1$. If $\mathfrak{m} = r = 1$, then $\varepsilon>0$ and we have $\log s = \tilde{L}(\varepsilon)\sim\varepsilon>0$. If $\mathfrak{m} = 1$ and $r>1$, then we have $\log s = \log r + \tilde{L}(\varepsilon)\sim\log r>0$, since $\tilde{L}(\varepsilon)\prec 1$. If $\mathfrak{m}\succ 1$, we have $\log s\sim\log\mathfrak{m}>\mathbb{R}$, so $\log s>0$. □

Proposition 3.1.11. *Under the same conditions, for all* $s\in\mathbb{T}^>$*, we have* $\log s\in\mathbb{T}_\succ$ *if and only if* s *is a monomial.*

Proof. Writing $s = r\,\mathfrak{d}_s\,(1+\varepsilon)$ and $\log s = L_1(\mathfrak{d}_s) + \log r + \tilde{L}(\varepsilon)$ as in (3.1.2), we have

$$(L_1(\mathfrak{d}_s), \log r, \tilde{L}(\varepsilon)) \in \mathbb{T}_{\succ} \times \mathbb{R} \times \mathbb{T}^{\prec}.$$

Therefore $\operatorname{supp} \log s = \operatorname{supp} L_1(\mathfrak{d}_s) \sqcup \operatorname{supp} \log r \sqcup \operatorname{supp} \tilde{L}(\varepsilon)$, and $\operatorname{supp} \log s \succ 1$ if and only if $\log r = \tilde{L}(\varepsilon) = 0$. Note that $\log r = 0 \Longleftrightarrow r = 1$. By Lemma 3.1.1, we have $\tilde{L}(\varepsilon) = 0 \Longleftrightarrow \tilde{E}(\tilde{L}(\varepsilon)) = 1 + \varepsilon = 1$. So $s \in \mathbb{T}_{\succ}$ if and only if $s = 1\,\mathfrak{d}_s\,(1+0)$, i.e. if and only if s is a monomial. □

Proposition 3.1.12. *Assume that L_1 satisfies the conditions of Proposition 3.1.10, and that moreover $L_1(\mathfrak{m}) \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}^{\succ}$. Then we have $\log s \leqslant s - 1$ for all $s \in \mathbb{T}^{>}$.*

Proof. Let $s = r\,\mathfrak{m}\,(1+\varepsilon) \in \mathbb{T}^{>}$, where $r \in \mathbb{R}^{>}$, $\mathfrak{m} := \mathfrak{d}_s$, and $\varepsilon \in \mathbb{T}^{\prec}$. If $\mathfrak{m} = 1$, then $\log \mathfrak{m} = 0$, so $\log s = \log r + \tilde{L}(\varepsilon)$. If $r = 1$, then $\log s = \tilde{L}(\varepsilon) \leqslant \varepsilon = s - 1$. If $r > 1$, then $\log r < (r-1)\,(1+\varepsilon)$, since $\log r, r - 1 \in \mathbb{R}$, $\varepsilon \prec 1$, and $\log r < r - 1$. Thus

$$\log s < (r-1)\,(1+\varepsilon) + \tilde{L}(\varepsilon) = r\,(1+\varepsilon) - (1+\varepsilon) + \tilde{L}(\varepsilon) \leqslant r\,(1+\varepsilon) - 1 = s - 1.$$

If $\mathfrak{m} \succ 1$, then $\log \mathfrak{m} = L_1(\mathfrak{m}) \prec \mathfrak{m}$ and $\log s - \log \mathfrak{m} = \log r + \tilde{L}(\varepsilon) \preccurlyeq 1$. Hence $\log s \prec s$ and $\log s \leqslant s - 1$. If $\mathfrak{m} \prec 1$, then $\log \mathfrak{m} = -L_1(\mathfrak{m}^{-1})$ is negative and infinite, so $\log s < -1 \leqslant s - 1$. □

Corollary 3.1.13. *Given L_1 as in Proposition 3.1.12, the structure $(\mathbb{T}, \log)$ is a transserial field. Conversely, given a transserial field $(\mathbb{T}, \log)$, the function $\log \restriction \mathfrak{M}$ satisfies the hypotheses of Proposition 3.1.12.*

3.1.3 Logarithmic transseries

Let us construct a transserial field. Recall that in the field $\mathbb{R}[[x^{\mathbb{Z}}]]$ of formal Laurent series, we have $s \asymp\!\!\!\!\asymp t$ whenever s and t are not finite. As a consequence of Lemma 3.1.7, there is no structure of transserial field on $\mathbb{R}[[x^{\mathbb{Z}}]]$. In fact, iterating Lemma 3.1.7 starting from a monomial $\mathfrak{m} \in \mathfrak{M}^{\succ}$, we have

$$\mathfrak{m} \gg \mathfrak{d}_{\log \mathfrak{m}} \gg \mathfrak{d}_{\log \mathfrak{d}_{\log \mathfrak{m}}} \gg \cdots,$$

which means that the ordered group $\mathfrak{M}$ must have infinite rank in the sense of [4, Section 2.4, p. 85]. The smallest linearly ordered Abelian group with infinite rank is the lexicographically ordered group

$$\mathbb{Z}^{(\mathbb{N})} = \{f \in \mathbb{Z}^{\mathbb{N}} : \exists n \in \mathbb{N}, \forall k > n, f(k) = 0\},$$

where $\alpha > 0$ if and only if $f \neq 0$ and $f(n_0) > 0$ for $n_0 = \min\{n \in \mathbb{N} : f(n) \neq 0\}$. The additive law is the pointwise sum. Thus $(\mathbb{Z}^{(\mathbb{N})}, +, 0, <)$ is naturally contained in the Hahn product group $H[\mathbb{N}^*, \mathbb{Z}]$ of Example 1.1.7.

We will consider a multiplicative copy $(\mathfrak{L}_{\log}, \cdot, 1, \prec)$ of $(\mathbb{Z}^{(\mathbb{N})}, +, 0, <)$. Any element $\mathfrak{l} = (\mathfrak{l}_n)_{n \in \mathbb{N}} \in \mathfrak{L}_{\log}$ is a formal product

$$\mathfrak{l} \eqqcolon \prod_{n \in \mathbb{N}} x_n^{\mathfrak{l}_n}$$

whose support is finite, and where each $x_n^{\mathfrak{l}_n}$ corresponds to $\mathfrak{l}_n\,\chi_n \in H[\mathbb{N}^*, \mathbb{Z}]$. Thus for $\mathfrak{l}, \mathfrak{m} \in \mathfrak{M}_{\log}$, we have $\mathfrak{l}\,\mathfrak{m} = \prod_{n \in \mathbb{N}} x_n^{\mathfrak{l}_n + \mathfrak{m}_n}$. We set $\mathbb{T}_{\log} := \mathbb{R}[[\mathfrak{L}_{\log}]]$, which is thus a field of well-based series.

Let us next define a strictly increasing group morphism $L_1 \colon \mathfrak{M}_{\log} \longrightarrow (\mathbb{T}_{\log})_{\succ}$ with $L_1(\mathfrak{m}) \prec \mathfrak{m}$, and in fact $L_1(\mathfrak{m}) \ll \mathfrak{m}$, for all $\mathfrak{m} \in \mathfrak{M}_{\log}^{\succ}$. Since the group $\mathfrak{M}_{\log}$ is generated by the set $\{x_n : n \in \mathbb{N}\}$ and L_1 is a group morphism, it is sufficient and necessary to define L_1 at each $x_n, n \in \mathbb{N}$, and then set

$$L_1(\mathfrak{l}) = \sum_{n \in \mathbb{N}} \mathfrak{l}_n\,L_1(x_n),$$

for each $\mathfrak{l} \in \mathfrak{M}_{\log}$ (recall that the sum above has finite support).

The condition $L_1(x_n) \ll x_n$ implies that the dominant monomial $\mathfrak{l}$ of $L_1(x_n)$ must satisfy $\mathfrak{l}_k = 0$ for all $k \leqslant n$. Thus $\mathfrak{l} = \prod_{k=n+1}^{+\infty} x_n^{\mathfrak{l}_n}$. It is then simplest to set $\mathfrak{l} = L_1(x_n) := x_{n+1}$. Let us check that the function

$$L_1 \colon \mathfrak{l} \longmapsto \sum_{n \in \mathbb{N}} \mathfrak{l}_n\,x_{n+1} \tag{3.1.3}$$

satisfies the desired conditions. Note that $\mathbb{T}_{\log}$ is a proper extension of the field of purely logarithmic transseries studied for instance in [52].

Proposition 3.1.14. *The function L_1 defined above satisfies the conditions of Proposition 3.1.12.*

Proof. The function L_1 is a group morphism by definition. It is clearly strictly increasing by definition of the lexicographic order on $\mathfrak{M}_{\log}$ and $\mathbb{T}_{\log}$. We have $L_1(\mathfrak{M}_{\log}) \subseteq (\mathbb{T}_{\log})_{\succ}$ since each x_n, $n \in \mathbb{N}$ is infinite. Finally, for $\mathfrak{l} \in \mathfrak{M}_{\log}$ with $\mathfrak{l} \succ 1$, setting $n_0 = \min \{n \in \mathbb{N} : \mathfrak{l}_n \neq 0\}$, we have

$$L_1(\mathfrak{l}) = \sum_{n \geqslant n_0} \mathfrak{l}_n \, x_{n+1} \asymp x_{n_0+1} \prec x_n.$$

This concludes the proof. ∎

With Corollary 3.1.13, we thus obtain a transserial field $(\mathbb{T}_{\log}, \log)$ which we call the field of *logarithmic transseries*. Note that writing $x := x_0$, we have

$$\mathfrak{l} = \prod_{n \in \mathbb{N}} (\log_n x)^{\mathfrak{l}_n}$$

for all $\mathfrak{l} \in \mathfrak{M}_{\log}$.

3.2 Exponentiation

Given a transserial field $(\mathbb{T}, \log)$, we now study the existence and properties of a functional inverse exp of $\log : \mathbb{T}^{>} \longrightarrow \mathbb{T}$. As in [60, 92, 72], we will see that $\mathbb{T}$ can be extended into a transserial field on which exp is totally defined.

3.2.1 The exponential

The logarithm $\log : \mathbb{T}^{>} \longrightarrow \mathbb{T}$ is strictly increasing, hence injective. We write exp for the partially defined functional left inverse of log, called the *exponential* on $\mathbb{T}$. That is, we have $\exp(\log s) = s$ for all $s \in \mathbb{T}^{>}$. By Proposition 3.1.11 and Corollary 3.1.13, we have $\mathbb{T}_{\succ} \cap \log \mathbb{T}^{>} = \log \mathfrak{M}$. We will sometimes write $\exp(\varphi) =: \mathrm{e}^{\varphi}$ when $\varphi \in \log \mathfrak{M}$. In other words the partial function $\varphi \mapsto \mathrm{e}^{\varphi}$ is the restriction of exp to purely large series in $\mathbb{T}_{\succ}$.

Proposition 3.2.1. [92, Proposition 2.3.8] *For $s \in \mathbb{T}^{>}$, we have $s \in \log \mathbb{T}^{>}$ if and only if $s_{\succ} \in \log \mathfrak{M}$. Thus, the function $\log : \mathbb{T}^{>} \longrightarrow \mathbb{T}$ is bijective if and only if $\mathbb{T}_{\succ} \subseteq \log \mathfrak{M}$.*

Proof. Let $s \in \mathbb{T}^{>}$ and write $s = s_{\succ} + r + s_{\prec}$ where $r \in \mathbb{R}$ and $s_{\prec} \in \mathbb{T}^{\prec}$. We have $r + s_{\prec} \in \log \mathbb{T}^{>}$, since $\exp(r + s_{\prec}) = \exp(r) \, E(s_{\prec}) \in \mathbb{T}^{>}$. Since $\log \mathbb{T}^{>}$ is an additive subgroup of $\mathbb{T}$, we have $s \in \log \mathbb{T}^{>}$ if and only if $s_{\succ} \in \log \mathbb{T}^{>}$. We deduce by the arguments above that $s \in \log \mathbb{T}^{>}$ if and only if $s_{\succ} \in \log \mathfrak{M}$. ∎

Let $\varphi \in \log \mathfrak{M}$. For all $\varepsilon \prec 1$ and $r \in \mathbb{R}$, the previous proposition gives us

$$\exp(\varphi + r + \varepsilon) = \exp(r) \, \mathrm{e}^{\varphi} \left(\sum_{k \in \mathbb{N}} \frac{1}{k!} \, \varepsilon^k \right), \tag{3.2.1}$$

In fact, this can be extended. Indeed we have $\mathbb{R} + \mathbb{T}^{\prec} = \tilde{L}(\mathbb{T}^{>} \cap \mathbb{T}^{\preccurlyeq}) \subseteq \log(\mathbb{T}^{>})$. Since exp is a morphism on its domain, we deduce that for $s \in \log \mathbb{T}^{>}$, $r \in \mathbb{R}$ and $\varepsilon \prec 1$, we have

$$\exp(s + r + \varepsilon) = \exp(r) \exp(s) \left(\sum_{k \in \mathbb{N}} \frac{1}{k!} \, \varepsilon^k \right). \tag{3.2.2}$$

As in Proposition 3.1.8, we see that exp is analytic at $s + r$ with $\mathrm{Conv}(\exp)_{s+r} = \mathbb{T}^{\prec}$ and

$$\exp_{s+r} = \sum_{k \in \mathbb{N}} \frac{\exp(r+s)}{k!} \, z^k.$$

We say that the exponential is *total* on $\mathbb{T}$ if $\log: \mathbb{T}^{>} \longrightarrow \mathbb{T}$ is surjective, hence if $\exp: \mathbb{T} \longrightarrow \mathbb{T}^{>}$ is totally defined. In that case, we expand the $\mathcal{L}_{\text{an}}$-structure $\mathbb{T}_{\text{an}}$ of Section 2.4.2 into an $\mathcal{L}_{\text{an,exp}}$-structure $\mathbb{T}_{\text{an,exp}}$ by interpreting exp as the exponential function and log as the logarithm extended to $\mathbb{T}$ by setting $\log(s) = 0$ for all $s \leqslant 0$. Since $\mathfrak{M} = \exp(\log \mathfrak{M}) = \exp(\mathbb{Q} \log \mathfrak{M}) = \mathfrak{M}^{\mathbb{Q}}$ is divisible, the field $\mathbb{T}$ must be real-closed. By [68] (see also [60, Proposition 2.2]), any transserial field with a total exponential is a proper class.

Remark 3.2.2. Here we note that the field $\mathbb{T}_{\text{LE}}$ of log-exp transseries, which has a total exponential and is set-sized, is not a transserial field. Nor is the field $\mathbb{T}$ of purely logarithmic transseries studied in particular by Allen Gehret [52]. In order to include such model theoretically interesting fields, a suggestion of Elliot Kaplan is to allow among fields of transseries those which can be written as unions of fields of well-based series within a transserial field, as long as log or exp turn out to be defined on the union. For instance, the field $\mathbb{T}_{\text{LE}}$ of logarithmic transseries is a directed union of fields of well-based series $\mathbb{T}_{m,n}$ where the exponential induces a strictly increasing morphism exp: $\mathbb{T}_{m,n} \hookrightarrow (\mathbb{T}_{m,n+1})^{>0}$ for all $m, n \in \mathbb{N}$, whence $\mathbb{T}_{\text{LE}}$ itself is closed under exp. Since we are mainly focused on constructing the ambient transserial and hyperserial fields, we will not consider such general notions.

Proposition 3.2.3. *Assume that* $\log: \mathbb{T}^{>} \longrightarrow \mathbb{T}$ *is surjective. Then* $\mathbb{T}_{\text{an,exp}}$ *is an elementary extension of* $\mathbb{R}_{\text{an,exp}}$.

Proof. By applying (3.2.1) for $\varphi = \varepsilon = 0$, we see that exp extends the real exponential function. Recall that $\log(s) \leqslant s - 1$ for all $s \in \mathbb{T}^{>}$. We claim that $\exp(s) > s^n$ for all $n \in \mathbb{N}$ and $s > n^2$. Indeed let $s \in \mathbb{T}$ and $n \in \mathbb{N}$ with $s > n^2$. First assume that $s \preccurlyeq 1$. So $s = r + s_{\prec}$ for a certain $r \in \mathbb{R}^{\geqslant 0}$ and a $s_{\prec} \in \mathbb{T}^{\prec}$. We have $r \geqslant n^2$ so $\exp(r) > r^n$ so $\exp(s) \sim \exp(r) > r^n \sim s^n$ so $\exp(s) > s^n$. Assume now that $s \succ 1$. We have $\exp\left(\frac{1}{n+1} s\right) \succ s$ so $\exp(s) > s^n$. This proves that exp satisfies Ressayre's axioms of [85]. Finally $\mathbb{T}$ is real-closed. By [34, Corollary 4.6], we deduce that $\mathbb{R}_{\text{an,exp}} \preccurlyeq \mathbb{T}_{\text{an,exp}}$. □

3.2.2 Exponential closure

Any transserial field $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ with logarithm log is contained in a minimal transserial field $\tilde{\mathbb{T}}$ on which the exponential is totally defined (see [92, Sections 2.3.2–2.3.4], where Schmeling's proof applies to transserial fields independently of the validity of the axiom **T4**). We will not study such extensions in this chapter, for we will prove their existence in the case of hyperserial fields in Section 6.1. However, we briefly state the definition for the reader's information. The following denotations are not binding since we will consider other constructions of exponential extensions later.

The field $\tilde{\mathbb{T}}$ is defined as an increasing union $\tilde{\mathbb{T}} = \bigcup_{\alpha \in \mathbf{On}} \mathbb{T}_{\alpha}$ of transserial fields $\mathbb{T}_{\alpha} = \mathbb{R}[[\mathfrak{M}_{\alpha}]]$, with partial logarithm $L_{\alpha,1}: \mathfrak{M}_{\alpha} \longrightarrow (\mathbb{T}_{\alpha})_{\succ}$. We have $\mathfrak{M}_0 := \mathfrak{M}$ and $L_{0,1} = \log \upharpoonright \mathfrak{M}$. Given $\alpha \in \mathbf{On}$ such that each $\mathfrak{M}_{\beta}, \beta < \alpha$ is defined, there are two cases.

- If $\alpha = \beta + 1$ for a certain β, then $\mathfrak{M}_{\alpha}$ is a multiplicative copy $\mathfrak{M}_{\alpha} = \mathrm{e}^{(\mathbb{T}_{\beta})_{\succ}}$ of $((\mathbb{T}_{\beta})_{\succ}, +, <)$, and $L_{\alpha,1}(\mathrm{e}^{\varphi}) := \varphi$ for all $\varphi \in (\mathbb{T}_{\beta})_{\succ}$. The inclusion $\mathfrak{M}_{\beta} \subseteq \mathfrak{M}_{\alpha}$ is given by $\mathfrak{M}_{\beta} \longrightarrow \mathfrak{M}_{\alpha}$; $\mathfrak{m} \mapsto \mathrm{e}^{L_{\beta,1}(\mathfrak{m})}$.
- If α is a non-zero limit, then $\mathfrak{M}_{\alpha} := \bigcup_{\beta < \alpha} \mathfrak{M}_{\alpha}$ and $L_{\alpha,1} = \bigcup_{\beta < \alpha} L_{\beta,1}$.

Note that setting $\tilde{\mathfrak{M}} := \bigcup_{\alpha \in \mathbf{On}} \mathfrak{M}_{\alpha}$, we have $\mathbb{T} = \mathbb{R}[[\tilde{\mathfrak{M}}]]$ by Lemma 1.1.9. It is easy to check that $\widetilde{L_1} := \bigcup_{\alpha \in \mathbf{On}} L_{\alpha,1}$ satisfies the conditions of Proposition 3.1.10, and that $\widetilde{L_1}(\tilde{\mathfrak{M}}) = \mathbb{T}_{\succ}$, so $\tilde{\mathbb{T}}$, equipped with the logarithm extending $\widetilde{L_1}$, is a transserial field with total exponential.

3.3 Transserial calculus

We now define a derivation and a composition law on logarithmic transseries. The results in this section are not new and the reader can find more details in [39, 60, 92, 36].

3.3.1 Transserial derivations

We fix a transserial field $(\mathbb{T},\log)$. A *transserial derivation* on $\mathbb{T}$ is a strongly linear function ∂: $\mathbb{T}\longrightarrow\mathbb{T}$ which satisfies the Leibniz rule

$$\forall s,t\in\mathbb{T},(\partial(s\,t)=\partial(s)\,t+s\,\partial(t))$$

and is compatible with the logarithm, i.e. satisfies

$$\forall s\in\mathbb{T}^{>},\left(\partial(\log s)=\frac{\partial(s)}{s}\right).$$

We next give a strengthening of [92, Proposition 4.1.5]:

Proposition 3.3.1. *Let $\partial:\mathbb{T}\longrightarrow\mathbb{T}$ be a strongly linear function with*

$$\partial(\log\mathfrak{m})=\frac{\partial(\mathfrak{m})}{\mathfrak{m}}\quad\text{for all }\mathfrak{m}\in\mathfrak{M}.$$

Then ∂ is a transserial derivation on $\mathbb{T}$.

Proof. We first prove that ∂ satisfies the Leibniz rule. Consider $\mathfrak{m},\mathfrak{n}\in\mathfrak{M}$. We have

$$\frac{\partial(\mathfrak{m}\,\mathfrak{n})}{\mathfrak{m}\,\mathfrak{n}}=\partial(\log\mathfrak{m}\,\mathfrak{n})=\partial(\log\mathfrak{m})+\partial(\log\mathfrak{n})=\frac{\partial(\mathfrak{m})}{\mathfrak{m}}+\frac{\partial(\mathfrak{n})}{\mathfrak{n}}.$$

We deduce that $\partial(\mathfrak{m}\,\mathfrak{n})=\partial(\mathfrak{m})\,\mathfrak{n}+\mathfrak{m}\,\partial(\mathfrak{n})$. Now let $s,t\in\mathbb{T}$. We have

$$\begin{aligned}
\partial(s\,t) &= \partial\left(\sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}}\,t_{\mathfrak{n}}\,\mathfrak{m}\,\mathfrak{n}\right)\\
&= \sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}}\,t_{\mathfrak{n}}\,\partial(\mathfrak{m}\,\mathfrak{n})\\
&= \sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}}\,t_{\mathfrak{n}}\,\partial(\mathfrak{m})\,\mathfrak{n}+\sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}}\,t_{\mathfrak{n}}\,\mathfrak{m}\,\partial(\mathfrak{n})\\
&= \left(\sum_{\mathfrak{m}} s_{\mathfrak{m}}\,\partial(\mathfrak{m})\right)\left(\sum_{\mathfrak{n}} t_{\mathfrak{n}}\,\mathfrak{n}\right)+\left(\sum_{\mathfrak{m}} s_{\mathfrak{m}}\,\mathfrak{m}\right)\left(\sum_{\mathfrak{n}} t_{\mathfrak{n}}\,\partial(\mathfrak{n})\right)\\
&= \partial(s)\,t+s\,\partial(t).
\end{aligned}$$

So the Leibniz rule holds for ∂. We deduce that we have $\partial(t^{k+1})=k\,\partial(t)\,t^k$ for all $t\in\mathbb{T}$ and $k\in\mathbb{N}$. Now let $t\in\mathbb{T}^{>}$ and write $t=r\,\mathfrak{d}_t\,(1+\varepsilon)$ where $r\in\mathbb{R}^{>}$ and $\varepsilon\prec 1$. Note that

$$\frac{\partial(t)}{t}=\frac{r\,\partial(\mathfrak{d}_t)\,(1+\varepsilon)+r\,\mathfrak{d}_t\,\partial(\varepsilon)}{r\,\mathfrak{d}_t\,(1+\varepsilon)}=\frac{\partial(\mathfrak{d}_t)}{\mathfrak{d}_t}+\frac{\partial(\varepsilon)}{(1+\varepsilon)}.$$

Recall that $\log t=\log\mathfrak{d}_t+\log r+\sum_{k\in\mathbb{N}}\frac{(-1)^k}{k+1}\,\varepsilon^{k+1}$ by **TF3**, so

$$\begin{aligned}
\partial(\log t) &= \partial(\log\mathfrak{d}_t)+\partial\left(\sum_{k\in\mathbb{N}}\frac{(-1)^k}{k+1}\,\varepsilon^{k+1}\right)\\
&= \frac{\partial(\mathfrak{d}_t)}{\mathfrak{d}_t}+\sum_{k\in\mathbb{N}}\frac{(-1)^k}{k+1}\,\partial(\varepsilon^{k+1})\\
&= \frac{\partial(\mathfrak{d}_t)}{\mathfrak{d}_t}+\partial(\varepsilon)\sum_{k\in\mathbb{N}}(-1)^k\,\varepsilon^k\\
&= \frac{\partial(\mathfrak{d}_t)}{\mathfrak{d}_t}+\frac{\partial(\varepsilon)}{(1+\varepsilon)}\\
&= \frac{\partial(t)}{t}.
\end{aligned}$$

Thus ∂ is a transserial derivation on $\mathbb{T}$. □

We next apply this to define a transserial derivation on the field $\mathbb{T}_{\log}$ of logarithmic transseries. Consider the function $\partial\colon \mathfrak{M}_{\log} \longrightarrow \mathbb{T}_{\log}$ given by

$$\forall \mathfrak{l} \in \mathfrak{M}_{\log}, \partial(\mathfrak{l}) := \mathfrak{l} \sum_{n\in\mathbb{N}} \mathfrak{l}_n \frac{1}{\prod_{k\leqslant n} \log_n x}.$$

In particular

$$\partial(\log_n x) = \frac{1}{\prod_{k<n} \log_n x}$$

for all $n \in \mathbb{N}$. Note that the set $\mathfrak{W} := \left\{ \frac{1}{\prod_{k\leqslant n} \log_n x} : n \in \mathbb{N} \right\}$ is well-based, with $\operatorname{supp} \partial = \mathfrak{W}$. We deduce with Propositions 1.3.6 and 1.3.1 that ∂ extends uniquely into a strongly linear function $\mathbb{T}_{\log} \longrightarrow \mathbb{T}_{\log}$ which we will still denote ∂.

Proposition 3.3.2. [92, Section 4.1.2] *The function $\partial\colon \mathbb{T}_{\log} \longrightarrow \mathbb{T}_{\log}$ is a transserial derivation on $\mathbb{T}_{\log}$.*

Proof. Given $\mathfrak{l} \in \mathfrak{M}_{\log}$, we have

$$\frac{\partial(\mathfrak{l})}{\mathfrak{l}} = \sum_{n\in\mathbb{N}} \mathfrak{l}_n \frac{1}{\prod_{k\leqslant n} \log_n x} = \sum_{n\in\mathbb{N}} \mathfrak{l}_n \, \partial(\log_{n+1} x) = \partial\left(\sum_{n\in\mathbb{N}} \mathfrak{l}_n \log_{n+1} x \right) = \partial(\log \mathfrak{l}).$$

By Proposition 3.3.1, we deduce that ∂ is a transserial derivation on $\mathbb{T}_{\log}$. □

3.3.2 A composition law

Let $(\mathbb{T}, \log)$ be a transserial field. We next define a function $\circ\colon \mathbb{T}_{\log} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ which will be our first example of composition law on fields of transseries. Again our results are slight generalizations (we do not impose Schmeling's axiom **T4**) of results in [92, Section 5.1.2].

Fix $s \in \mathbb{T}^{>,\succ}$. For $\mathfrak{l} \in \mathfrak{M}_{\log}$, we set

$$\mathfrak{l} \circ s := \prod_{n\in\mathbb{N}} (\log_n s)^{\mathfrak{l}_n}, \tag{3.3.1}$$

and write $\triangle_s$ for the map $\triangle_s\colon \mathfrak{M}_{\log} \longrightarrow \mathbb{T}_{\log}; \mathfrak{l} \mapsto \mathfrak{l} \circ s$.

Proposition 3.3.3. *The mapping $\triangle_s$ is well-based.*

Proof. We will use Proposition 1.3.7. Since $s > \mathbb{R}$, we have $\log_n s > 0$ for all $n \in \mathbb{N}$, whence $\triangle_s(\mathfrak{M}_{\log}) > 0$ by (3.3.1). Consider a logarithmic transmonomial $\mathfrak{l} \in \mathfrak{M}_{\log}$, and set

$$t := \log(\triangle_s(\mathfrak{l})) = \sum_{n\in\mathbb{N}} \mathfrak{l}_n \log_{n+1} s$$

And write $t = t_{\succ} + t_{\asymp} + t_{\prec}$ as in (1.2.3). By (3.2.1), we have

$$\triangle_s(\mathfrak{l}) = \exp(t) = \exp(t_{\asymp}) \, \mathrm{e}^{t_{\succ}} \sum_{k\in\mathbb{N}} \frac{1}{k!} (t_{\prec})^k,$$

with $\mathrm{e}^{t_{\succ}} = \mathfrak{d}_{\exp(t)} = \mathfrak{d}_{\triangle_s(\mathfrak{l})}$. Thus $\operatorname{supp} \triangle_s(\mathfrak{l}) \subseteq \mathfrak{d}_{\triangle_s(\mathfrak{l})} \cdot (\operatorname{supp} t_{\prec})^{\infty}$ where the set $(\operatorname{supp} t_{\prec})^{\infty}$ is well-based since $t_{\prec} \prec 1$. We deduce that $\operatorname{supp}_{\odot} \triangle_s \subseteq (\operatorname{supp} t_{\prec})^{\infty}$ is well-based. Note that $\triangle_s$ preserves products, so it suffices to prove that $\triangle_s(\mathfrak{l}) \succ 1$ whenever $\mathfrak{l} \succ 1$ in order to show that $\mathfrak{d} \circ \triangle_s$ is strictly $\prec$-increasing. Assume that $\mathfrak{l} \succ 1$. Then

$$\log \triangle_s(\mathfrak{l}) = \mathfrak{l}_{n_0} \log_{n_0}(s) + \mathfrak{l}_{n_0+1} \log_{n_0+1}(s) + \cdots$$

where $n_0 = \min \operatorname{supp} \mathfrak{l}$ and $\mathfrak{l}_{n_0} > 0$. By Lemma 3.1.7, we have $\log_{n_0}(s) \gg \log_{n_0+1}(s) \gg \cdots$, whence $\log_{n_0}(s) \succ \log_{n_0+1}(s) \succ \cdots$, so $\log \triangle_s(\mathfrak{l}) \sim \mathfrak{l}_{n_0} \log_{n_0}(s)$ is positive infinite. We deduce since exp is strictly increasing that that $\triangle_s(\mathfrak{l}) \succ 1$. By Proposition 1.3.7, the map $\triangle_s$ is well-based. □

Note that by definition, the map $\triangle_s$ preserves products. Thus by Proposition 1.3.2, the mapping $\triangle_s$ extends into a strongly linear morphism of ordered rings $\mathbb{T}_{\log} \longrightarrow \mathbb{T}$. Finally, it is easy to see that $\triangle_s$ satisfies the conditions of Proposition 3.1.9. In summary, the function $\circ : \mathbb{T}_{\log} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ satisfies the following axioms:

LT1. For all $t \in \mathbb{T}^{>,\succ}$, the function $\mathbb{T}_{\log} \longrightarrow \mathbb{T}; s \mapsto s \circ t$ is a strongly linear morphism of ordered rings.

LT2. For all $s \in \mathbb{T}_{\log}^{>}$ and $t \in \mathbb{T}^{>,\succ}$, we have $(\log s) \circ t = \log(s \circ t)$.

LT3. For all $t \in \mathbb{T}^{>,\succ}$, we have $x \circ t = t$.

Taking $\mathbb{T} = \mathbb{T}_{\log}$, this yields an internal composition law $\circ : \mathbb{T}_{\log} \times \mathbb{T}_{\log}^{>,\succ} \longrightarrow \mathbb{T}_{\log}$. It will follow from Section 7.3.1 that $\circ$ is, among other properties, associative, i.e. that we have

$$(s \circ t) \circ u = s \circ (t \circ u)$$

for all $s \in \mathbb{T}_{\log}$, $t \in \mathbb{T}_{\log}^{>,\succ}$ and $u \in \mathbb{T}_{\log}^{>,\succ}$.

In fact, it is known that the operations ∂ and $\circ$ can be further extended respectively into a transserial derivation ∂ and a composition law $\circ : \mathbb{T} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ satisfying **LT1**, **LT2** and **LT3**, where $\mathbb{T}$ is a transserial field with a total exponential. In the case when $\mathbb{T} = \widetilde{\mathbb{T}_{\log}}$, the extensions are unique. See [60, Theorem 2.3, Corollary 2.5 and Proposition 2.6] for an example, [92, Theorems 4.4.2 and 5.3.2] for general examples in the cases of transseries fields, [70, Theorem 5.2] for the extension of ∂ in general, and [18, Theorem 6.30] and [19, Theorem 6.3] in the case of surreal numbers. Hyperseries will allow us to extend this type of result to even larger transserial fields.

Part II

Hyperseries

Hyperseries arise

Part II is dedicated to the introduction of one of the two main objects of this thesis: *hyperserial fields*. Those fields are to hyperseries what transserial fields are to transseries. It is possible to work with hyperseries as if they were a peculiar kind of transseries, as Schmeling's work [92, Chapters 7–9] illustrates. Nevertheless, it becomes necessary, in order to understand the precise asymptotics of hyperseries compared to transseries and to be able to construe surreal numbers as hyperseries, to extend some of Schmeling and van der Hoeven's notions in the specific context of hyperexponential and hyperlogarithmic functions. The move from transseries to hyperseries is multi-directional, as it affects several features of the behavior of transseries. New asymptotics must be introduced that reflect that of very fast or slowly growing (e.g. sublogarithmic or transexponential) functions, new formulas must be derived for how those functions interact with derivations and compositions in the formal realm, and new ways must be found to represent hyperseries. The ideas supporting those moves are to be found in substance in the existing literature. This includes inequalities involving transexponential or sublogarithmic growth rates [57, 66, 88, 25, 39, 40] and methods to construct fields of formal series mimicking those growth rates [39, 92, 33]. It will be our goal to gather those insights and to implement them in the coherent setting of hyperserial fields.

From monomials to log-atomic monomials

In fields of well-based series, and perhaps in an easier and more spectacular way in the field of surreal numbers, "simplest" elements in each exp-log class (see Section 1.4) can be picked in a canonical way. This is the purpose of *log-atomic* monomials. Given a transserial field $(\mathbb{T}, \log)$, Van der Hoeven defined log-atomic series as infinite monomials $\mathfrak{m}$ for which each $\log_n \mathfrak{m}$ for $n \in \mathbb{N}$ is also a monomial. By definition, each exp-log class in $\mathbb{T}^{>,\succ}$ may contain at most one log-atomic series. In certain cases, each class contains exactly one log-atomic series. It was shown by Schmeling [92, Section 7.3.4] that in that case, defining a formal version L_ω of $\log_\omega$ as a function on $\mathbb{T}^{>,\succ}$ reduces to defining L_ω at log-atomic series. In that sense and in view of the definition of log from its definition at monomials (see (3.1.2)), log-atomic series play the same role for L_ω as monomials do for the logarithm.

Write E_ω for the partially defined left inverse of L_ω. Likewise, we can pick, in each convex hull of $s \pm \frac{1}{\log_n E_\omega(s)}$ for $s \in \mathbb{R}^{>,\succ}$, the simplest (in fact, the shortest, see Proposition 5.3.7) series φ, called an *ω-truncated* series. Defining E_ω on $\mathbb{T}^{>,\succ}$ reduces to defining it at each ω-truncated series. In view of the definition of exp from its value at purely large series (see (3.2.1)), ω-truncated series play the same role for L_ω as purely large series do for exp. We will see that the correspondences

$$\forall s \in \mathbb{T}^{>}, \log(\mathfrak{d}_s) = (\log s)_{\succ}, \qquad \forall t \in \mathbb{T}, \mathfrak{d}_{\exp(t)} = \exp(t_{\succ})$$

extend, once appropriately formulated, to the hyperserial case (see Corollaries 5.3.10 and 5.3.12).

Defining fields of hyperseries containing large Hardy fields with composition requires to be able to describe, in a precise manner, the structures of log-atomic and ω-truncated series (and their generalization to larger ordinals than ω). This will occupy us for most of Chapters 4 and 5.

From power series to hyperseries

The largest field of hyperseries that we will construct in Part II will be denoted $\tilde{\mathbb{L}}$ and be called the field of finitely nested hyperseries. It can be obtained by closing van den Dries, van der Hoeven and Kaplan's field $\mathbb{L}$ of logarithmic hyperseries [33] under hyperexponential functions. We write ℓ_γ for the formal γ-th iterate of the logarithm for any $\gamma \in \mathbf{On}$. So the natural inclusion $\mathbb{T}_{\log} \longrightarrow \mathbb{L}$ sends x to ℓ_0 and $\log_n x$ to ℓ_n for each $n < \omega$. Conversely $\mathrm{e}_\gamma^{\ell_0}$ denotes the formal functional inverse of ℓ_γ (we will intrduce the notation $\mathrm{e}_\gamma^{\varphi}$ for more general hyperseries φ in Chapter 6). There are conceptually simple routes that go from regular power series

$$p = \sum_{k=0}^{+\infty} p_k \, \ell_0^{-k}$$

to convoluted hyperseries such as the element

$$f := \sum_{v \in \mathbb{N}^n} v! \, \mathrm{e}^{-v_{[0]}\sqrt{\ell_0} - v_{[1]}\sqrt{\ell_1} - \cdots} \left(\sum_{n,p \in \mathbb{N}} \ell_{v!} \circ \mathrm{e}_{\omega^2}^{2^{-n}\ell_\omega - p\ell_{\omega^\omega}} \right) \tag{1}$$

of $\tilde{\mathbb{L}}$. We can indeed build up to f within $\tilde{\mathbb{L}}$ by starting with the simple power series ℓ_0, proceeding as in the following sequence of steps, where $\alpha \leqslant \omega^\omega$ is an ordinal:

Series	**Type of series**	**Class**	**Operations**	**Result**
ℓ_0	power series	$\mathbb{R}[[\ell_0]]$	$^{-1}$	ℓ_0^{-1}
ℓ_0^{-1}	formal Laurent series	$\mathbb{R}[[\ell_0^{\mathbb{Z}}]]$	$\int$	ℓ_1
ℓ_0^{-1}, ℓ_1^{-1}	formal Laurent series	$\mathbb{R}[[\ell_0^{\mathbb{Z}}]]$	$\times$	$\ell_0^{-1}\ell_1^{-1}$
$\ell_0^{-1}\ell_1^{-1}$	logarithmic transseries	$\mathbb{T}_{\log}$	$\int$	ℓ_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ell_n, n < \omega$	logarithmic transseries	$\mathbb{T}_{\log}$	inv	$\mathrm{e}_n^{\ell_0}$
$\ell_0, \ell_1, \ldots$	logarithmic transseries	$\mathbb{T}_{\log}$	$\sum$	$\sum_{n<\omega} \ell_n$
$\mathrm{e}_1^{\ell_0}, -\sum_{n<\omega} \ell_n$	log-exp, logarithmic transseries	$\mathbb{T}_{\mathrm{LE}}, \mathbb{T}_{\log}$	$\circ$	$\frac{1}{\prod_{n<\omega} \ell_n}$
$\frac{1}{\prod_{n<\omega} \ell_n}$	general transseries	$(\mathbb{T}_{\log})_{(<1)}$	$\int$	ℓ_ω
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ell_0, \ldots, \ell_\gamma, \gamma < \alpha$	logarithmic hyperseries	$\mathbb{L}$	$\sum$	$\sum_{\gamma<\alpha} \ell_\gamma$
$\mathrm{e}_1^{\ell_0}, -\sum_{\gamma<\alpha} \ell_\gamma$	finitely nested hyperseries	$\tilde{\mathbb{L}}$	$\circ$	$\frac{1}{\prod_{\gamma<\alpha} \ell_\gamma}$
$\frac{1}{\prod_{\gamma<\alpha} \ell_\gamma}$	logarithmic hyperseries	$\mathbb{L}$	$\int$	ℓ_α
ℓ_α	logarithmic hyperseries	$\mathbb{L}$	inv	$\mathrm{e}_\alpha^{\ell_0}$
$r \in \mathbb{R}, \ell_\gamma, \mathrm{e}_\gamma^{\ell_0}, \gamma < \omega^\omega$	finitely nested hyperseries	$\tilde{\mathbb{L}}$	$\sum, \times, \circ$	f

From skeletons to hyperserial calculi

In order to construct $\tilde{\mathbb{L}}$ and similar fields, we rely on analytic calculi $(\circ_{\mathbb{T}}, \partial, \circ)$ on fields of well-based series $\mathbb{T}$. That is, we work with structures

$$\begin{aligned} \circ_{\mathbb{T}} \colon \mathbb{F} \times \mathbb{T}^{>,\succ} &\longrightarrow \mathbb{T}, \\ \circ \colon \mathbb{F} \times \mathbb{F}^{>,\succ} &\longrightarrow \mathbb{F} \\ \partial \colon \mathbb{F} &\longrightarrow \mathbb{F}, \end{aligned}$$

where $(\mathbb{F}, \partial, \circ)$ should contain formal counterparts $\ell_\gamma, \mathrm{e}_\gamma, \gamma \in \mathbf{On}$ to hyperlogarithmic or hyperexponential functions, and which satisfy formal versions of the aforementioned differential and functional equations. As in the case of transserial fields, it is easier to work with hyperlogarithms first and define hyperexponential extensions afterwards, in order to take hyperexponentials into account. For instance, differentiating Abel's equation

$$\ell_\omega - 1 = \ell_\omega \circ \ell_1$$

gives a tentative equation

$$\begin{aligned} \ell_\omega' &= \frac{\ell_\omega' \circ \ell_1}{\ell_0} \\ &= \frac{\ell_\omega' \circ \ell_2}{\ell_0 \ell_1} \\ &= \ldots \\ &\overset{?}{=} \frac{1}{\ell_0 \ell_1 \ell_2 \cdots} \end{aligned}$$

which suggests that ℓ_ω should be differentially algebraic over a transserial field containing $\mathbb{T}_{\log}$ and closed under certain transfinite products (which are nothing more than exponentials transfinite sums). Furthermore, since the formal inverse $\mathrm{e}_\alpha^{\ell_0}$ of ℓ_α should have derivative $(\mathrm{e}_\alpha^{\ell_0})' = \frac{1}{\ell'_\alpha} \circ \mathrm{e}_\alpha^{\ell_0}$, it is necessary when working with fields of hyperseries $\mathbb{T}$ equipped with hyperexponentials and derivations to have at hand a composition law $\circ: \mathbb{F} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ where $\mathbb{F}$ should contain all derivatives ℓ'_α for $\alpha \in \mathbf{On}$.

This line of thought motivated the definition of the field $\mathbb{L}$ of logarithmic hyperseries by van den Dries, van der Hoeven and Kaplan [33] as a candidate for $\mathbb{F}$. Indeed $\mathbb{L}$ is the field of well-based series over $\mathbb{R}$ whose monomial group is that of formal transfinite products (i.e. multiplicatively denoted Hahn products)

$$\prod_{\gamma<\alpha} \ell_\gamma^{\mathfrak{l}_\gamma}, \quad \text{for } \alpha \in \mathbf{On} \text{ and } (\mathfrak{l}_\gamma)_{\gamma<\alpha} \in \mathbb{R}^\alpha$$

of the formal hyperlogarithmic terms ℓ_γ. We have

$$\partial(\ell_\mu) = \frac{1}{\prod_{\gamma<\mu} \ell_\gamma} \qquad \text{and} \qquad \ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} - 1$$

for all $\mu \in \mathbf{On}$.

Hyperserial fields will be fields of well-based series equipped with an external composition law $\circ_{\mathbb{T}}: \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$. Our work in Part II is mostly contained in the pre-print [14] written with Elliot Kaplan and Joris van der Hoeven, and we follow a similar route as that which we took in that paper. In Section 2.5, we saw that the definition of the real-powered calculus $\mathbb{R}[[x^{\mathbb{R}}]] \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ depended only on the existence of a well-behaved real powering operation $\mathbb{R} \times \mathfrak{M} \longrightarrow \mathfrak{M}$. In Section 3.3, we saw that the definition of the logarithmic transserial calculus $\mathbb{T}_{\log} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ depended only on the existence of a well-behaved partial logarithm function $L_1: \mathfrak{M} \longrightarrow \mathbb{T}_{\succ}$. Here in Part II, logarithmic hyperseries $\mathbb{L}$ will play the roles played by real-powered series $\mathbb{R}[[x^{\mathbb{R}}]]$ and logarithmic transseries $\mathbb{T}_{\log}$ in Part I. In Chapter 7, we will give likewise designed conditions on a reduced calculus, i.e. on a list of partial hyperlogarithms $L_{\omega^\mu}: \mathfrak{M}_{\omega^\mu} \longrightarrow \mathbb{T}$ where each $\mathfrak{M}_{\omega^\mu}$, $\mu \in \mathbf{On}$ is a (rather small) subclass of $\mathfrak{M}$. For $\mu = 1$, this is the class of log-atomic monomials mentioned earlier. The reduced structure $(\mathbb{T}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ is called a *hyperserial skeleton*. We will then move on to study hyperexponentials on $\mathbb{T}$ (Chapter 5), and show that $\mathbb{T}$ enjoys a closure under hyperexponentials (Chapter 6). Finally, in Chapter 7, we will define hyperserial fields, prove that they do arise from hyperserial skeletons, and give examples.

Chapter 4

Hyperserial skeletons

Hyperserial fields, as we will see in more detail in Chapter 7, are fields of well-based series $\mathbb{T}$ equipped with a calculus of monotone and analytic hyperlogarithmic functions, i.e. with a composition law

$$\circ : \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$$

where each logarithmic hyperseries f induces an analytic and monotone function

$$\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}; s \mapsto f \circ s.$$

The defining axioms for hyperserial fields (see Section 4.2 and Chapter 7) are numerous and involved. Furthermore, the existence of strongly linear maps $\mathbb{L} \longrightarrow \mathbb{T}; f \mapsto f \circ s$ can be difficult to prove because of the relative complexity of logarithmic hyperseries. Nonetheless, our task requires us to work with several types of hyperserial fields. In order to simplify their definition, we decided to look at "reduced" calculi. Instead of a full composition law $\circ : \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$, one works with a partial law $\mathbf{L} \times \mathbf{M} \longrightarrow \mathbb{T}$ where $\mathbf{L} \subset \mathbb{L}$ and $\mathbf{M} \subseteq \mathbb{T}^{>,\succ}$ are subclasses which represent "small" portions of their ambient field. That is, we will focus on the action of certain logarithmic hyperseries $f \in \mathbb{L}$, namely, the hyperlogarithmic terms $\ell_{\omega^\mu}, \mu \in \mathbf{On}$ on certain monomials $\mathfrak{a} \in \mathfrak{M}^{\succ}$, which we call $L_{<\omega^\mu}$-atomic. The structure $\mathbb{T}$ together with the list of partial functions $\mathfrak{a} \mapsto \ell_{\omega^\mu} \circ \mathfrak{a}$ will be called the *skeleton* of $(\mathbb{T}, \circ)$.

Conversely, given a list of partial functions $L_{\omega^\mu}, \mu \in \mathbf{On}$ on a field of well-based series $\mathbb{T}$, we are looking for conditions so that there exist a composition law $\circ : \mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ for which each L_{ω^μ} is the previously described partial function $\mathfrak{a} \mapsto \ell_{\omega^\mu} \circ \mathfrak{a}$, and for which $(\mathbb{T}, \circ)$ is a hyperserial field. This search motivates the axioms to be found in Section 4.2. Sections 4.3 and 4.4 are dedicated to the recovery of the composition law from those partial functions.

This method will be applied prominently twice in the thesis: when defining the hyperserial structure of (hyper)exponential extensions of hyperserial fields (Chapter 6), and defining the hyperserial structure on surreal numbers (Chapter 12).

We will see in Chapter 7 that the method is sound, i.e. that under a condition of so-called confluence (see Subsection 4.2.3), we can go from hyperserial fields to hyperserial skeletons, and back.

4.1 Logarithmic hyperseries

A central object in our work is the field $\mathbb{L}$ of *logarithmic hyperseries* of [33], equipped with its natural derivation $\partial : \mathbb{L} \longrightarrow \mathbb{L}$ and composition law $\circ : \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$. Here, we recall its definition and some of its properties.

4.1.1 Useful ordinal notations

If μ is a successor ordinal, then we define μ_- to be the unique ordinal with $\mu = \mu_- + 1$. If μ is a limit ordinal, then we define $\mu_- := \mu$. For $\beta = \omega^\mu$, we write $\beta_{/\omega} := \omega^{\mu_-}$.

Recall that every ordinal γ has a unique *Cantor normal form*

$$\gamma = \sum_{\eta \in \mathbf{On}} \omega^\eta \gamma_\eta$$

where the set $\operatorname{supp}\gamma := \{\omega^\eta : \eta \in \mathbf{On} \wedge \gamma_\eta \neq 0\}$ is finite. So

$$\gamma = \omega^{\eta_1}\gamma_{\eta_1} + \cdots + \omega^{\eta_r}\gamma_{\eta_r}$$

where $r \in \mathbb{N}$, $\gamma_{\eta_1}, \ldots, \gamma_{\eta_r} \in \mathbb{N}^>$ and $\eta_1, \ldots, \eta_r \in \mathbf{On}$ with $\eta_1 > \cdots > \eta_r$. The ordinals η with $\omega^\eta \in \operatorname{supp}\gamma$ are called the *exponents* of the Cantor normal form and the integers γ_η, $\eta \in \operatorname{supp}\gamma$ its *coefficients*. We write $\gamma <_o \rho$ (resp. $\gamma \leqslant_o \rho$) if $\gamma < \operatorname{supp}\rho$ (resp. $\gamma < (\operatorname{supp}\rho)\,\omega$), e.g.

$$\begin{aligned} \omega^3 + 2\,\omega + 3 <_o \omega^4 \;\; &<_o \;\; \omega^4 \quad \text{and} \\ \omega^2\, 2 + \omega \;\; &\leqslant_o \;\; \omega^2. \end{aligned}$$

We also define $\gamma_{\geqslant \omega^\eta}$ (resp. $\gamma_{>\omega^\eta}$)to be the unique ordinal with $\omega^\eta \leqslant_o \gamma_{\geqslant\omega^\eta}$ (resp. $\omega^\eta <_o \gamma_{>\omega^\eta}$) and with $\gamma = \gamma_{\geqslant\omega^\eta} + \iota$ for some $\iota \prec \omega^\eta$ (resp. $\gamma = \gamma_{>\omega^\eta} + \iota$ for some $\iota \leqslant \omega^{\eta+1}$). In other terms

$$\left(\sum_{\eta \in \mathbf{On}} \omega^\eta \gamma_\eta\right)_{\geqslant \omega^\iota} = \sum_{\eta \geqslant \iota} \omega^\eta \gamma_\eta \qquad \text{and} \qquad \left(\sum_{\eta \in \mathbf{On}} \omega^\eta \gamma_\eta\right)_{> \omega^\iota} = \sum_{\eta > \iota} \omega^\eta \gamma_\eta.$$

Note that $\gamma_{\geqslant\omega^\eta} = 0$ if and only if $\gamma < \omega^\eta$.

4.1.2 Definition and examples

The field of logarithmic hyperseries will be an extension of the field of logarithmic transseries, where instead of formal products

$$\prod_{n \in \mathbb{N}} x_n^{\mathfrak{l}_n},$$

indexed by (a finite subset of) $\mathbb{N}$ we will consider for monomials formal products

$$\prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma}.$$

indexed by arbitrary ordinals α. Indeed, for each ordinal γ, there is an element $\ell_\gamma \in \mathbb{L}$ which we call the *γ-th iterated hyperlogarithm*, or *formal hyperlogarithm of strength* γ. Intuitively speaking, we have

$$\ell_0 = x,\ \ell_1 = \log x,\ \ell_2 = \log\log x,\ \ldots,\ \ell_\omega = L_\omega(x),\ \ell_{\omega+1} = \log L_\omega(x),\ \mathit{etc.}$$

Let α be an ordinal. We write $\mathfrak{L}_{<\alpha}$ for the Hahn product group of $(\mathbb{Z}, +, <)$ to the power $(\alpha, \ni)$ (note the reversed ordering $\ni$). Thus elements of $\mathfrak{L}_{<\alpha}$ are formal products

$$\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma}$$

with $(\mathfrak{l}_\gamma)_{\gamma<\alpha} \in \mathbb{R}^\alpha$ and where $\ell_\gamma := \chi_\gamma$ as per Section 1.1.3 is the monomial $\mathfrak{l}$ with $\mathfrak{l}_\gamma = 1$ and $\mathfrak{l}_\beta = 0$ for all $\beta < \alpha$ with $\beta \neq \gamma$. We recall that $\mathfrak{L}_{<\alpha}$ is ordered by setting $\mathfrak{l} \succ 1$ if $\mathfrak{l}_\gamma > 0$ for some $\gamma < \alpha$ with $\mathfrak{l}_\beta = 0$ for all $\beta < \gamma$. We also have a real power operation on $\mathfrak{L}_{<\alpha}$ given by setting

$$\left(\prod_{\gamma<\alpha} \ell_\gamma^{\mathfrak{l}_\gamma}\right)^r := \prod_{\gamma<\alpha} \ell_\gamma^{r\,\mathfrak{l}_\gamma}$$

for $r \in \mathbb{R}$. This operation extends to all of $\mathbb{L}_{<\alpha}$ as described in Section 2.5.

We call $\mathbb{L}_{<\alpha} := \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ the field of logarithmic hyperseries of *strength* α. If β, γ are ordinals with $\gamma < \beta \leqslant \alpha$, then we let $[\gamma, \beta)$ denote the interval $\{\rho \in \mathbf{On} : \gamma \leqslant \rho < \beta\}$ and we let $\mathfrak{L}_{[\gamma,\beta)}$ denote the subgroup

$$\{\mathfrak{l} \in \mathfrak{L}_{<\alpha} : \mathfrak{l}_\rho = 0 \ \text{ whenever } \ \rho \notin [\gamma, \beta)\}.$$

As in [33], we write $\mathbb{L}_{[\gamma,\beta)} := \mathbb{R}[[\mathfrak{L}_{[\gamma,\beta)}]]$, $\mathfrak{L} := \bigcup_{\alpha\in\mathbf{On}} \mathfrak{L}_{<\alpha}$ and

$$\mathbb{L} := \mathbb{R}[[\mathfrak{L}]] = \bigcup_{\alpha \in \mathbf{On}} \mathbb{L}_{<\alpha}.$$

We will sometimes write $\mathfrak{L}_{<\mathbf{On}} = \mathfrak{L}$ and $\mathbb{L}_{<\mathbf{On}} = \mathbb{L}$. We have natural inclusions $\mathfrak{L}_{[\gamma,\beta)} \subseteq \mathfrak{L}_{<\alpha} \subseteq \mathfrak{L}$, which give natural inclusions $\mathbb{L}_{[\gamma,\beta)} \subseteq \mathbb{L}_{<\alpha} \subset \mathbb{L}$.

4.1.3 Derivation on $\mathbb{L}$

The field $\mathbb{L}$ is equipped with a strongly linear derivation $\partial: \mathbb{L} \longrightarrow \mathbb{L}$. Given $\alpha \in \mathbf{On}$ and a logarithmic hypermonomial $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$, we define the derivation of $\mathfrak{l}$ by

$$\partial \mathfrak{l} := \left(\sum_{\gamma<\alpha} \mathfrak{l}_\gamma (\ell_\gamma)^\dagger \right) \mathfrak{l}, \tag{4.1.1}$$

where $(\ell_\gamma)^\dagger = \prod_{\iota \leqslant \gamma} \ell_\iota^{-1} \in \mathfrak{L}_{<\alpha}$. Note that $\partial \ell_\gamma = (\ell_\gamma)^\dagger \ell_\gamma = \prod_{\iota<\gamma} \ell_\iota^{-1}$. For $f \in \mathbb{L}$ and $k \in \mathbb{N}$, we sometimes write $f^{(k)} := \partial^k f$. Equipped with its derivation, the field $\mathbb{L}$ is an H-field with small derivation, so for $f, g \in \mathbb{L}$, we have

$$f > \mathbb{R} \Longrightarrow f' > 0, \qquad f \prec 1 \Longrightarrow f' \prec 1, \qquad f \prec g \not\asymp 1 \Longrightarrow f' \prec g'.$$

Moreover, $\operatorname{supp} \partial \preccurlyeq \ell_0^{-1}$ is well-based, which implies the following variant of [33, Lemma 2.13]:

Lemma 4.1.1. *Let $\alpha = \omega^\nu$, let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series, and let $\Phi: \mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}$ be a strongly linear field embedding. For $f \in \mathbb{L}_{<\alpha}$ and $s \in \mathbb{T}$ with $s \prec \Phi(\ell_0)$, the family $(\Phi(f^{(n)})\, s^n)_{n\in\mathbb{N}}$ is well-based. Moreover, the map $\Psi: \mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}; f \mapsto \sum_{n\in\mathbb{N}} \frac{\Phi(f^{(n)})}{n!} s^n$ is a strongly linear ordered field embedding.*

Proof. Since $\operatorname{supp} \partial \preccurlyeq \ell_0^{-1}$ is well-based and Φ is a strongly linear field embedding, the set

$$\mathfrak{S} := \bigcup_{\mathfrak{l} \in \operatorname{supp}_* \partial} \operatorname{supp} \Phi(\mathfrak{l}) \preccurlyeq \Phi(\ell_0)^{-1}$$

is well-based. Thus $\mathfrak{S} \cdot (\operatorname{supp} s)$ is well-based. Since $s \prec \Phi(\ell_0)$, we have $\mathfrak{S} \cdot (\operatorname{supp} s) \prec 1$. Let $f \in \mathbb{L}$. For each $n \in \mathbb{N}^>$, we have

$$\operatorname{supp} (\Phi(f^{(n)})\, s^n) \;\subseteq\; (\operatorname{supp} \Phi(f)) \cdot (\mathfrak{S} \cdot (\operatorname{supp} s))^n.$$

Since $\operatorname{supp} \Phi(f)$ is well-based and $\mathfrak{S} \cdot (\operatorname{supp} s) \prec 1$, it follows that $(\Phi(f^{(n)})\, s^n)_{n\in\mathbb{N}}$ is well-based and that the map Ψ is well-defined and strongly linear. For all $f, g \in \mathbb{L}_{<\alpha}$, we also have

$$\sum_{n\in\mathbb{N}} \frac{\Phi((f g)^{(n)})}{n!} s^n = \sum_{n\in\mathbb{N}} \sum_{i+j=n} \frac{\Phi(f^{(i)})\, \Phi(g^{(j)})}{i!} s^n = \left(\sum_{i\in\mathbb{N}} \frac{\Phi(f^{(i)})}{i!} s^i \right) \left(\sum_{j\in\mathbb{N}} \frac{\Phi(f^{(j)})}{j!} s^j \right),$$

which shows that Ψ preserves multiplication. Finally Ψ is strictly increasing by Proposition 2.5.3. □

4.1.4 Composition on $\mathbb{L}$

In addition to its derivation, the field $\mathbb{L}$ comes equipped with a composition law $\circ: \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$ which is unique to satisfy the following list of properties.

For $g \in \mathbb{L}^{>,\succ}$, the map $\circ_g: \mathbb{L} \rightarrow \mathbb{L}; f \mapsto f \circ g$ is a strongly linear embedding of ordered fields. As a consequence this map preserves the relations $<$ and $\prec$ [33, Lemma 6.6].

For $f \in \mathbb{L}$ and $g, h \in \mathbb{L}^{>,\succ}$, we have $f \circ (g \circ h) = (f \circ g) \circ h$ [33, Proposition 7.14].

For $g \in \mathbb{L}^{>,\succ}$ and $r \in \mathbb{R}$, we have $\ell_0^r \circ g = g^r$ [33, Corollary 7.5].

For $g, h \in \mathbb{L}^{>,\succ}$ and $r \in \mathbb{R}^>$, we have $\ell_1 \circ (g h) = \ell_1 \circ g + \ell_1 \circ h$ and $\ell_1 \circ (r h) = \log r + \ell_1 \circ h$ [33, Section 1.4].

For ordinals $\sigma \leqslant_o \rho$, we have $\ell_\sigma \circ \ell_\rho = \ell_{\rho+\sigma}$ [33, Corollary 5.11].

For each successor ordinal μ, we have $\ell_{\omega^\mu} \circ \ell_{\omega^{\mu_-}} = \ell_{\omega^\mu} - 1$ [33, Lemma 5.8].

The constant term of $\ell_{\omega^\mu} \circ \ell_{\omega^\gamma}$ vanishes if $\mu > \gamma$ is a limit ordinal [33, Lemma 5.8].

For $f, h \in \mathbb{L}$ and $g \in \mathbb{L}^{>,\succ}$ with $h \prec g$, the family $((f^{(k)} \circ g)\, h^k)_{k\in\mathbb{N}}$ is well-based, with

$$f \circ (g + h) \;=\; \sum_{k\in\mathbb{N}} \frac{f^{(k)} \circ g}{k!} h^k. \qquad \text{[33, Proposition 8.1]}$$

By Corollary 2.3.7, the last property implies that each logarithmic hyperseries acts as an analytic function on $\mathbb{L}^{>,\succ}$, and that the derivation $f \mapsto f'$ corresponds to the differentiation of Section 2.1.2.

The uniqueness follows from [33, Theorem 1.3]. By Proposition 2.1.3, the derivation also satisfies the chain rule: for all $f\in\mathbb{L}$ and $g\in\mathbb{L}^{>,\succ}$, we have

$$(f\circ g)' \;=\; (f'\circ g)\,g'. \qquad \text{[33, Proposition 7.8]}$$

For $\alpha=\omega^{\nu}$, the unique composition $\circ$ from above restricts to a composition $\mathbb{L}_{<\alpha}\times\mathbb{L}_{<\alpha}^{>,\succ}\longrightarrow\mathbb{L}_{<\alpha}$. For $\gamma<\alpha$, the map $\circ_{\ell_\gamma}:\mathbb{L}_{<\alpha}\longrightarrow\mathbb{L}_{<\alpha}$ defined by $\circ_{\ell_\gamma}(f):=f\circ\ell_\gamma$ is a strongly linear field embedding with image $\mathbb{L}_{[\gamma,\alpha)}$ by [33, Lemma 5.13]. Accordingly, for $g\in\mathbb{L}_{[\gamma,\alpha)}$, we let $g^{\uparrow\gamma}$ denote the unique series in $\mathbb{L}_{<\alpha}$ with $g^{\uparrow\gamma}\circ\ell_\gamma=g$. Note that $\ell_{\omega^{\mu+1}}^{\uparrow\omega^\mu}=\ell_{\omega^{\mu+1}}+1$ for all μ and that, more generally, $\ell_{\omega^{\mu+1}}^{\uparrow\omega^\mu n+\gamma}=\ell_{\omega^{\mu+1}}^{\uparrow\gamma}+n$ for $\gamma<\omega^{\mu+1}$ and $n\in\mathbb{N}$. For $\mu<\nu$ and $f\in\mathbb{L}_{[\omega^{\mu+1},\alpha)}$ we have

$$f\circ\ell_{\omega^\mu} \;=\; \sum_{k\in\mathbb{N}}\frac{(-1)^k}{k!}\boldsymbol{\delta}^k(f) \;=\; \mathrm{e}^{-\delta}f \tag{4.1.2}$$

where $\boldsymbol{\delta}$ is the derivation $\frac{1}{\ell'_{\omega^{\mu+1}}}\partial$ on $\mathbb{L}_{<\alpha}$ (see [33, Section 5.1]). Let $R(f):=\sum_{k\in\mathbb{N}^>}\frac{(-1)^k}{k!}\boldsymbol{\delta}^k(f)$. Then $R:\mathbb{L}_{[\omega^{\mu+1},\alpha)}\longrightarrow\mathbb{L}_{[\omega^{\mu+1},\alpha)}$ is strongly linear and $R(f)\prec f$, so, by [33, Lemma 2.2],

$$f^{\uparrow\omega^\mu} \;=\; f-R(f)+R^2(f)-\cdots \;=\; \mathrm{e}^{\delta}f. \tag{4.1.3}$$

In particular,

$$f^{\uparrow\omega^\mu}-f \;\sim\; -R(f) \;\sim\; \frac{1}{\ell'_{\omega^{\mu+1}}}f'. \tag{4.1.4}$$

Lemma 4.1.2. *For each $\mu<\nu$, each $\gamma<\beta\leqslant\omega^\mu$, and each $k\in\mathbb{N}^>$, we have $(\ell_\beta^{\uparrow\gamma})^{(k)}\in\mathbb{L}_{<\omega^\mu}^{\prec}$.*

Proof. Since $\mathbb{L}_{<\omega^\mu}$ is closed under taking derivatives and the derivation preserves infinitesimals, it suffices to prove the lemma for $k=1$. We have $\ell_\beta^{\uparrow\gamma}\circ\ell_\gamma=\ell_\beta$, so

$$(\ell_\beta^{\uparrow\gamma}\circ\ell_\gamma)'=((\ell_\beta^{\uparrow\gamma})'\circ\ell_\gamma)\,\ell'_\gamma=\ell'_\beta.$$

Since $\ell'_\gamma,\ell'_\beta\in\mathbb{L}_{<\omega^\mu}$ and $\ell'_\beta\prec\ell'_\gamma$, this yields $(\ell_\beta^{\uparrow\gamma})'\circ\ell_\gamma\in\mathbb{L}_{<\omega^\mu}^{\prec}$. Since $(\ell_\beta^{\uparrow\gamma})'\circ\ell_\gamma\in\mathbb{L}_{[\gamma,\alpha)}$ as well, we have $(\ell_\beta^{\uparrow\gamma})'\circ\ell_\gamma\in\mathbb{L}_{[\gamma,\omega^\mu)}^{\prec}$. Since the map $f\longmapsto f^{\uparrow\gamma}$ maps $\mathbb{L}_{[\gamma,\omega^\mu)}^{\prec}$ onto $\mathbb{L}_{<\omega^\mu}^{\prec}$, we conclude that $(\ell_\beta^{\uparrow\gamma})'=((\ell_\beta^{\uparrow\gamma})'\circ\ell_\gamma)^{\uparrow\gamma}\in\mathbb{L}_{<\omega^\mu}^{\prec}$. □

4.2 Hyperserial skeletons

The definition of hyperserial skeletons involves several sets of axioms that we will now introduce. All the results in this section can be found in [14, Section 3]. Here is a summary:

Section 4.2.1. The Domain of definition axioms state rules on the domain of partial functions in hyperserial skeletons;

Section 4.2.2. A range of axioms constrain the way those functions interact with themselves (Functional equations), the ordering (Asymptotics and Monotonicity), the structure of well-based series (Regularity) and the partial exponential on $\mathbb{L}$ (Products);

Section 4.2.3. An axiom further is designed to impose that the hyperserial skeleton is sufficiently rich to extend into a composition law;

Section 4.2.4. Shows that $\mathbb{L}$ is naturally equipped with a structure of hyperserial skeleton respecting our axioms.

4.2.1 Domain of definition

We let $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ be an ordered field of well-based series with real powers. Let $\boldsymbol{\nu}\leqslant\mathbf{On}$ be an ordinal with $\boldsymbol{\nu}>0$. Given a structure $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ where L_{ω^μ} are partial functions on $\mathbb{T}$, we consider the following axioms for $\mu<\boldsymbol{\nu}$:

Domain of definition:

DD$_\mathbf{0}$. $\operatorname{dom}L_1=\mathfrak{M}^{\succ}$.

DD$_{\boldsymbol{\mu}}$. $\operatorname{dom}L_{\omega^\mu}=\begin{cases}\bigcap_{\eta<\mu}\operatorname{dom}L_{\omega^\eta} & \text{if } \mu \text{ is a non-zero limit}\\ \bigcap_{n\in\mathbb{N}}\operatorname{dom}L_{\omega^{\mu_-}}^{\circ n} & \text{if } \mu=\mu_-+1.\end{cases}$

Suppose $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ satisfies all axioms $\mathbf{DD}_{\boldsymbol{\mu}}$ for $\mu<\boldsymbol{\nu}$. We set $\mathfrak{M}_{\omega^\mu} := \operatorname{dom} L_{\omega^\mu} \subseteq \mathfrak{M}^{\succ}$ for all $\mu<\boldsymbol{\nu}$ and we extend this notation to the case when $\mu=\boldsymbol{\nu}$, by setting

$$\mathfrak{M}_{\omega^\nu} = \begin{cases} \bigcap_{\eta<\boldsymbol{\nu}} \operatorname{dom} L_{\omega^\eta} & \text{if } \boldsymbol{\nu} \text{ is a non-zero limit} \\ \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\nu_-}}^{\circ n} & \text{if } \boldsymbol{\nu}=\boldsymbol{\nu}_-+1. \end{cases} \tag{4.2.1}$$

For $\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$, we call $\mathfrak{M}_{\omega^{\boldsymbol{\mu}}}$ the class of $L_{<\omega^\mu}$*-atomic* elements. Note that $\mathfrak{M}_{\omega^{\boldsymbol{\mu}}} \subseteq \mathfrak{M}_{\omega^{\boldsymbol{\eta}}}$ for all $\boldsymbol{\eta}\leqslant\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$. We let L_0 be the identity function with $\operatorname{dom} L_0 := \mathfrak{M}^{\succ}$ and, for $\beta<\omega^{\boldsymbol{\nu}}$ with Cantor normal form $\beta=\omega^{\gamma_1} n_1+\cdots+\omega^{\gamma_k} n_k$, we define

$$L_\beta := L_{\omega^{\gamma_k}}^{\circ n_k} \circ \cdots \circ L_{\omega^{\gamma_1}}^{\circ n_1},$$

where $f^{\circ n}$ for $n\in\mathbb{N}$ denotes the n-fold iterate of a given partial function f.

Here we understand that $x\in\operatorname{dom} L_\beta$ whenever $x\in\operatorname{dom} L_{\omega^{\gamma_1}}^{\circ n_1}$, $L_{\omega^{\gamma_1}}^{\circ n_1} x\in\operatorname{dom} L_{\omega^{\gamma_2}}^{\circ n_2}$, and so on until $L_{\omega^{\gamma_{k-1}}}^{\circ n_{k-1}} \circ\cdots\circ L_{\omega^{\gamma_1}}^{\circ n_1} x\in\operatorname{dom} L_{\omega^{\gamma_k}}^{\circ n_k}$. Note that $\mathfrak{M}_\omega$ is the class of infinite monomials $\mathfrak{m}\in\mathfrak{M}^{\succ}$ such hat $L_1^{\circ n}(\mathfrak{m})$ is an infinite monomial for all $n\in\mathbb{N}$. This generalizes as follows.

Proposition 4.2.1. *For $\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$ with $\boldsymbol{\mu}>0$, we have*

$$\mathfrak{M}_{\omega^{\boldsymbol{\mu}}} \;=\; \{s\in\mathbb{T}^{>,\succ} : s\in\operatorname{dom} L_\beta \text{ and } L_\beta(s)\in\mathfrak{M}^{\succ},\ \text{for all } \beta<\omega^{\boldsymbol{\mu}}\}.$$

Proof. Given $\mathfrak{a}\in\mathfrak{M}_{\omega^{\boldsymbol{\mu}}}$ and $\beta<\omega^{\boldsymbol{\mu}}$, let us first show by induction on $\boldsymbol{\mu}$ that $L_\beta(\mathfrak{a})$ is defined and in $\mathfrak{M}^{\succ}$. This holds for $\boldsymbol{\mu}=0$ by definition. Let $0<\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$ and assume that the assertion holds strictly below $\boldsymbol{\mu}$. If $\beta=0$, then $L_0(\mathfrak{a})=\mathfrak{a}\in\mathfrak{M}^{\succ}$. Assume $\beta>0$ and let $\eta<\boldsymbol{\mu}$, $n\in\mathbb{N}^{>}$ and $\iota<\omega^\eta$ be such that $\beta=\omega^\eta n+\iota$. We have $\mathfrak{a}\in\mathfrak{M}_{\omega^{\eta+1}}$ so $L_{\omega^\eta n}(\mathfrak{a})\in\mathfrak{M}_{\omega^{\eta+1}}$ by definition. In particular $L_{\omega^\eta n}(\mathfrak{a})\in\mathfrak{M}_{\omega^\eta}$, so our inductive hypothesis on $\boldsymbol{\mu}$ applied to η gives that $L_\iota(L_{\omega^\eta n}(\mathfrak{a}))=L_\beta(\mathfrak{a})$ is a monomial.

Given $\mathfrak{a}\in\mathbb{T}^{>,\succ}$ such that $\mathfrak{a}\in\operatorname{dom} L_\beta$ and $L_\beta(\mathfrak{a})\in\mathfrak{M}^{\succ}$ for all $\beta<\omega^{\boldsymbol{\mu}}$, let us next show by induction that $\mathfrak{a}\in\mathfrak{M}_{\omega^{\boldsymbol{\mu}}}$. This is clear if $\boldsymbol{\mu}=0$. Let $1<\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$ be such that the statement holds strictly below $\boldsymbol{\mu}$. If $\boldsymbol{\mu}$ is a successor, then for $\iota<\omega^{\boldsymbol{\mu}_-}$ and $n\in\mathbb{N}$, we have $L_{\omega^{\boldsymbol{\mu}_-}n+\iota}(\mathfrak{a})=L_\iota(L_{\omega^{\boldsymbol{\mu}_-}}(\mathfrak{a}))\in\mathfrak{M}^{\succ}$ so for all $n\in\mathbb{N}$, $L_{\omega^{\boldsymbol{\mu}_-}n}(\mathfrak{a})\in\mathfrak{M}_{\omega^{\boldsymbol{\mu}_-}}$, whence $\mathfrak{a}\in\mathfrak{M}_{\omega^{\boldsymbol{\mu}}}$. Assume now that μ is a limit and let $\eta<\boldsymbol{\mu}$. Then $L_\beta(\mathfrak{a})\in\mathfrak{M}^{\succ}$ for all $\beta<\omega^\eta$, so the induction hypothesis yields $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$. We again conclude that $\mathfrak{a}\in\mathfrak{M}_{\omega^{\boldsymbol{\mu}}}$. ☐

4.2.2 Axioms for the hyperlogarithms

Let $\mathbb{T}$ be an ordered field of well-based series with real powers, let $\boldsymbol{\nu}\leqslant\mathbf{On}$, and let $(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}}$ be partial functions $(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}}$ on $\mathbb{T}$ which satisfy the axioms $\mathbf{DD}_{\boldsymbol{\mu}}$ for all $\mu<\boldsymbol{\nu}$. We consider the following axioms, where μ is an ordinal with $0<\mu<\boldsymbol{\nu}$.

Functional equations:

$\mathbf{FE_0}$. $L_1(\mathfrak{m}^r)=r\,L_1(\mathfrak{m})$ and $L_1(\mathfrak{m}\,\mathfrak{n})=L_1(\mathfrak{m})+L_1(\mathfrak{n})$ for all $r\in\mathbb{R}^{>}$ and all $\mathfrak{m},\mathfrak{n}\in\mathfrak{M}_1$.

$\mathbf{FE}_{\boldsymbol{\mu}}$. For $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$, we have $L_{\omega^\mu}(L_{\omega^{\mu_-}}(\mathfrak{a}))=L_{\omega^\mu}(\mathfrak{a})-1$ if μ is a successor ($\mathbf{FE}_\mu$ holds trivially if μ is a limit).

Asymptotics:

$\mathbf{A_0}$. $L_1(\mathfrak{m})\prec\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{M}_1$.

$\mathbf{A}_{\boldsymbol{\mu}}$. $L_{\omega^\mu}(\mathfrak{a})\prec L_{\omega^\eta}(\mathfrak{a})$ for all $\eta<\mu$ and all $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$.

Monotonicity:

$\mathbf{M_0}$. $L_1(\mathfrak{m})>0$ for all $\mathfrak{m}\in\mathfrak{M}_1$.

$\mathbf{M}_{\boldsymbol{\mu}}$. $L_{\omega^\mu}(\mathfrak{a})+L_{\omega^\eta n}(\mathfrak{a})^{-1}<L_{\omega^\mu}(\mathfrak{b})-L_{\omega^\eta n}(\mathfrak{b})^{-1}$ for all $\eta<\mu$, $n\in\mathbb{N}$ and $\mathfrak{a}\prec\mathfrak{b}$ in $\mathfrak{M}_{\omega^\mu}$.

Regularity:

$\mathbf{R_0}$. $\operatorname{supp} L_1(\mathfrak{m})\succ 1$ for all $\mathfrak{m}\in\mathfrak{M}_1$.

$\mathbf{R}_{\boldsymbol{\mu}}$. $\operatorname{supp} L_{\omega^\mu}(\mathfrak{a})\succ L_{\omega^\eta n}(\mathfrak{a})^{-1}$ for all $\eta<\mu$, $n\in\mathbb{N}$, and $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$.

The axiom $\mathbf{M}_\mu$ implies in particular that L_{ω^μ} is strictly increasing, hence injective. We define a logarithm $\log: \mathfrak{M} \longrightarrow \mathbb{T}$ as follows:

$$\log \mathfrak{m} = \begin{cases} L_1(\mathfrak{m}) & \text{if } \mathfrak{m} \in \mathfrak{M}^{\succ} \\ -L_1(\mathfrak{m}^{-1}) & \text{if } \mathfrak{m} \in \mathfrak{M}^{\prec} \\ 0 & \text{if } \mathfrak{m} = 1. \end{cases} \tag{4.2.2}$$

Then $\log \mathfrak{M}$ is an ordered $\mathbb{R}$-vector subspace of $\mathbb{T}$. By $\mathbf{FE_0}$, $\mathbf{A_0}$, $\mathbf{M_0}$ and $\mathbf{R_0}$, the structure $(\mathbb{T}, \log)$ is a transserial field. For $\mu \in \mathbf{On}$ with $0 \leqslant \mu \leqslant \boldsymbol{\nu}$, we consider the following axiom:

Infinite products:

$\mathbf{P}_{\boldsymbol{\mu}}$. $\sum_{\gamma < \omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{a}) \in \log \mathfrak{M}$ for all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and all sequences $(r_\gamma)_{\gamma < \omega^\mu}$ of real numbers.

Remark 4.2.2. The axiom $\mathbf{P}_{\boldsymbol{\mu}}$ allows us to define the infinite product $\prod_{\gamma < \omega^\mu} L_{\gamma+1}(\mathfrak{a})^{r_\gamma}$ for $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ to be the unique monomial $\mathfrak{m} \in \mathfrak{M}$ with $\log \mathfrak{m} = \sum_{\gamma < \omega^\mu} r_\gamma L_{\gamma+1}(\mathfrak{a})$, hence the name. Note that the axiom $\mathbf{P}_0$ is a consequence of $\mathbf{FE_0}$: if $\mathbf{FE_0}$ holds, then for $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}^{\succ}$, we have $r L_1(\mathfrak{m}) = \log \mathfrak{m}^r$.

Definition 4.2.3. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$. A* **hyperserial skeleton of force $\boldsymbol{\nu}$** *is a structure $(\mathbb{T}, (L_{\omega^\mu})_{\mu < \boldsymbol{\nu}})$ where $\mathbb{T}$ is an ordered field of well-based series with real powers and $(L_{\omega^\mu})_{\mu < \boldsymbol{\nu}}$ are partial functions on $\mathbb{T}$ which satisfy $\mathbf{DD}_{\boldsymbol{\mu}}$, $\mathbf{FE}_{\boldsymbol{\mu}}$, $\mathbf{A}_{\boldsymbol{\mu}}$, $\mathbf{M}_{\boldsymbol{\mu}}$, and $\mathbf{R}_{\boldsymbol{\mu}}$ for all $\mu < \boldsymbol{\nu}$, as well as $\mathbf{P}_{\boldsymbol{\mu}}$ for all $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$.*

Note that a hyperserial skeleton of force 0 is just a field of well-based series with real powers and that $(\mathbb{T}, (L_{\omega^\mu})_{\mu < \mathbf{On}})$ is a hyperserial skeleton of force $\mathbf{On}$ if and only if $(\mathbb{T}, (L_{\omega^\mu})_{\mu < \nu})$ is a hyperserial skeleton of force ν for each ordinal ν. We will often write $\mathbb{T}$ to denote a hyperserial skeleton (of force $\boldsymbol{\nu} \leqslant \mathbf{On}$), where it is implied that for $\mu < \boldsymbol{\nu}$, the term L_{ω^μ} refers to the ω^μ-th hyperlogarithm on $\mathbb{T}$.

Definition 4.2.4. *Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ and $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ be hyperserial skeletons of force $\boldsymbol{\nu} \leqslant \mathbf{On}$. We say that a function $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ is an* **embedding** *of force $\boldsymbol{\nu}$ if it is a strongly linear strictly increasing ring morphism with $\Phi(\mathfrak{M}_{\omega^\mu}) \subseteq \mathfrak{N}_{\omega^\mu}$ for each $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ such that*

$$\Phi(\mathfrak{m}^r) = \Phi(\mathfrak{m})^r \text{ for all } \mathfrak{m} \in \mathfrak{M} \text{ and } r \in \mathbb{R},$$

and such that

$$\Phi(L_{\omega^\mu}(\mathfrak{a})) = L_{\omega^\mu}(\Phi(\mathfrak{a})) \text{ for all } \mu < \boldsymbol{\nu} \text{ and } \mathfrak{a} \in \mathfrak{M}_{\omega^\mu}.$$

If $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ is a hyperserial embedding of force $\boldsymbol{\nu}$, then we say that $\mathbb{U}$ is an **extension** *of $\mathbb{T}$ of force $\boldsymbol{\nu}$. If $\mathbb{T} \subseteq \mathbb{U}$ and $\mathrm{Id}_{\mathbb{S}}$ is an embedding of force $\boldsymbol{\nu}$, then we say that $\mathbb{T}$ is a* **hyperserial subskeleton** *of $\mathbb{T}$ of force $\boldsymbol{\nu}$.*

4.2.3 Confluence

In this subsection, let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$. We inductively define the notion of $\boldsymbol{\mu}$*-confluence* in conjunction with functions $\mathfrak{d}_{\omega^\mu}$: $\mathbb{T}^{>,\succ} \longrightarrow \mathfrak{M}_{\omega^\mu}$ and the classes $\mathcal{E}_{\omega^\mu}[s] \subseteq \mathbb{T}^{>,\succ}$, as follows:

Definition 4.2.5. *The field $\mathbb{T}$ is 0-**confluent** if $\mathfrak{M}$ is not trivial. The function $\mathfrak{d}_1$ maps each $s \in \mathbb{T}^{>,\succ}$ to its dominant monomial $\mathfrak{d}_s$. For each $s \in \mathbb{T}^{>,\succ}$, we set*

$$\mathcal{E}_1[s] := \{t \in \mathbb{T}^{>,\succ} : t \asymp s\}.$$

Let $\mu \in \mathbf{On}$ with $0 < \mu \leqslant \boldsymbol{\nu}$, let $s \in \mathbb{T}^{>,\succ}$, and suppose $\mathbb{T}$ is η-confluent for all $\eta < \mu$.

- *If μ is a successor, then we define $\mathcal{E}_{\omega^\mu}[s]$ to be the class of series t with*

$$(L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(s) \asymp (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(t)$$

for some $n \in \mathbb{N}$.

- *If μ is a limit, then we define $\mathcal{E}_{\omega^\mu}[s]$ to be the class of series t with*

$$L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(s)) \asymp L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t))$$

for some $\eta < \mu$.

*If each class $\mathcal{E}_{\omega^\mu}[s]$ contains an $L_{<\omega^\mu}$-atomic element, then we say that $\mathbb{T}$ is μ-**confluent**. We will see that each class $\mathcal{E}_{\omega^\mu}[s]$ contains at most one $L_{<\omega^\mu}$-atomic element, which we denote by $\mathfrak{d}_{\omega^\mu}(s)$.*

Remark 4.2.6. We note that μ-confluence is somewhat stronger than the similar notion of $\log_{\omega^\mu}$-confluence from [92], due to the extra requirement that we have maps $\mathfrak{d}_{\omega^\mu}$.

Lemma 4.2.7. *Let $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$ and suppose $\mathbb{T}$ is μ-confluent. Then the function $\mathfrak{d}_{\omega^\mu}$ is well-defined. Moreover, we have $\mathcal{E}_{\omega^\eta}[t] \subseteq \mathcal{E}_{\omega^\mu}[t]$ for all $\eta \leqslant \mu$ and $t \in \mathbb{T}^{>,\succ}$.*

Proof. We prove this by induction on μ, noticing that the case $\mu = 0$ is trivial. Assume that this is the case for all ordinals $\eta < \mu$ and let $s \in \mathbb{T}^{>,\succ}$. To see that $\mathfrak{d}_{\omega^\mu}$ is well-defined, let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_{\omega^\mu}$ with $\mathfrak{a}, \mathfrak{b} \in \mathcal{E}_{\omega^\mu}[s]$. We need to show that $\mathfrak{a} = \mathfrak{b}$.

Assume that μ is a successor. Take $m, n \in \mathbb{N}$ with

$$\begin{aligned} (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ m}(\mathfrak{a}) &\asymp (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ m}(s) \quad \text{and} \\ (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(\mathfrak{b}) &\asymp (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(s). \end{aligned}$$

We may assume for instance that $m \leqslant n$. The inductive assumption that $\mathcal{E}_1[t] \subseteq \mathcal{E}_{\omega^{\mu_-}}[t]$ for all $t \in \mathbb{T}^{>,\succ}$ gives

$$\mathfrak{d}_{\omega^{\mu_-}}((L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ m}(\mathfrak{a})) = \mathfrak{d}_{\omega^{\mu_-}}((L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ m}(s)),$$

whence $(L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ k}(\mathfrak{a}) = (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ k}(s)$ for all $k > m$. In particular $(L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(\mathfrak{a}) \asymp (L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(\mathfrak{b})$.

Since $L_{\omega^{\mu_-}k}(\mathfrak{a})$ is $L_{<\omega^\mu}$-atomic for each k and since $\mathfrak{d}_{\omega^{\mu_-}}$ is well-defined by our induction hypothesis, we have $\mathfrak{d}_{\omega^{\mu_-}}(L_{\omega^{\mu_-}k}(\mathfrak{a})) = L_{\omega^{\mu_-}k}(\mathfrak{a})$ for each k. It follows by induction on k that $(L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ k}(\mathfrak{a}) = L_{\omega^{\mu_-}k}(\mathfrak{a})$ for each k and, likewise, $(L_{\omega^{\mu_-}} \circ \mathfrak{d}_{\omega^{\mu_-}})^{\circ k}(\mathfrak{b}) = L_{\omega^{\mu_-}k}(\mathfrak{b})$ for each k, so $L_{\omega^{\mu_-}n}(\mathfrak{a}) \asymp L_{\omega^{\mu_-}n}(\mathfrak{b})$. As both $L_{\omega^{\mu_-}n}(\mathfrak{a})$ and $L_{\omega^{\mu_-}n}(\mathfrak{b})$ are monomials, we have $L_{\omega^{\mu_-}n}(\mathfrak{a}) = L_{\omega^{\mu_-}n}(\mathfrak{b})$. Recall that $L_{\omega^{\mu_-}}$ is injective by $\mathbf{M}_{\mu_-}$, so $\mathfrak{a} = \mathfrak{b}$.

Assume now that μ is a limit and take $\eta, \rho < \mu$ with $L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{a})) \asymp L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(s))$ and $L_{\omega^\rho}(\mathfrak{d}_{\omega^\rho}(\mathfrak{b})) \asymp L_{\omega^\rho}(\mathfrak{d}_{\omega^\rho}(s))$. We may assume for instance that $\eta \leqslant \rho$. Note that we have $\mathfrak{a} = \mathfrak{d}_{\omega^{\eta+1}}(s)$ by definition and by the induction hypothesis. Recall that $\mathfrak{M}_{\omega^\mu} \subseteq \mathfrak{M}_{\omega^{\rho+1}}$ by definition. The inclusion $\mathcal{E}_{\omega^{\eta+1}}[s] \subseteq \mathcal{E}_{\omega^{\rho+1}}[s]$ and the fact that $\mathfrak{d}_{\omega^{\rho+1}}$ is well-defined thus give $\mathfrak{a} = \mathfrak{d}_{\omega^{\rho+1}}(s) = \mathfrak{b}$.

As to our second assertion, consider $t \in \mathbb{T}^{>,\succ}$ and $u \in \mathcal{E}_{\omega^\eta}[t]$ with $\eta < \mu$. If μ is a successor, then the induction hypothesis $\mathcal{E}_{\omega^\eta}[t] \subseteq \mathcal{E}_{\omega^{\mu_-}}[t]$ implies that $\mathfrak{d}_{\omega^{\mu_-}}(u) = \mathfrak{d}_{\omega^{\mu_-}}(t)$, so $L_{\omega^{\mu_-}}(\mathfrak{d}_{\omega^{\mu_-}}(u)) \asymp L_{\omega^{\mu_-}}(\mathfrak{d}_{\omega^{\mu_-}}(t))$, whence $u \in \mathcal{E}_{\omega^\mu}[t]$. Assume that μ is a limit. We have $\mathfrak{d}_{\omega^\eta}(u) = \mathfrak{d}_{\omega^\eta}(t)$ because $\mathfrak{d}_{\omega^\eta}$ is well-defined. In particular $L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(u)) \asymp L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t))$, so $u \in \mathcal{E}_{\omega^\mu}[t]$. This concludes the inductive proof. □

Remark 4.2.8. We see that the conditions "for some $n \in \mathbb{N}$" and "for some $\eta < \mu$" in Definition 4.2.5 can be replaced by "for large enough $n \in \mathbb{N}$" and "for large enough $\mu < \nu$" respectively.

Corollary 4.2.9. *Let $\mu, \eta \in \mathbf{On}$ with $\eta \leqslant \mu \leqslant \boldsymbol{\nu}$. If $\mathbb{T}$ is μ-confluent, then $\mathfrak{d}_{\omega^\mu}(s) = \mathfrak{d}_{\omega^\mu}(\mathfrak{d}_{\omega^\eta}(s))$ for all $s \in \mathbb{T}^{>,\succ}$.*

Proposition 4.2.10. *Let $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$. If $\mathbb{T}$ is η-confluent for all $\eta < \mu$, then the class $\mathcal{E}_{\omega^\mu}[s]$ is convex for each $s \in \mathbb{T}^{>,\succ}$. Moreover, if $\mathbb{T}$ is μ-confluent, then $\mathfrak{d}_{\omega^\mu} \colon \mathbb{T}^{>,\succ} \longrightarrow \mathfrak{M}_{\omega^\mu}$ is non-decreasing.*

Proof. We prove this by induction on μ. Let $s \in \mathbb{T}^{>,\succ}$. It is clear that $\mathcal{E}_1[s]$ is convex and that $\mathfrak{d}_1$ is increasing. Let $\mu > 0$ and assume that the result holds for all $\eta < \mu$. By the monotonicity axioms, each function L_{ω^η} is strictly increasing on $\mathfrak{M}_{\omega^\eta}$ (when $\eta = 0$, one also needs to use $\mathbf{FE_0}$ to see that $L_1(\mathfrak{m}/\mathfrak{n}) = L_1(\mathfrak{m}) - L_1(\mathfrak{n}) > 0$ for $\mathfrak{m} \succ \mathfrak{n} \in \mathfrak{M}_1$). As the composition of non-decreasing functions is non-decreasing, the function $(L_{\omega^\eta} \circ \mathfrak{d}_{\omega^\eta})^{\circ n}$ is non-decreasing for each $\eta < \mu$ and each $n \in \mathbb{N}$. We deduce that $\mathfrak{d}_{\omega^\mu}$ is non-decreasing and that the classes $\mathcal{E}_{\omega^\mu}[s], s \in \mathbb{T}^{>,\succ}$ are convex. □

If $\mathbb{T}$ is η-confluent for all $\eta<\mu$, then the proposition implies that the classes $\mathcal{E}_{\omega^\eta}[s]$ with $s\in\mathbb{T}^{>,\succ}$ form a partition of $\mathbb{T}^{>,\succ}$ into convex subclasses. If $\mathbb{T}$ is also μ-confluent, then we have the following explicit decomposition for all $\eta\leqslant\mu$:

$$\mathbb{T}^{>,\succ}=\bigsqcup_{\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}}\mathcal{E}_{\omega^\eta}[\mathfrak{a}].$$

Definition 4.2.11. *$\mathbb{T}$ is said to be* **confluent** *if it is μ-confluent for each $\mu\in\mathbf{On}$ with $\mu\leqslant\boldsymbol{\nu}$. An extension/embedding $\Psi:\mathbb{T}\longrightarrow\mathbb{U}$ of force $\boldsymbol{\nu}$ is* ***confluent*** *if $\mathbb{U}$ is confluent.*

Note that if $\boldsymbol{\nu}\in\mathbf{On}$, then $\mathbb{T}$ is confluent if and only if it is $\boldsymbol{\nu}$-confluent.

4.2.4 The skeleton of logarithmic hyperseries

Let ν be an ordinal and set $\alpha:=\omega^\nu$. The goal of this section is to check that $\mathbb{L}_{<\alpha}$ is a confluent hyperserial skeleton of force ν. This is immediate for $\nu=0$, so we assume that $\nu>0$.

Definition 4.2.12. *Let $\operatorname{dom}L_1:=\mathfrak{L}_{<\alpha}^{\succ}$ and for $0<\mu<\nu$, let $\operatorname{dom}L_{\omega^\mu}:=\{\ell_\sigma:\omega^{\mu_-}\leqslant_o\sigma<\alpha\}$. Given $\mathfrak{l}\in\operatorname{dom}L_{\omega^\mu}$, set*

$$L_{\omega^\mu}(\mathfrak{l}):=\ell_{\omega^\mu}\circ\mathfrak{l}.$$

We will show that $(\mathbb{L}_{<\alpha},(L_{\omega^\mu})_{\mu<\nu})$ is a hyperserial skeleton by checking that the axioms are satisfied. We begin with the domain of definition axioms.

Lemma 4.2.13. *$(\mathbb{L}_{<\alpha},(L_{\omega^\mu})_{\mu<\nu})$ satisfies $\mathbf{DD}_{\boldsymbol{\mu}}$ and $(\mathfrak{L}_{<\alpha})_{\omega^\mu}=\{\ell_\sigma:\omega^{\mu_-}\leqslant_o\sigma<\alpha\}$, for all $\mu\leqslant\nu$.*

Proof. We prove this by induction on μ. The case when $\mu=0$ is immediate. For $\mu=1$, consider an infinite monomial $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\alpha}$. We have $L_1(\mathfrak{l})=\sum_{\gamma<\alpha}\mathfrak{l}_\gamma\,\ell_{\gamma+1}$, which is a monomial if and only $\mathfrak{l}=\ell_\gamma$ for some $\gamma<\alpha$. Conversely, for each $\gamma<\alpha$ we have $L_n(\ell_\gamma)=\ell_{\gamma+n}\in\mathfrak{L}_{<\alpha}$. Now let $1<\mu\leqslant\nu$ and suppose that the lemma holds for all non-zero ordinals less than μ. Assume that μ is a limit. We have $\mu_-=\mu$, whence

$$\bigcap_{\eta<\mu}\operatorname{dom}L_{\omega^\eta}=\bigcap_{\eta<\mu_-}\operatorname{dom}L_{\omega^{\eta+1}}=\bigcap_{\eta<\mu_-}\{\ell_\gamma:\omega^\eta\leqslant_o\gamma<\alpha\}=\{\ell_\gamma:\omega^{\mu_-}\leqslant_o\gamma<\alpha\}=\operatorname{dom}L_{\omega^\mu}.$$

Assume now that μ is a successor. If $\mathfrak{l}\in\operatorname{dom}L_{\omega^\mu}$, then $\mathfrak{l}=\ell_\sigma$ where $\omega^{\mu_-}\leqslant_o\sigma<\alpha$, and we clearly have $L_{\omega^{\mu_-}}^{\circ n}(\mathfrak{l})=\ell_{\sigma+\omega^{\mu_-}n}\in\operatorname{dom}L_{\omega^{\mu_-}}$ for all $n\in\mathbb{N}$, whence $\mathfrak{l}\in\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_{\omega^{\mu_-}}^{\circ n}$. Conversely, let $\mathfrak{l}\in\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_{\omega^{\mu_-}}^{\circ n}$. Then $\mathfrak{l}=\ell_\sigma$ where $\omega^{\mu_{--}}\leqslant_o\sigma<\alpha$. If μ_- is a limit, then $\mu_{--}=\mu_-$, whence $\omega^{\mu_-}\leqslant_o\sigma<\alpha$ and $\mathfrak{l}\in\operatorname{dom}L_{\omega^\mu}$. If μ_- is a successor, then $\sigma=\gamma+\omega^{\mu_{--}}m$ for some $\gamma\geqslant_o\omega^{\mu_-}$ and some $m\in\mathbb{N}$, so

$$\ell_\sigma=\ell_{\gamma+\omega^{\mu_{--}}m}=\ell_{\omega^{\mu_{--}}m}\circ\ell_\gamma.$$

Since $L_{\omega^{\mu_-}}(\ell_{\omega^{\mu_{--}}m})=\ell_{\omega^{\mu_-}}-m$, we see that

$$L_{\omega^{\mu_-}}(\mathfrak{l})=L_{\omega^{\mu_-}}(\ell_\sigma)=(\ell_{\omega^{\mu_-}}-m)\circ\ell_\gamma=\ell_{\gamma+\omega^{\mu_-}}-m.$$

Since $L_{\omega^{\mu_-}}(\mathfrak{l})\in\operatorname{dom}L_{\omega^{\mu_-}}$, we must have $m=0$, so $\mathfrak{l}=\ell_\sigma\in\operatorname{dom}L_{\omega^\mu}$. □

For $\beta<\omega^\nu$ and $\mathfrak{l}\in\operatorname{dom}L_\beta$, note that $L_\beta(\mathfrak{l})=\ell_\beta\circ\mathfrak{l}$. Note also that the notions of $L_{<\omega^{\mu+1}}$-atomicity and $L_{<\omega^\mu}$-atomicity coincide in $\mathbb{L}_{<\alpha}$ whenever μ is a limit with $\mu+1\leqslant\nu$. This will not be the case in general.

Proposition 4.2.14. *The field $\mathbb{L}_{<\alpha}$ satisfies $\mathbf{P}_{\boldsymbol{\mu}}$ for all $\mu\leqslant\nu$.*

Proof. Let $\mu\leqslant\nu$ and let $\mathfrak{l}\in(\mathfrak{L}_{<\alpha})_{\omega^\mu}$. By Remark 4.2.2, we may assume $\mu>0$. We have $\mathfrak{l}=\ell_\sigma$ for some $\omega^{\mu_-}\leqslant_o\sigma<\alpha$. Let $(r_\gamma)_{\gamma<\omega^\mu}$ be a sequence of real numbers. We have

$$\sum_{\gamma<\omega^\mu}r_\gamma\,L_{\gamma+1}(\mathfrak{l})=\sum_{\gamma<\omega^\mu}r_\gamma\,\ell_{\gamma+1}\circ\ell_\sigma=\sum_{\gamma<\omega^\mu}r_\gamma\,\ell_{\sigma+\gamma+1}.$$

This sum coincides with $\log \mathfrak{m}$ where $\mathfrak{m} := \prod_{\gamma<\omega^\mu} \ell_{\sigma+\gamma}^{r_\gamma} \in \mathfrak{L}_{<\alpha}$. □

Proposition 4.2.15. *The field* $\mathbb{L}_{<\alpha}$ *satisfies* $\mathbf{R}_\mu$, $\mathbf{A}_\mu$, *and* $\mathbf{M}_\mu$, *for all* $0<\mu<\nu$.

Proof. Let $0<\mu<\nu$ and let $\mathfrak{l} \in (\mathfrak{L}_{<\alpha})_{\omega^\mu}$. We have $\mathfrak{l} = \ell_\sigma$ for some $\omega^{\mu_-} \leqslant_o \sigma < \alpha$. Write $\sigma = \gamma + \omega^{\mu_-} n$ where $\gamma = \sigma_{\geqslant \omega^\mu}$, $n \in \mathbb{N}$, and $n=0$ if μ is a limit. We claim $L_{\omega^\mu}(\mathfrak{l}) = \ell_{\gamma+\omega^\mu} - n$. If μ is a successor, then since $\ell_{\omega^\mu} \circ \ell_{\omega^{\mu_-} n} = \ell_{\omega^\mu} - n$, we have

$$L_{\omega^\mu}(\mathfrak{l}) = \ell_{\omega^\mu} \circ \ell_\sigma = \ell_{\omega^\mu} \circ \ell_{\gamma+\omega^{\mu_-} n} = \ell_{\omega^\mu} \circ (\ell_{\omega^{\mu_-} n} \circ \ell_\gamma) = (\ell_{\omega^\mu} - n) \circ \ell_\gamma = \ell_{\gamma+\omega^\mu} - n.$$

If μ is a limit, then $\mathfrak{l} = \ell_\gamma$, so

$$L_{\omega^\mu}(\mathfrak{l}) = \ell_{\omega^\mu} \circ \ell_\gamma = \ell_{\gamma+\omega^\mu}.$$

Now we move on to verification of $\mathbf{R}_\mu$, $\mathbf{A}_\mu$, and $\mathbf{M}_\mu$. The only elements in $\operatorname{supp} L_{\omega^\mu}(\mathfrak{l})$ are $\ell_{\gamma+\omega^\mu}$ and possibly 1 (if $n \neq 0$), so $\operatorname{supp} L_{\omega^\mu}(\mathfrak{l}) \succcurlyeq 1 \succ L_{\omega^\eta n}(\mathfrak{l})^{-1}$ for all $\eta<\mu$ and $n \in \mathbb{N}$, which proves $\mathbf{R}_\mu$. For $\eta<\mu$, we have

$$L_{\omega^\eta}(\mathfrak{l}) = \ell_{\omega^\eta} \circ \ell_\sigma = \ell_{\sigma+\omega^\eta} \prec \ell_{\gamma+\omega^\mu} \asymp L_{\omega^\mu}(\mathfrak{l}),$$

so $\mathbf{A}_\mu$ holds as well.

As to $\mathbf{M}_\mu$, take $\mathfrak{l}' \in (\mathfrak{L}_{<\alpha})_{\omega^\mu}$ with $\mathfrak{l}' \succ \mathfrak{l}$. We have $\mathfrak{l}' = \ell_{\sigma'}$ for some σ' with $\omega^{\mu_-} \leqslant_o \sigma' < \alpha$. Write $\sigma' = \gamma' + \omega^{\mu_-} n'$ where $\gamma' = \sigma'_{\geqslant \omega^\mu}$, $n' \in \mathbb{N}$, and $n'=0$ if μ is a limit. The argument above gives $L_{\omega^\mu}(\mathfrak{l}') = \ell_{\gamma'+\omega^\mu} - n'$. If $\gamma'<\gamma$, then $L_{\omega^\mu}(\mathfrak{l}') \succ L_{\omega^\mu}(\mathfrak{l})$ and if $\gamma'=\gamma$, then $n'<n$ and $L_{\omega^\mu}(\mathfrak{l}') - L_{\omega^\mu}(\mathfrak{l}) = n - n' \geqslant 1$. In either case, $\mathbf{M}_\mu$ is satisfied. □

Recall that for $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$ and $\gamma<\alpha$, we write $\mathfrak{l}_\gamma$ for the real exponent of ℓ_γ in $\mathfrak{l}$. Given $f \in \mathbb{L}_{<\alpha}^{>,\succ}$, we define λ_f to be the least ordinal with $(\mathfrak{d}_f)_{\lambda_f} \neq 0$; see also [33, p. 23].

Proposition 4.2.16. $\mathbb{L}_{<\omega^\nu}$ *is ν-confluent. More precisely, for* $0<\mu\leqslant\nu$ *and* $f \in \mathbb{L}_{<\alpha}$, *we have*

$$\mathfrak{d}_{\omega^\mu}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}}. \tag{4.2.3}$$

Proof. We first note that $\mathbb{L}_{<\alpha}$ is 0-confluent as $\mathfrak{L}_{<\alpha}$ is not trivial. We proceed by induction on $0<\mu\leqslant\nu$. Take $f \in \mathbb{L}_{<\alpha}^{>,\succ}$. If $\mu=1$, then we have $L_1(\mathfrak{d}_1(f)) \asymp \ell_{\lambda_f+1} = L_1(\ell_{\lambda_f})$ where ℓ_{λ_f} is $L_{<\omega}$-atomic, so $\mathfrak{d}_\omega(f) = \ell_{\lambda_f}$ and $\mathbb{L}_{<\alpha}$ is 1-confluent. It remains to note that $(\lambda_f)_{\geqslant 1} = \lambda_f$.

Now suppose that $\mu>1$ and assume that $\mathbb{L}_{<\alpha}$ is η-confluent and satisfies (4.2.3) for all $\eta<\mu$. Suppose μ is a successor, so $\mathfrak{d}_{\omega^{\mu_-}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_{--}}}}$. Write $(\lambda_f)_{\geqslant \omega^{\mu_{--}}} = (\lambda_f)_{\geqslant \omega^{\mu_-}} + \omega^{\mu_{--}} n$ with $n \in \mathbb{N}$ and with $n=0$ if μ_- is a limit. We have $\ell_{(\lambda_f)_{\geqslant \omega^{\mu_{--}}}} = \ell_{\omega^{\mu_{--}} n} \circ \ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}}$, so

$$L_{\omega^{\mu_-}}(\mathfrak{d}_{\omega^{\mu_-}}(f)) = (\ell_{\omega^{\mu_-}} \circ \ell_{\omega^{\mu_{--}} n}) \circ \ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}} = L_{\omega^{\mu_-}}(\ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}}) - n \asymp L_{\omega^{\mu_-}}(\ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}})$$

and $\mathfrak{d}_{\omega^\mu}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}}$.

Now suppose μ is a limit, so there is $\eta<\mu$ with $(\lambda_f)_{\geqslant \omega^\eta} = (\lambda_f)_{\geqslant \omega^\mu} = (\lambda_f)_{\geqslant \omega^{\mu_-}}$. By hypothesis, we have that $\mathfrak{d}_{\omega^{\eta+1}}(f) = \ell_{(\lambda_f)_{\geqslant \omega^\eta}}$ and so

$$L_{\omega^{\eta+1}}(\mathfrak{d}_{\omega^{\eta+1}}(f)) = L_{\omega^{\eta+1}}(\ell_{(\lambda_f)_{\geqslant \omega^\eta}}) = L_{\omega^{\eta+1}}(\ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}}).$$

Again, this yields $\mathfrak{d}_{\omega^\mu}(f) = \ell_{(\lambda_f)_{\geqslant \omega^{\mu_-}}}$. □

Theorem 4.2.17. $\mathbb{L}_{<\alpha}$ *is a confluent hyperserial skeleton of force* ν.

Proof. Using the identity

$$\ell_1 \circ \mathfrak{l} = \sum_{\gamma<\alpha} \mathfrak{l}_\gamma \ell_{\gamma+1}$$

for $\mathfrak{l} = \prod_{\gamma<\alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$, the field $\mathbb{L}_{<\alpha}$ is easily seen to satisfy $\mathbf{FE}_0$, $\mathbf{A}_0$, $\mathbf{M}_0$, and $\mathbf{R}_0$. Moreover, $\mathbb{L}_{<\alpha}$ satisfies $\mathbf{FE}_\mu$ for all $0<\mu<\nu$ by [33, Lemma 5.6]. Using Propositions 4.2.14, 4.2.15 and 4.2.16, we conclude that $\mathbb{L}_{<\alpha}$ is a confluent hyperserial skeleton of force ν. □

Corollary 4.2.18. $\mathbb{L}$ *is a confluent hyperserial skeleton of force* **On**.

4.3 Extending the partial hyperlogarithms

Let $\boldsymbol{\nu} \leqslant \mathbf{On}$. We will follow [14, Section 4] while simplifying some proofs by using the material of Chapters 2 and 3. The purpose of the next two sections is to prove the following theorem:

Theorem 4.3.1. *Let* $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ *be a confluent hyperserial skeleton of force* $\boldsymbol{\nu}$*. There is a unique function* $\circ: \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ *satisfying:*

C1$_{\boldsymbol{\nu}}$. $\mathbb{L}_{<\omega^\nu} \longrightarrow \mathbb{T}; f \longmapsto f \circ s$ *is a strongly* $\mathbb{R}$*-linear ordered field embedding for each* $s \in \mathbb{T}^{>,\succ}$*;*

C2$_{\boldsymbol{\nu}}$. $\ell_0^r \circ \mathfrak{m} = \mathfrak{m}^r$ *for all* $\mathfrak{m} \in \mathfrak{M}$ *and* $r \in \mathbb{R}$*;*
$\ell_{\omega^\mu} \circ \mathfrak{a} = L_{\omega^\mu}(\mathfrak{a})$ *for all* $\mu < \boldsymbol{\nu}$ *and* $\mathfrak{a} \in \operatorname{dom} L_{\omega^\mu}$*;*

C3$_{\boldsymbol{\nu}}$. $f \circ (g \circ s) = (f \circ g) \circ s$ *for all* $f \in \mathbb{L}_{<\omega^\nu}$*,* $g \in \mathbb{L}_{<\omega^\nu}^{>,\succ}$*, and* $s \in \mathbb{T}^{>,\succ}$*;*

C4$_{\boldsymbol{\nu}}$. $f \circ (t + \delta) = \sum_{k\in\mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ *for all* $f \in \mathbb{L}_{<\omega^\nu}$*,* $t \in \mathbb{T}^{>,\succ}$*, and* $\delta \in \mathbb{T}$ *with* $\delta \prec t$*.*

We will start by extending each partial hyperlogarithm L_{ω^μ} for $\mu < \boldsymbol{\nu}$ to $\mathbb{T}^{>,\succ}$, and then define the corresponding composition laws using rules of strong linearity. However, our proof will be an induction where the definition of L_{ω^μ} reauires the validity of Theorem 4.3.1 for all ordinals $< \mu$.

We claim that it suffices to prove the theorem in the case when $\boldsymbol{\nu} \in \mathbf{On}$. The case when $\boldsymbol{\nu} = \mathbf{On}$ can then be proved as follows: let $(\mathbb{T}, (L_{\omega^\mu})_{\mu\in\mathbf{On}})$ be a confluent hyperserial skeleton of force $\mathbf{On}$. Then for every $\nu < \mathbf{On}$, there exists a unique composition $\circ_\nu: \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ that satisfies **C1$_{\boldsymbol{\nu}}$**, **C2$_{\boldsymbol{\nu}}$**, **C3$_{\boldsymbol{\nu}}$**, and **C4$_{\boldsymbol{\nu}}$**. For $\mu < \nu$, the composition $\circ_\nu$ extends $\circ_\mu$, by uniqueness. For any $f \in \mathbb{L}$ and $s \in \mathbb{T}^{>,\succ}$, we have $f \in \mathbb{L}_{<\nu}$ for some $\nu < \mathbf{On}$, so we may define $f \circ s := f \circ_\nu s$ and this definition does not depend on ν. It is straightforward to check that this defines the unique composition $\circ$: $\mathbb{L} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ which satisfies **C1$_{\mathbf{On}}$**, **C2$_{\mathbf{On}}$**, **C3$_{\mathbf{On}}$**, and **C4$_{\mathbf{On}}$**.

Throughout this section, we fix an ordinal ν and a hyperserial skeleton $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of force ν. We fix also $\mu < \nu$ such that $\mathbb{T}$ is μ-confluent and we set

$$\beta := \omega^\mu.$$

We assume that Theorem 4.3.1 holds for μ, so we have a unique composition $\circ: \mathbb{L}_{<\beta} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ satisfying **C1$_\mu$**, **C2$_\mu$**, **C3$_\mu$**, and **C4$_\mu$**. For $\gamma < \beta$ and $s \in \mathbb{T}^{>,\succ}$, we write $L_\gamma(s) := \ell_\gamma \circ s$. In light of Lemma 4.1.2, the expression $(\ell_\beta^{\uparrow\gamma})^{(k)} \circ s$ makes sense for each $k > 0$. Moreover, **C4$_\mu$** and Propositions 2.3.5 and 2.3.6 together imply that:

Lemma 4.3.2. *Each* $f \in \mathbb{L}_{<\beta}$ *induces an analytic function*

$$\begin{aligned} \mathcal{A}_f \colon \mathbb{T}^{>,\succ} &\longrightarrow \mathbb{T} \\ s &\longmapsto f \circ s, \quad \textit{with} \\ \operatorname{Conv}(\mathcal{A}_f)_s &\supseteq \mathbb{T}^{\prec s} \quad \textit{and} \\ \mathcal{A}_f^{(k)} &= \mathcal{A}_{f^{(k)}} \end{aligned}$$

for all $s \in \mathbb{T}^{>,\succ}$ *and* $k \in \mathbb{N}$*.*

Recall that as a hyperserial skeleton of force 1, the structure $(\mathbb{T}, L_1)$ already induces a logarithm $\log: \mathbb{T}^> \longrightarrow \mathbb{T}$ such that $(\mathbb{T}, \log)$ is a transserial field and for which the content of Section 3.1 applies. Which means that we can focus on hyperlogarithms. Our main goal in this section is to prove the following result:

Proposition 4.3.3. *Assume that* $\mu > 0$*. There is an extension of* L_β *to* $\mathbb{T}^{>,\succ}$ *such that for all* $s \in \mathbb{T}^{>,\succ}$*,* $\mathfrak{a} \in \mathfrak{M}_\beta$*, and* $\gamma < \beta$ *with* $\varepsilon := L_\gamma(s) - L_\gamma(\mathfrak{a}) \prec 1$*, we have*

$$L_\beta(s) = L_\beta(\mathfrak{a}) + \sum_{k\in\mathbb{N}^>} \frac{(\ell_\beta^{\uparrow\gamma})^{(k)} \circ L_\gamma(\mathfrak{a})}{k!} \varepsilon^k.$$

It is crucial here that this identity is valid for all $\gamma < \beta$. We will also prove that L_β satisfies the extension of **FE$_{\boldsymbol{\mu}}$** to $\mathbb{T}^{>,\succ}$ (Proposition 4.3.11), that L_β has Taylor expansions around every point (Theorem 4.3.10) and that it is strictly increasing on $\mathbb{T}^{>,\succ}$ (Lemma 4.3.12).

Our extension will heavily depend on Taylor series expansions, so it is convenient to introduce some notation for that. Let $f\in\mathbb{L}_{<\alpha}$ be such that $f^{(k)}\in\mathbb{L}_{<\beta}$ for all $k>0$. Let $t\in\mathbb{T}^{>,\succ}$ and $\delta\in\mathbb{T}$ with $\delta\prec t$. By Lemma 4.1.1 with $\alpha=\beta$, f' in place of f, and $\Phi\colon\mathbb{L}_{<\beta}\longrightarrow\mathbb{T}; g\mapsto g\circ t$, we see that the family $((f^{(k)}\circ t)\,\delta^k)_{k\in\mathbb{N}^>}$ is well-based. We define

$$\mathcal{T}_f(t,\delta):=\sum_{k\in\mathbb{N}^>}\frac{f^{(k)}\circ t}{k!}\,\delta^k\in\mathbb{T}.$$

4.3.1 Confluence revisited

Assume now that $\mu>0$. Let us revisit the notion of confluence.

Lemma 4.3.4. *Let $s,t\in\mathbb{T}^{>,\succ}$ and suppose that $L_\gamma(s)\asymp L_\gamma(t)$ for some $\gamma<\beta$. Then $L_\sigma(s)-L_\sigma(t)\prec 1$ for all $\sigma<\beta$ with $\sigma\geqslant\gamma+2$.*

Proof. We first show that $L_{\gamma+2}(s)-L_{\gamma+2}(t)\prec 1$. Take $c\in\mathbb{R}^>$ and $\varepsilon\prec 1$ such that $L_\gamma(s)=L_\gamma(t)(c+\varepsilon)$. We have

$$L_{\gamma+1}(s)=L_1(L_\gamma(s))=L_{\gamma+1}(t)+\log(c+\varepsilon),$$

where $\log(c+\varepsilon)\preccurlyeq 1$. Set $\delta:=L_{\gamma+1}(t)^{-1}\log(c+\varepsilon)\prec 1$, so $L_{\gamma+1}(s)=L_{\gamma+1}(t)\,(1+\delta)$. We have

$$L_{\gamma+2}(s)=\log L_{\gamma+1}(s)=L_{\gamma+2}(t)+\log(1+\delta),$$

where $\log(1+\delta)=\tilde{L}(\delta)\sim\delta\prec 1$. Thus, $L_{\gamma+2}(s)-L_{\gamma+2}(t)\prec 1$.

Now, fix σ with $\gamma+2\leqslant\sigma<\beta$ and set $\delta:=L_{\gamma+2}(s)-L_{\gamma+2}(t)$. By $\mathbf{C3}_\mu$ and $\mathbf{C4}_\mu$, we have

$$L_\sigma(s)=\ell_\sigma^{\uparrow\gamma+2}\circ L_{\gamma+2}(s)=\ell_\sigma^{\uparrow\gamma+2}\circ L_{\gamma+2}(t)+\mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(L_{\gamma+2}(t),\delta)=L_\sigma(t)+\mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(L_{\gamma+2}(t),\delta).$$

Lemma 4.1.2 in conjunction with the fact that $\delta\prec 1$ gives us that $\mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(L_{\gamma+2}(t),\delta)\prec 1$, so $L_\sigma(s)-L_\sigma(t)\prec 1$. □

Proposition 4.3.5. *For all $s\in\mathbb{T}^{>,\succ}$, we have*

$$\mathcal{E}_\beta[s]=\{t\in\mathbb{T}^{>,\succ}:L_\gamma(s)-L_\gamma(t)\prec 1\text{ for some }\gamma<\beta\}.$$

Proof. We fix $s\in\mathbb{T}^{>,\succ}$. Since $\mu>0$, we know by Lemma 4.3.4 that it is enough to show that $\mathcal{E}_\beta[s]=\{t\in\mathbb{T}^{>,\succ}:L_\gamma(s)\asymp L_\gamma(t)\text{ for some }\gamma<\beta\}$. We proceed by induction on μ. If $\mu=1$, then $\beta=\omega$ and

$$\mathcal{E}_\omega[s]=\{t\in\mathbb{T}^{>,\succ}:(L_1\circ\mathfrak{d}_1)^{\circ n}(t)\asymp(L_1\circ\mathfrak{d}_1)^{\circ n}(s)\text{ for some }n\in\mathbb{N}\}.$$

An easy induction on n yields $(L_1\circ\mathfrak{d}_1)^{\circ n}(t)\asymp L_n(t)$ for each $t\in\mathbb{T}^{>,\succ}$, whence the result.

Now suppose that $\mu>1$. If μ is a successor, then for each $t\in\mathcal{E}_\beta[s]$ there is some $n\in\mathbb{N}$ with $(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(t)\asymp L_{\omega^{\mu_-}n}(\mathfrak{d}_\beta(t))$. By our inductive assumption applied to μ_-, we have that $L_\gamma(t)-L_\gamma(\mathfrak{d}_{\omega^{\mu_-}}(t))\prec 1$ for some $\gamma<\omega^{\mu_-}$. By Lemma 4.3.4, we have $L_{\omega^{\mu_-}}(t)-L_{\omega^{\mu_-}}(\mathfrak{d}_{\omega^{\mu_-}}(t))\prec 1$ and an easy induction on n gives us that $(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(t)-L_{\omega^{\mu_-}n}(t)\prec 1$. Thus, we have that $L_{\omega^{\mu_-}n}(t)\asymp L_{\omega^{\mu_-}n}(\mathfrak{d}_\beta(t))$ for some $n\in\mathbb{N}$. Likewise, $L_{\omega^{\mu_-}m}(s)\asymp L_{\omega^{\mu_-}m}(\mathfrak{d}_\beta(s))$ for some $m\in\mathbb{N}$. By replacing m and n with $\max\{m,n\}$ and invoking Lemma 4.3.4, we may assume that $m=n$. Since $\mathfrak{d}_\beta(s)=\mathfrak{d}_\beta(t)$, we have $L_{\omega^{\mu_-}n}(s)\asymp L_{\omega^{\mu_-}n}(t)$. On the other hand, given $t\in\mathbb{T}^{>,\succ}$, if $L_\gamma(s)\asymp L_\gamma(t)$ for some $\gamma<\beta$, then take some $n\in\mathbb{N}$ with $\gamma+2\leqslant\omega^{\mu_-}n<\beta$. By Lemma 4.3.4, we have $(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(s)\asymp L_{\omega^{\mu_-}n}(s)\asymp L_{\omega^{\mu_-}n}(t)\asymp(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(t)$, so $t\in\mathcal{E}_\beta[s]$.

If μ is a limit, then for each $t\in\mathcal{E}_\beta[s]$ there is $\eta<\mu$ with $L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t))\asymp L_{\omega^\eta}(\mathfrak{d}_\beta(t))$. By our inductive assumption applied to η, we have that $L_\gamma(t)-L_\gamma(\mathfrak{d}_{\omega^\eta}(t))\prec 1$ for some $\gamma<\omega^\eta$, so $L_{\omega^\eta}(t)-L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(t))\prec 1$ by Lemma 4.3.4. Thus $L_{\omega^\eta}(t)\asymp L_{\omega^\eta}(\mathfrak{d}_\beta(t))$ and likewise, $L_{\omega^\sigma}(s)\asymp L_{\omega^\sigma}(\mathfrak{d}_\beta(s))$ for some $\sigma<\mu$. By replacing η and σ with $\max\{\eta,\sigma\}$ and invoking Lemma 4.3.4, we may assume that $\eta=\sigma$. Since $\mathfrak{d}_\beta(s)=\mathfrak{d}_\beta(t)$, we have $L_{\omega^\eta}(s)\asymp L_{\omega^\eta}(t)$. On the other hand, given $t\in\mathbb{T}^{>,\succ}$, if $L_\gamma(s)\asymp L_\gamma(t)$ for some $\gamma<\beta$, then take some η with $\gamma\leqslant\omega^\eta<\beta$. By Lemma 4.3.4, we have $L_{\omega^\eta}(\mathfrak{d}_\beta(s))\asymp L_{\omega^\eta}(s)\asymp L_{\omega^\eta}(t)\asymp L_{\omega^\eta}(\mathfrak{d}_\beta(t))$, so $t\in\mathcal{E}_\beta[s]$. □

Proposition 4.3.5 in conjunction with Lemma 4.3.4 gives us the two following corollaries:

Corollary 4.3.6. *Let $\beta = \omega^\eta < \alpha$ and let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$. For $\gamma < \alpha$ with $\gamma\omega < \beta$, we have $\mathfrak{d}_\beta(L_\gamma(\mathfrak{a})) = \mathfrak{a}$.*

Proof. Note that $\gamma\omega = \omega^{\iota+1}$ for a certain ordinal ι. Given $n \in \mathbb{N}^>$ with $\omega^\iota n \geqslant \gamma$, we have

$$L_{\gamma\omega}(L_{\omega^\iota n}(\mathfrak{a})) \leqslant L_{\gamma\omega}(L_\gamma(\mathfrak{a})) \leqslant L_{\gamma\omega}(\mathfrak{a}),$$

where $L_{\gamma\omega}(L_{\omega^\iota n}(\mathfrak{a})) = L_{\gamma\omega}(\mathfrak{a}) - n \asymp L_{\gamma\omega}(\mathfrak{a})$. So $L_{\gamma\omega}(L_\gamma(\mathfrak{a})) \asymp L_{\gamma\omega}(\mathfrak{a})$. We deduce since $\gamma\omega < \beta$ that $L_\gamma(\mathfrak{a}) \in \mathcal{E}_\beta[\mathfrak{a}]$, whence $\mathfrak{d}_\beta(L_\gamma(\mathfrak{a})) = \mathfrak{a}$. □

Corollary 4.3.7. *For each $s \in \mathbb{T}^{>,\succ}$ there is $\gamma < \beta$ such that*

$$L_\rho(s) - L_\rho(\mathfrak{d}_\beta(s)) \prec 1,$$

for all $\gamma \leqslant \rho < \beta$. Moreover, if $L_\gamma(s) - L_\gamma(\mathfrak{a}) \prec 1$ for some $\mathfrak{a} \in \mathfrak{M}_\beta$ and some $\gamma < \beta$, then $\mathfrak{a} = \mathfrak{d}_\beta(s)$.

4.3.2 Definition of the extended hyperlogarithms

Definition 4.3.8. *Let $s \in \mathbb{T}^{>,\succ}$ and let $\gamma < \beta$ with $\varepsilon := L_\gamma(s) - L_\gamma(\mathfrak{d}_\beta(s)) \prec 1$. We define*

$$L_\beta(s) := L_\beta(\mathfrak{d}_\beta(s)) + \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon).$$

As discussed at the beginning of the section, the series $\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon)$ exists in $\mathbb{T}$ by Lemmas 4.1.1 and 4.1.2. To prove Proposition 4.3.3 all that remains is to show:

Lemma 4.3.9. *The above definition does not depend on the choice of γ.*

Proof. Let s, γ, ε be as in Definition 4.3.8 and suppose that $L_\sigma(s) - L_\sigma(\mathfrak{d}_\beta(s)) \prec 1$ for some $\sigma < \beta$. Set $\delta := L_\sigma(s) - L_\sigma(\mathfrak{d}_\beta(s))$. We need to show that

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon) = \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta).$$

Without loss of generality, we may assume that $\sigma \leqslant \gamma$. Now

$$\begin{aligned} L_\gamma(\mathfrak{d}_\beta(s)) + \varepsilon \;=\; L_\gamma(s) &= \ell_\gamma^{\uparrow\sigma} \circ L_\sigma(s) \\ &= \ell_\gamma^{\uparrow\sigma} \circ (L_\sigma(\mathfrak{d}_\beta(s)) + \delta) \\ &= \ell_\gamma^{\uparrow\sigma} \circ L_\sigma(\mathfrak{d}_\beta(s)) + \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta). \end{aligned}$$

Since $\ell_\gamma^{\uparrow\sigma} \circ L_\sigma(\mathfrak{d}_\beta(s)) = L_\gamma(\mathfrak{d}_\beta(s))$, this yields $\varepsilon = \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta)$. Set

$$F := \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}\big(\ell_\gamma, \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(\ell_\sigma, z)\big), \quad G := \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(\ell_\sigma, z),$$

considered as formal power series $F = \sum_{i \in \mathbb{N}} F_i z^i$ and $G = \sum_{j \in \mathbb{N}} G_j z^j$ in $\mathbb{L}_{<\alpha}[[z]]$. Then

$$\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{d}_\beta(s)), \varepsilon) = \sum_{i \in \mathbb{N}} (F_i \circ \mathfrak{d}_\beta(s))\, \delta^i \quad \text{and} \quad \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(L_\sigma(\mathfrak{d}_\beta(s)), \delta) = \sum_{j \in \mathbb{N}} (G_j \circ \mathfrak{d}_\beta(s))\, \delta^j,$$

so it suffices to show that $F = G$. For each $h \in \mathbb{L}_{<\alpha}^{\preceq}$, we have

$$\begin{aligned} \tilde{F}(h) &= \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}\big(\ell_\gamma, \mathcal{T}_{\ell_\gamma^{\uparrow\sigma}}(\ell_\sigma, h)\big) = \mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(\ell_\gamma, \ell_\gamma^{\uparrow\sigma} \circ (\ell_\sigma + h) - \ell_\gamma) = \ell_\beta^{\uparrow\gamma} \circ (\ell_\gamma + \ell_\gamma^{\uparrow\sigma} \circ (\ell_\sigma + h) - \ell_\gamma) - \ell_\beta \\ &= (\ell_\beta^{\uparrow\gamma} \circ \ell_\gamma^{\uparrow\sigma}) \circ (\ell_\sigma + h) - \ell_\beta = \ell_\beta^{\uparrow\sigma} \circ (\ell_\sigma + h) - \ell_\beta = \mathcal{T}_{\ell_\beta^{\uparrow\sigma}}(\ell_\sigma, h) = \tilde{G}(h), \end{aligned}$$

so $(\widetilde{F - G})(h) = 0$ for all $h \in \mathbb{L}_{<\alpha}^{\preceq}$, and we conclude that $F = G$ by Corollary 2.2.14. □

Theorem 4.3.10. *The function L_β is analytic on $\mathbb{T}^{>,\succ}$ with*

$$\begin{aligned} \operatorname{Conv}(L_\beta)_s &\supseteq \mathbb{T}^{\prec s} \quad \textit{and} \\ L_\beta^{(k)}(s) &= \ell_\beta^{(k)} \circ s \end{aligned}$$

for all $k \in \mathbb{N}$ and $s \in \mathbb{T}^{>,\succ}$.

Proof. Let $\gamma<\beta$. For all $\mathfrak{a},\mathfrak{b}\in\mathfrak{M}_\beta$ with $\mathfrak{a}\prec\mathfrak{b}$, we have $L_\gamma(\mathfrak{a})\prec L_\gamma(\mathfrak{b})$ so

$$L_\gamma(\mathfrak{a})+\mathbb{T}^{\prec L_\gamma(\mathfrak{a})}\cap L_\gamma(\mathfrak{b})+\mathbb{T}^{\prec L_\gamma(\mathfrak{b})}=\varnothing.$$

In particular have a well-defined function

$$\begin{array}{rccc} L_\beta^{\uparrow\gamma}\colon & \bigsqcup_{\mathfrak{a}\in\mathfrak{M}_\beta}(L_\gamma(\mathfrak{a})+\mathbb{T}^{\prec}) & \longrightarrow & \mathbb{T}^{>,\succ} \\ & L_\gamma(\mathfrak{a})+\varepsilon & \longmapsto & L_\beta(\mathfrak{a})+\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(\mathfrak{a}),\varepsilon) \end{array}$$

By Corollary 2.3.7, the function $L_\beta^{\uparrow\gamma}$ is analytic. We also have

$$(L_\beta^{\uparrow\gamma})^{(k)}\colon s\longmapsto(\ell_\beta^{\uparrow\gamma})^{(k)}\circ s$$

for all $k\in\mathbb{N}$ by Proposition 2.3.6. Since $\mathfrak{M}$ is (divisible hence) densely ordered, we may apply Proposition 2.3.8 to $(L_\beta^{\uparrow\gamma})$ and L_γ at each s in the class

$$\mathbf{O}_\gamma:=\bigsqcup_{\mathfrak{a}\in\mathfrak{M}_\beta}\{t\in\mathbb{T}^{>,\succ}:L_\gamma(t)-L_\gamma(\mathfrak{a})\prec 1\}.$$

Indeed, combining Lemma 4.3.2 and the identity $L_\beta\upharpoonright\mathbf{O}_\gamma=((L_\beta^{\uparrow\gamma})\circ L_\gamma)\upharpoonright\mathbf{O}_\gamma$ of Lemma 4.3.9, we obtain that L_β is analytic on $\mathbf{O}_\gamma$ and that for all $s\in\mathbf{O}_s$, we have

$$\begin{array}{rcll} \operatorname{Conv}(L_\beta)_s & \supseteq & \mathbb{T}^{\prec s}, \quad \text{and} & \\ L_\beta'(s) & = & \ell_\gamma'\circ s\times((\ell_\beta^{\uparrow\gamma})'\circ\ell_\gamma\circ s) & \text{(by Proposition 2.1.3)} \\ & = & \ell_\beta'\circ s. & \text{(by the chain rule in } \mathbb{L}) \end{array}$$

Recall that $\mathcal{A}_f^{(k)}=\mathcal{A}_{f^{(k)}}$ for all $f\in\mathbb{L}_{<\beta}$ and $k\in\mathbb{N}$. We deduce that $L_\beta^{(k)}(s)=\ell_\beta^{(k)}\circ s$ for all $k\in\mathbb{N}$ and $s\in\mathbf{O}_\gamma$. For $\mathfrak{a}\in\mathfrak{M}_\beta$, we have $\mathcal{E}_\beta[\mathfrak{a}]=\bigcup_{\gamma<\beta}\mathbf{O}_\gamma$ by Proposition 4.3.5. It follows that $\mathbb{T}^{>,\succ}=\bigcup_{\gamma<\beta}\mathbf{O}_\gamma$, thus concluding the proof. □

4.3.3 Properties of extended hyperlogarithms

We end this section with extensions of our monotonicity and functional equations axioms.

Proposition 4.3.11. *Assume μ is a successor. For $s\in\mathbb{T}^{>,\succ}$, we have $L_\beta(L_{\omega^{\mu_-}}(s))=L_\beta(s)-1$.*

Proof. By Corollary 4.3.7, there is some $n\in\mathbb{N}^>$ such that $\varepsilon:=L_{\omega^{\mu_-}n}(s)-L_{\omega^{\mu_-}n}(\mathfrak{d}_\beta(s))\prec 1$. We may write

$$L_{\omega^{\mu_-}(n-1)}(L_{\omega^{\mu_-}}(s))=L_{\omega^{\mu_-}(n-1)}(L_{\omega^{\mu_-}}(\mathfrak{d}_\beta(s)))+\varepsilon.$$

Note that $L_{\omega^{\mu_-}}(\mathfrak{d}_\beta(s))$ is $L_{<\beta}$-atomic, so $\mathfrak{d}_\beta(L_{\omega^{\mu_-}}(s))=L_{\omega^{\mu_-}}(\mathfrak{d}_\beta(s))$. For $k\in\mathbb{N}^>$ we have

$$\left(\ell_\beta^{\uparrow\omega^{\mu_-}(n-1)}\right)^{(k)}=(\ell_\beta+(n-1))^{(k)}=\ell_\beta^{(k)}=(\ell_\beta+n)^{(k)}=(\ell_\beta^{\uparrow\omega^{\mu_-}n})^{(k)},$$

so $\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu_-}(n-1)}}(\mathfrak{a},\varepsilon)=\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu_-}n}}(\mathfrak{a},\varepsilon)$ for all $\mathfrak{a}\in\mathfrak{M}_\beta$. It follows that

$$\begin{array}{rcll} L_\beta(L_{\omega^{\mu_-}}(s)) & = & L_\beta(L_{\omega^{\mu_-}}(\mathfrak{d}_\beta(s)))+\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu_-}(n-1)}}(L_{\omega^{\mu_-}n}(\mathfrak{d}_\beta(s)),\varepsilon) & \text{(by Definition 4.3.8)} \\ & = & L_\beta(\mathfrak{d}_\beta(s))-1+\mathcal{T}_{\ell_\beta^{\uparrow\omega^{\mu_-}n}}(L_{\omega^{\mu_-}n}(\mathfrak{d}_\beta(s)),\varepsilon) & \text{(by } \mathbf{FE}_{\boldsymbol{\mu}}) \\ & = & L_\beta(s)-1 & \text{(by Definition 4.3.8)} \end{array}$$

This concludes the proof. □

Lemma 4.3.12. *The function L_β is strictly increasing on $\mathbb{T}^{>,\succ}$.*

Proof. By induction on μ, we may assume that L_{ω^η} is strictly increasing on $\mathbb{T}^{>,\succ}$ for all $\eta<\mu$ (the $\eta=0$ case follows from Proposition 3.1.10). As a composition of strictly increasing functions is strictly increasing, the function L_γ is strictly increasing on $\mathbb{T}^{>,\succ}$ for all $\gamma<\beta$. Given $s<t\in\mathbb{T}^{>,\succ}$, let us show that $L_\beta(s)<L_\beta(t)$. We start with the case when $\mathfrak{d}_\beta(s)=\mathfrak{d}_\beta(t)=:\mathfrak{a}$ and take $\gamma<\beta$ with $L_\gamma(s)-L_\gamma(\mathfrak{a})\prec 1$ and $L_\gamma(t)-L_\gamma(\mathfrak{a})\prec 1$. Then $\varepsilon:=L_\gamma(t)-L_\gamma(s)$ is infinitesimal and positive by our induction hypothesis. By Theorem 4.3.10 we have

$$L_\beta(t)-L_\beta(s)=\mathcal{T}_{\ell_\beta^{\uparrow\gamma}}(L_\gamma(s),\varepsilon)\sim((\ell_\beta^{\uparrow\lambda})'\circ L_\lambda(s))\,\varepsilon.$$

Since $\ell_\beta^{\uparrow\lambda} > \mathbb{R}$, we have $(\ell_\beta^{\uparrow\lambda})' > 0$, so $L_\beta(t) - L_\beta(s) > 0$.

Now we turn to the case when $\mathfrak{d}_\beta(s) \prec \mathfrak{d}_\beta(t)$. Set $\mathfrak{a} := \mathfrak{d}_\beta(s)$ and $\mathfrak{b} := \mathfrak{d}_\beta(t)$ and take an ordinal $\lambda := \omega^\eta n < \beta$ with

$$L_\lambda(s) - L_\lambda(\mathfrak{a}) \prec 1 \quad \text{and} \quad L_\lambda(t) - L_\lambda(\mathfrak{b}) \prec 1.$$

Set $\delta := L_\lambda(s) - L_\lambda(\mathfrak{a})$, so

$$L_\beta(s) - L_\beta(\mathfrak{a}) = \mathcal{T}_{\ell_\beta^{\uparrow\lambda}}(L_\lambda(\mathfrak{a}), \delta) \sim ((\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(\mathfrak{a}))\, \delta \prec (\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(\mathfrak{a})$$

Repeated applications of (4.1.4) with η in place of μ gives $\ell_\beta^{\uparrow\lambda} \sim \ell_\beta$, so $(\ell_\beta^{\uparrow\lambda})' \sim \ell_\beta'$ and

$$(\ell_\beta^{\uparrow\lambda})' \circ L_\lambda(\mathfrak{a}) \sim \ell_\beta' \circ L_\lambda(\mathfrak{a}).$$

Since $\beta > 1$, we have $\ell_\beta \prec \ell_1$ so $\ell_\beta' \prec \ell_1' = \ell_0^{-1}$. Thus, $\ell_\beta' \circ L_\lambda(\mathfrak{a}) \prec L_\lambda(\mathfrak{a})^{-1}$. All together, this shows that $L_\beta(s) - L_\beta(\mathfrak{a}) \prec L_\lambda(\mathfrak{a})^{-1}$. Likewise, we have $L_\beta(t) - L_\beta(\mathfrak{b}) \prec L_\lambda(\mathfrak{b})^{-1}$. By the monotonicity axiom $\mathbf{M}_\mu$, we have $L_\beta(\mathfrak{a}) + L_\lambda(\mathfrak{a})^{-1} < L_\beta(\mathfrak{b}) - L_\lambda(\mathfrak{b})^{-1}$, so $L_\beta(s) < L_\beta(t)$. □

4.4 Defining the external composition law

Throughout this section, ν stands for a fixed ordinal and $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ for a fixed confluent hyperserial skeleton of force ν. We now set

$$\alpha := \omega^\nu.$$

Our aim is to construct a well-behaved external composition $\mathbb{L}_{<\alpha} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ that satisfies $\mathbf{C1}_\nu$, $\mathbf{C2}_\nu$, $\mathbf{C3}_\nu$, and $\mathbf{C4}_\nu$ from Theorem 4.3.1. We will also prove that the mapping $\mathfrak{L}_{<\alpha} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ s$ has relatively well-based support for all $s \in \mathbb{T}^{>,\succ}$. Throughout the section, we make the inductive assumption that Theorem 4.3.1 holds for all $\mu < \nu$ and that the mapping $\mathfrak{L}_{<\omega^\mu} \longrightarrow \mathbb{T}; \mathfrak{l} \mapsto \mathfrak{l} \circ s$ has relatively well-based support for all $\mu < \nu$ and $s \in \mathbb{T}^{>,\succ}$.

4.4.1 The case when $\nu = 0$

Here $\mathbb{T}$ is a 0-confluent hyperserial skeleton of force 0. The field $\mathbb{L}_{<1} = \mathbb{R}[[x^{\mathbb{R}}]] \cong \mathbb{R}[[\ell_0^{\mathbb{R}}]]$ is the field of well-based series of real powers of the variable ℓ_0, with real coefficients. We have already seen (see Theorem 2.5.12) that the real powering operation on $\mathbb{T}^>$ extends into a calculus of power series $\mathbb{R}[[x^{\mathbb{R}}]] \times \mathbb{T}^> \longrightarrow \mathbb{T}$ which in particular restricts to a composition law satisfying $\mathbf{C1}_0$, $\mathbf{C2}_0$, $\mathbf{C3}_0$, and $\mathbf{C4}_0$. In order to complete the proof of Theorem 4.3.1 for $\nu = 0$, it remains to show uniqueness.

Proposition 4.4.1. *The function $\circ$ is unique to satisfy $\mathbf{C1}_0$, $\mathbf{C2}_0$, $\mathbf{C3}_0$, and $\mathbf{C4}_0$.*

Proof. Let $\bullet$ be a composition satisfying conditions $\mathbf{C1}_0$, $\mathbf{C2}_0$, $\mathbf{C3}_0$, and $\mathbf{C4}_0$. Write $s = c\,\mathfrak{m}\,(1 + \varepsilon) \in \mathbb{T}^{>,\succ}$, where $c \in \mathbb{R}^{\neq}$, $\mathfrak{m} := \mathfrak{d}_s$, and $\varepsilon \prec 1$. By strong linearity, it suffices to verify that $\ell_0^r \bullet s = s^r$ for any monomial in $\mathfrak{L}_{<1}$. Given $r \in \mathbb{R}$, the condition $\mathbf{C4}_0$ implies

$$\ell_0^r \bullet s = \ell_0^r \bullet (c\,\mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \frac{(\ell_0^r)^{(k)} \bullet c\,\mathfrak{m}}{k!} (c\,\mathfrak{m}\,\varepsilon)^k.$$

We have $(\ell_0^r)^{(k)} = k! \binom{r}{k} \ell_0^{r-k}$, so

$$\ell_0^r \bullet s = \ell_0^r \bullet (c\,\mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \binom{r}{k} (\ell_0^{r-k} \bullet c\,\mathfrak{m})(c\,\mathfrak{m}\,\varepsilon)^k.$$

We have $\ell_0^r \bullet (c\,\mathfrak{m}) = \ell_0^r \bullet (c\,\ell_0 \bullet \mathfrak{m}) = (\ell_0^r \circ (c\,\ell_0)) \bullet \mathfrak{m} = c^r (\ell_0^r \bullet \mathfrak{m})$ by $\mathbf{C3}_0$ and $\ell_0^r \bullet \mathfrak{m} = \mathfrak{m}^r$ by $\mathbf{C2}_0$, so $\ell_0^r \bullet (c\,\mathfrak{m}) = (c\,\mathfrak{m})^r$. Likewise, $\ell_0^{r-k} \bullet (c\,\mathfrak{m}) = (c\,\mathfrak{m})^{r-k}$, so

$$\ell_0^r \bullet s = (c\,\mathfrak{m})^r + \sum_{k \in \mathbb{N}^>} \binom{r}{k} (c\,\mathfrak{m})^{r-k} (c\,\mathfrak{m}\,\varepsilon)^k = (c\,\mathfrak{m})^r \left(1 + \sum_{k \in \mathbb{N}^>} \binom{r}{k} \varepsilon^k\right) = s^r. \qquad \square$$

4.4.2 $\mathbf{C1}_\nu$ and $\mathbf{C2}_\nu$ for $\nu > 0$

For the remainder of this section, we assume that $\nu > 0$. By the results in Section 4.3.2, we have a well-defined extension of L_γ to all of $\mathbb{T}^{>,\succ}$ for each $\gamma < \alpha$. Indeed, for $s \in \mathbb{T}^{>,\succ}$ and $\gamma < \alpha$, take n with $\gamma = \omega^{\nu_-} n + \sigma$ with $\sigma < \omega^{\nu_-}$ (so $n = 0$ if ν is a limit). Then we may set $L_\gamma(s) := L_\sigma(L_{\omega^{\nu_-}}^{\circ n}(s))$.

Given $\mathfrak{a} \in \mathfrak{M}_\alpha$ and $\mathfrak{l} = \prod_{\gamma<\alpha} \ell_\gamma^{r_\gamma} \in \mathfrak{L}_{<\alpha}$, we have by $\mathbf{P}_\nu$ that $\sum_{\gamma<\alpha} r_\gamma L_{\gamma+1}(\mathfrak{a}) \in \log \mathfrak{M}$, so we set $\mathfrak{l} \circ \mathfrak{a} := \exp(\sum_{\gamma<\alpha} r_\gamma L_{\gamma+1}(\mathfrak{a})) \in \mathfrak{M}$. Clearly, the map $\mathfrak{L}_{<\alpha} \longrightarrow \mathfrak{M}; \mathfrak{l} \mapsto \mathfrak{l} \circ \mathfrak{a}$ is an embedding of monomial groups which preserves real powers, and by $\mathbf{A}_\nu$, this embedding is order-preserving as well. For $f \in \mathbb{L}_{<\alpha}$, we set $f \circ \mathfrak{a} := \sum_{\mathfrak{l} \in \mathfrak{L}_{<\alpha}} f_{\mathfrak{l}} (\mathfrak{l} \circ \mathfrak{a})$. By Proposition 1.3.2, we have:

Lemma 4.4.2. *The map $\mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}; f \mapsto f \circ \mathfrak{a}$ is a strongly linear ordered field embedding.*

Proposition 4.4.3. *For $\rho < \alpha$ and $\mathfrak{a} \in \mathfrak{M}_\beta$, the function L_ρ is analytic at $\mathfrak{a}$ with*

$$\begin{aligned} \operatorname{Conv}(L_\rho)_{\mathfrak{a}} &\supseteq \mathfrak{a} + \mathbb{T}^{\prec \mathfrak{a}} \quad \textit{and} \\ L_\rho^{(k)}(\mathfrak{a}) &= \ell_\rho^{(k)} \circ \mathfrak{a} \end{aligned}$$

for all $k \in \mathbb{N}$.

Proof. If ν is a limit ordinal, then this follows from $\mathbf{C4}_\mu$ for any ordinal μ with $\rho < \omega^\mu$, so we may assume that ν is a successor. The lemma is immediate when $\rho = 0$, so suppose $\rho > 0$ and take $n \in \mathbb{N}$ and $0 < \gamma \leqslant \omega^{\nu_-}$ with $\rho = \omega^{\nu_-} n + \gamma$. We have

$$L_\rho = L_\gamma \circ L_{\omega^{\nu_-}}^{\circ n}.$$

Now Lemma 4.3.2 on the one hand, and Proposition 3.1.8 Theorem 4.3.10 on the other, give that L_γ and $L_{\omega^{\nu_-}}$ are analytic with $\operatorname{Conv}(L_{\omega^{\nu_-}})_s \supseteq s + \mathbb{T}^{\prec s}$ and $\operatorname{Conv}(L_\gamma)_s \supseteq s + \mathbb{T}^{\prec s}$ for all $s \in \mathbb{T}^{>,\succ}$. Since moreover we have

$$L_{\omega^{\nu_-}}(\mathfrak{a} + \delta) \sim L_{\omega^{\nu_-}}(\mathfrak{a}) \qquad \text{and} \qquad L_\gamma(\mathfrak{a} + \delta) \sim L_\gamma(\mathfrak{a})$$

for all $\delta \in \mathbb{T}^{\prec \mathfrak{a}}$, we may iteratively apply Proposition 2.3.8 and deduce that the same is true of L_ρ. By the chain rule in $\mathbb{L}$ and in ordered fields (Proposition 2.1.3), we also obtain that $L_\rho^{(k)}(\mathfrak{a}) = \ell_\rho^{(k)} \circ \mathfrak{a}$ for all $s \in \mathbb{T}^{>,\succ}$. □

In the general situation when $s \in \mathbb{T}^{>,\succ}$, our next goal is to show that the family $(L_{\gamma+1}(s))_{\gamma<\alpha}$ is well-based. For the remainder of this subsection, we fix $s \in \mathbb{T}^{>,\succ}$. By ν-confluence and Corollary 4.3.7, take $n \in \mathbb{N}$ and $\mu < \nu$ such that $L_\gamma(s) - L_\gamma(\mathfrak{d}_\alpha(s)) \prec 1$ for all $\omega^\mu n \leqslant \gamma < \alpha$. If ν is a successor, we can arrange that $\mu = \nu_-$. Set $\mathfrak{a} := \mathfrak{d}_\alpha(s)$, set $\varepsilon := L_\gamma(s) - L_\gamma(\mathfrak{d}_\alpha(s))$, and set $\beta := \omega^\mu$.

Lemma 4.4.4. *Let $f \in \mathbb{L}_{<\alpha}$ and let $m \in \mathbb{N}$. If ν is a successor or $f \in \bigcup_{\eta<\nu} \mathbb{L}_{<\omega^\eta}$, then the expression $f \circ L_{\beta m}(\mathfrak{a})$ is defined and equal to $(f \circ \ell_{\beta m}) \circ \mathfrak{a}$.*

Proof. Suppose ν is a successor ordinal, so $\beta = \omega^{\nu_-}$. Then $L_{\beta m}(\mathfrak{a}) \in \mathfrak{M}_\alpha$, so $f \circ L_{\beta m}(\mathfrak{a})$ is defined. As the maps $f \longmapsto (f \circ \ell_{\beta m}) \circ \mathfrak{a}$ and $f \longmapsto f \circ L_{\beta m}(\mathfrak{a})$ are strongly linear, we may assume that f is a monomial $\mathfrak{l} = \prod_{\gamma<\alpha} \ell_\gamma^{\mathfrak{l}_\gamma}$. Since $\ell_\gamma \circ \ell_{\beta m} = \ell_{\beta m + \gamma}$ for $\gamma < \alpha$, we have

$$(\mathfrak{l} \circ \ell_{\beta m}) \circ \mathfrak{a} = \left(\prod_{\gamma<\alpha} \ell_{\beta m+\gamma}^{\mathfrak{l}_\gamma} \right) \circ \mathfrak{a} = \exp\left(\sum_{\gamma<\alpha} \mathfrak{l}_\gamma L_{\beta m+\gamma}(\mathfrak{a}) \right) = \exp\left(\sum_{\gamma<\alpha} \mathfrak{l}_\gamma L_\gamma(L_{\beta m}(\mathfrak{a})) \right) = \mathfrak{l} \circ L_{\beta m}(\mathfrak{a}).$$

Now suppose that ν is a limit and that $f \in \mathbb{L}_{<\omega^\eta}$ for some $\eta < \nu$. By increasing η, we may assume that $\beta m < \omega^\eta$, so $f, \ell_{\beta m} \in \mathbb{L}_{<\omega^\eta}$. Then $\mathbf{C2}_\eta$ and $\mathbf{C3}_\eta$ give

$$(f \circ \ell_{\beta m}) \circ \mathfrak{a} = f \circ (\ell_{\beta m} \circ \mathfrak{a}) = f \circ L_{\beta m}(\mathfrak{a}).$$ □

We rely on the following technical lemma in order to have a precise description of the support of $L_\rho(s)$ for certain ordinals ρ.

Lemma 4.4.5. *There is a well-based family* $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ *from* $\mathbb{L}_{[\beta n,\alpha)}^{\prec}$ *such that*

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k$$

for each γ *with* $\beta n\leqslant\gamma<\alpha$.

Proof. Fix γ with $\beta n\leqslant\gamma<\alpha$. We first claim that

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon).$$

If ν is a limit, then take η with $\gamma<\omega^\eta<\alpha$. Then $\ell_{\gamma+1}^{\uparrow\beta n},\ell_{\beta n}\in\mathbb{L}_{<\omega^\eta}$, so $\mathbf{C4}_\eta$ gives

$$L_{\gamma+1}(s)=\ell_{\gamma+1}^{\uparrow\beta n}\circ L_{\beta n}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon)$$

and $\mathbf{C3}_\eta$ gives $\ell_{\gamma+1}^{\uparrow\beta n}\circ L_{\beta n}(\mathfrak{a})=L_{\gamma+1}(\mathfrak{a})$, thereby proving the claim. If ν is a successor, then take $\rho<\alpha$ with $\gamma+1=\beta\,n+\rho$. Since $L_{\beta n}(\mathfrak{a})$ is $L_{<\omega^\nu}$-atomic, Proposition 4.4.3 and the fact that $\ell_{\gamma+1}^{\uparrow\beta n}=\ell_\rho$, yield

$$\begin{aligned}L_{\gamma+1}(s)&=L_\rho(L_{\beta n}(s))=L_\rho(L_{\beta n}(\mathfrak{a}))+\mathcal{T}_{\ell_\rho}(L_{\beta n}(\mathfrak{a}),\varepsilon)\\&=L_{\gamma+1}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon).\end{aligned}$$

Having proved our claim, let $k>0$ be given and set $f_{\gamma,k}:=\frac{1}{k!}(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\circ\ell_{\beta n}\in\mathbb{L}_{[\beta n,\alpha)}$. Lemma 4.1.2 yields $(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\prec 1$, whence $f_{\gamma,k}\prec 1$. If ν is a limit, then $(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\in\mathbb{L}_{<\omega^\eta}$, where η is as above. So in both the successor and limit cases, we may apply Lemma 4.4.4 with $(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}$ in place of f to get

$$f_{\gamma,k}\circ\mathfrak{a}=\frac{1}{k!}(\ell_{\gamma+1}^{\uparrow\beta n})^{(k)}\circ L_{\beta n}(\mathfrak{a}).$$

This implies that

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\mathcal{T}_{\ell_{\gamma+1}^{\uparrow\beta n}}(L_{\beta n}(\mathfrak{a}),\varepsilon)=L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k.$$

It remains to show that the family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is well-based. Since $(\ell_{\gamma+1})_{\beta n\leqslant\gamma<\alpha}$ is a well-based family in $\mathbb{L}_{<\alpha}$ and $\mathbb{L}_{[\beta n,\alpha)}\longrightarrow\mathbb{L}_{<\alpha};f\mapsto f^{\uparrow\beta n}$ is strongly linear, the family $(\ell_{\gamma+1}^{\uparrow\beta n})_{\beta n\leqslant\gamma<\alpha}$ is well-based. Since $\operatorname{supp}_*\partial$ is well-based and infinitesimal, the family $((\ell_{\gamma+1}^{\uparrow\beta n})^{(k)})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is well-based. We conclude that the family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is well-based. □

Proposition 4.4.6. *Let* $(r_\gamma)_{\gamma<\alpha}$ *be a sequence of real numbers. Then the family* $(L_{\gamma+1}(s))_{\gamma<\alpha}$ *is well-based and the series* $\sum_{\gamma<\alpha}r_\gamma L_{\gamma+1}(s)$ *lies in* $\log\mathbb{T}^>$.

Proof. We will show the following:

a) For each $k<n$, the family $(L_{\gamma+1}(s))_{\beta k\leqslant\gamma<\beta(k+1)}$ is well-based and

$$\sum_{\beta k\leqslant\gamma<\beta(k+1)}r_\gamma L_{\gamma+1}(s)\ \in\ \log\mathbb{T}^>.$$

b) The family $(L_{\gamma+1}(s))_{\beta n\leqslant\gamma<\alpha}$ is well-based and

$$\sum_{\beta n\leqslant\gamma<\alpha}r_\gamma L_{\gamma+1}(s)\ \in\ \log\mathbb{T}^>.$$

The proposition follows from (a) and (b), since the union of finitely many well-based families is well-based and $\log\mathbb{T}^>$ is closed under finite sums.

To see why (a) holds, let $k<n$ and note that

$$(L_{\gamma+1}(s))_{\beta k\leqslant\gamma<\beta(k+1)}=(L_{\rho+1}(L_{\beta k}(s)))_{\rho<\beta}.$$

Since $(\ell_{\rho+1})_{\rho<\beta}$ is well-based, $\mathbf{C1}_\mu$ gives that $(L_{\rho+1}(L_{\beta k}(s)))_{\rho<\beta}=(\ell_{\rho+1}\circ L_{\beta k}(s))_{\rho<\beta}$ is well-based. We have

$$\sum_{\beta k\leqslant\gamma<\beta(k+1)}r_\gamma L_{\gamma+1}(s)=\sum_{\rho<\beta}r_{\beta k+\rho}L_{\rho+1}(L_{\beta k}(s)).$$

Set $\mathfrak{l}:=\prod_{\rho<\beta}\ell_\rho^{r_{\beta k+\rho}}\in\mathfrak{L}_{<\beta}$. We claim that $\sum_{\rho<\beta} r_{\beta k+\rho}\,L_{\rho+1}(L_{\beta k}(s))=\log(\mathfrak{l}\circ L_{\beta k}(s))$. If $\mu=0$, then $\mathfrak{l}=\ell_0^{r_k}$ and

$$\sum_{\rho<\beta} r_{\beta k+\rho}\,L_{\rho+1}(L_{\beta k}(s))=r_k\,L_1(L_k(s))=\log(L_k(s)^{r_k})=\log(\mathfrak{l}\circ L_{\beta k}(s)).$$

If $\mu>0$, then $\mathbf{C3}_\mu$ gives

$$\sum_{\rho<\beta} r_{\beta k+\rho}\,L_{\rho+1}(L_{\beta k}(s))=\log(\mathfrak{l})\circ L_{\beta k}(s)=\log(\mathfrak{l}\circ L_{\beta k}(s)).$$

As for (b), let $\varepsilon:=L_{\beta n}(s)-L_{\beta n}(\mathfrak{a})$. By Lemma 4.4.5, there exists a well-based family $(f_{\gamma,k})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ from $\mathbb{L}_{[\beta n,\alpha)}^{\prec}$ such that

$$L_{\gamma+1}(s)=L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k.$$

The families $(L_{\gamma+1}(\mathfrak{a}))_{\beta n\leqslant\gamma<\alpha}$ and $(f_{\gamma,k}\circ\mathfrak{a})_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ are well-based by Lemma 4.4.2 and the fact that $\mathfrak{a}\in\mathfrak{M}_\alpha$. Since the family $(\varepsilon^k)_{k\in\mathbb{N}}$ is also well-based, it follows that $((f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k)_{\beta n\leqslant\gamma<\alpha,k\in\mathbb{N}^>}$ is again well-based. In particular,

$$(L_{\gamma+1}(s))_{\beta n\leqslant\gamma<\alpha}=\left(L_{\gamma+1}(\mathfrak{a})+\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k\right)_{\beta n\leqslant\gamma<\alpha}$$

is well-based. Now

$$\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\,L_{\gamma+1}(s)=\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\,L_{\gamma+1}(\mathfrak{a})+\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\sum_{k\in\mathbb{N}^>}(f_{\gamma,k}\circ\mathfrak{a})\,\varepsilon^k.$$

Since $f_{\gamma,k}$ and ε^k are infinitesimal for all $k>0$, we may write

$$\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\,L_{\gamma+1}(s)=\left(\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\,L_{\gamma+1}(\mathfrak{a})\right)+\delta,$$

where $\delta\in\mathbb{T}^{\prec}$. By (3.1.1), we have $\delta=L(E(\delta))\in\log\mathbb{T}^>$. Furthermore, P_ν implies

$$\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\,L_{\gamma+1}(\mathfrak{a})\in\log\mathfrak{M}\subseteq\log\mathbb{T}^>.$$

We conclude that $\sum_{\beta n\leqslant\gamma<\alpha} r_\gamma\,L_{\gamma+1}(s)\in\log\mathbb{T}^>$. □

Let $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\alpha}$. In light of Proposition 4.4.6, we define

$$\mathfrak{l}\circ s\ :=\ \exp\left(\sum_{\gamma<\alpha}\mathfrak{l}_\gamma\,L_{\gamma+1}(s)\right).$$

We note that the map $\mathfrak{L}_{<\alpha}\longrightarrow\mathbb{T}^>;\mathfrak{l}\longmapsto\mathfrak{l}\circ s$ is an embedding of ordered multiplicative groups for each $s\in\mathbb{T}^{>,\succ}$.

Our next objective is to show that the map $\mathfrak{L}_{<\alpha}\longrightarrow\mathbb{T};\mathfrak{l}\longmapsto\mathfrak{l}\circ s$ extends by strong linearity to a map $\mathbb{L}_{<\alpha}\longrightarrow\mathbb{T}$ which satisfies $\mathbf{C1}_\nu$ and $\mathbf{C2}_\nu$. For this, we will show that $\mathfrak{l}\longmapsto\mathfrak{l}\circ s$ is a relatively well-based mapping, by using a similar "gluing" technique as for Proposition 4.4.6. Recall that our second induction hypothesis from the beginning of this section stipulated that the mapping $\mathfrak{L}_{<\omega^\mu}\longrightarrow\mathbb{T};\mathfrak{l}\longmapsto\mathfrak{l}\circ s$ is relatively well-based for all $\mu<\nu$ and $s\in\mathbb{T}^{>,\succ}$.

Proposition 4.4.7. *Let $\Phi:\mathfrak{L}_{<\alpha}\longrightarrow\mathbb{T}$ be the map $\Phi(\mathfrak{l}):=\mathfrak{l}\circ s$. Then Φ is relatively well-based.*

Proof. Let $\Phi_{\geqslant n}$ be the restriction of Φ to $\mathfrak{L}_{[\beta n,\alpha)}$ and for $k<n$, let Φ_k be the restriction of Φ to $\mathfrak{L}_{[\beta k,\beta(k+1))}$. Since

$$\operatorname{supp}_\odot\Phi\subseteq(\operatorname{supp}_\odot\Phi_0)\cdots(\operatorname{supp}_\odot\Phi_{n-1})\,(\operatorname{supp}_\odot\Phi_{\geqslant n}),$$

it suffices to show that each Φ_k and $\Phi_{\geqslant n}$ are relatively well-based. For the Φ_k, fix $k<n$. Our induction hypothesis implies that the map $\Psi_k: \mathfrak{L}_{[0,\beta)} \longrightarrow \mathbb{T}; \mathfrak{l} \longmapsto \mathfrak{l} \circ L_{\beta k}(s)$ is relatively well-based. By Lemma 4.4.4 with $\mathfrak{l}$ in place of f, we have

$$\Phi_k(\mathfrak{l} \circ \ell_{\beta k}) = (\mathfrak{l} \circ \ell_{\beta k}) \circ s = \mathfrak{l} \circ L_{\beta k}(s) = \Psi_k(\mathfrak{l}).$$

It follows that Φ_k is also relatively well-based with $\operatorname{supp}_{\odot} \Phi_k = \operatorname{supp}_{\odot} \Psi_k$.

Now for $\Phi_{\geqslant n}$. Let $\mathfrak{l} = \prod_{\beta n < \gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{[\beta n, \alpha)}$. By Lemma 4.4.5, we have a well-based family $(f_{\gamma,k})_{\beta n \leqslant \gamma < \alpha, k \in \mathbb{N}^>}$ from $\mathbb{L}^{\prec}_{[\beta n, \alpha)}$ such that

$$\log(\Phi_{\geqslant n}(\mathfrak{l})) = \sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(s) = \sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(\mathfrak{a}) + \sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma \sum_{k \in \mathbb{N}^>} (f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k.$$

Exponentiating both sides, we obtain

$$\Phi_{\geqslant n}(\mathfrak{l}) = (\mathfrak{l} \circ \mathfrak{a})\, E\left(\sum_{\beta n \leqslant \gamma < \alpha} \mathfrak{l}_\gamma \sum_{k \in \mathbb{N}^>} (f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k \right)$$

so $\mathfrak{d}_{\Phi_{\geqslant n}(\mathfrak{l})} = \mathfrak{l} \circ \mathfrak{a}$. The set

$$\mathfrak{E} := \bigcup_{\beta n \leqslant \gamma < \alpha, k \in \mathbb{N}^>} \operatorname{supp}\left((f_{\gamma,k} \circ \mathfrak{a})\, \varepsilon^k \right)$$

is well-based, infinitesimal and does not depend on $\mathfrak{l}$. Since

$$\frac{\operatorname{supp} \Phi_{\geqslant n}(\mathfrak{l})}{\mathfrak{d}_{\Phi_{\geqslant n}(\mathfrak{l})}} \subseteq \mathfrak{E}^\infty$$

for all $\mathfrak{l} \in \mathfrak{L}_{[\beta n, \alpha)}$, we conclude that $\operatorname{supp}_{\odot} \Phi_{\geqslant n} \subseteq \mathfrak{E}^\infty$ is well-based. □

We already noted that the map Φ from Proposition 4.4.7 is an order-preserving multiplicative embedding. By Proposition 1.3.7, it follows that Φ is well-based, so it extends uniquely into an order-preserving and strongly linear embedding $\hat{\Phi}: \mathbb{L}_{<\alpha} \longrightarrow \mathbb{T}$. Taking $f \circ s := \hat{\Phi}(f)$ for all $f \in \mathbb{L}_{<\alpha}$, this proves **C1$_\nu$**. By construction, we also have **C2$_\nu$**. Note that $\circ$ extends the unique composition $\mathbb{L}_{<\omega^\eta} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ of Theorem 4.3.1 for $\eta < \nu$.

4.4.3 Properties C3$_\nu$ and C4$_\nu$ and uniqueness for $\nu > 0$

Let $\varepsilon \in \mathbb{T}^{\prec}$. By [62, Corollary 16], we have

$$\tilde{E}(r\, \tilde{L}(\varepsilon)) = \sum_{k \in \mathbb{N}} \binom{r}{k} \varepsilon^k = (1+\varepsilon)^r \quad \text{for all } r \in \mathbb{R}. \tag{4.4.1}$$

Proposition 4.4.8. *For $s \in \mathbb{T}^>$, and $r \in \mathbb{R}$, we have $\log s^r = r \log s$.*

Proof. First, note that $\log \mathfrak{m}^r = r \log \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{M}$: if $\mathfrak{m} \succ 1$, then this is just axiom **FE$_0$**; if $\mathfrak{m} \prec 1$, then $\log \mathfrak{m}^r = -\log \mathfrak{m}^{-r} = r \log \mathfrak{m}$; if $\mathfrak{m} = 1$, then $\log \mathfrak{m}^r = 0 = r \log \mathfrak{m}$. Now, writing $s = c\, \mathfrak{m}\, (1+\varepsilon)$ with $c \in \mathbb{R}^>$, $\mathfrak{m} := \mathfrak{d}_s$, and $\varepsilon \prec 1$, we have

$$\begin{aligned} \log(s^r) &= \log(\mathfrak{m}^r) + \log c^r + \tilde{L}((\varepsilon)^r) \\ &= \log(\mathfrak{m}^r) + \log c^r + \tilde{L}(\tilde{E}(r\, \tilde{L}(\varepsilon))) && \text{(by (4.4.1))} \\ &= r \log \mathfrak{m} + r \log c + r\, \tilde{L}(\varepsilon) && \text{(by (3.1.1))} \\ &= r \log(s). \end{aligned}$$

□

Proposition 4.4.9. *For $r \in \mathbb{R}, g \in \mathbb{L}^{>,\succ}_{<\alpha}$ and $s \in \mathbb{T}^{>,\succ}$, we have $(\ell_0^r \circ g) \circ s = \ell_0^r \circ (g \circ s)$.*

Proof. As in Proposition 2.5.7, it suffices to prove that $(\mathfrak{l} \circ s)^r = \mathfrak{l}^r \circ s$ holds for each $\mathfrak{l} = \prod_{\gamma < \alpha} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\alpha}$. For such $\mathfrak{l}$, we have

$$\log(\mathfrak{l}^r \circ s) = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma\, r\, L_{\gamma+1}(s).$$

By Proposition 4.4.8, we also have $\log((\mathfrak{l}\circ s)^r) = r\log(\mathfrak{l}\circ s) = r\sum_{\gamma<\omega^\eta}\mathfrak{l}_\gamma L_{\gamma+1}(s)$. By injectivity of the logarithm, we conclude that $(\mathfrak{l}\circ s)^r = \mathfrak{l}^r\circ s$. □

Lemma 4.4.10. *For all $h\in\mathbb{L}^{>}_{<\alpha}$ and all $s\in\mathbb{T}^{>,\succ}$, we have $\log(h\circ s) = (\log h)\circ s$.*

Proof. First, we note that for $\mathfrak{l} = \prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\alpha}$, we have

$$(\log\mathfrak{l})\circ s = \left(\sum_{\gamma<\alpha}\mathfrak{l}_\gamma\ell_{\gamma+1}\right)\circ s = \sum_{\gamma<\alpha}\mathfrak{l}_\gamma L_{\gamma+1}(s) = \log(\mathfrak{l}\circ s),$$

where the last equality uses the definition of $\mathfrak{l}\circ s$. Now, let $h\in\mathbb{L}^{>}_{<\alpha}$ and write $h = c\,\mathfrak{m}\,(1+\varepsilon)$ with $c\in\mathbb{R}^{>}$, $\mathfrak{m} := \mathfrak{d}_h$, and $\varepsilon\prec 1$. Then $h\circ s = c\,(\mathfrak{m}\circ s)(1+\varepsilon\circ s)$ and

$$\begin{aligned}(\log h)\circ s &= (\log\mathfrak{m})\circ s + \log c + \sum_{k\in\mathbb{N}^{>}}\frac{(-1)^{k-1}}{k}\varepsilon^k\circ s\\ &= \log(\mathfrak{m}\circ s) + \log c + \sum_{k\in\mathbb{N}^{>}}\frac{(-1)^{k-1}}{k}(\varepsilon\circ s)^k,\\ &= \log(c\,(\mathfrak{m}\circ s)(1+\varepsilon\circ s)) = \log(h\circ s).\end{aligned}$$

Here we used the facts that $(\log c)\circ s = \log c$ and that composition with s commutes with powers and infinite sums. □

Proposition 4.4.11. *The function $\circ$ satisfies **C3$_\nu$**, i.e. for all $f\in\mathbb{L}_{<\alpha}$, $g\in\mathbb{L}^{>,\succ}_{<\alpha}$, and $s\in\mathbb{T}^{>,\succ}$ we have $f\circ(g\circ s) = (f\circ g)\circ s$.*

Proof. We will show by induction on $\mu\leqslant\nu$ that $f\circ(g\circ s) = (f\circ g)\circ s$ for all $f\in\mathbb{L}_{<\omega^\mu}$, all $g\in\mathbb{L}^{>,\succ}_{<\alpha}$, and all $s\in\mathbb{T}^{>,\succ}$. If $\mu = 0$, then this follows from Proposition 4.4.9 and strong linearity.

Let $\mu > 0$, let g and s be fixed, and assume that the proposition holds whenever $f\in\mathbb{L}_{<\omega^\eta}$ for some $\eta<\mu$. By strong linearity, it suffices to prove that $\mathfrak{l}\circ(g\circ s) = (\mathfrak{l}\circ g)\circ s$ for all $\mathfrak{l} = \prod_{\gamma<\omega^\mu}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\omega^\mu}$. Lemma 4.4.10 gives

$$\begin{aligned}\log(\mathfrak{l}\circ(g\circ s)) &= (\log\mathfrak{l})\circ(g\circ s) = \sum_{\gamma<\omega^\mu}\mathfrak{l}_\gamma\ell_{\gamma+1}\circ(g\circ s),\\ \log((\mathfrak{l}\circ g)\circ s) &= (\log(\mathfrak{l}\circ g))\circ s = ((\log\mathfrak{l})\circ g)\circ s = \left(\sum_{\gamma<\omega^\mu}\mathfrak{l}_\gamma\ell_{\gamma+1}\circ g\right)\circ s.\end{aligned}$$

Using the injectivity of log and strong linearity, we may thus reduce to the case when $\mathfrak{l} = \ell_\gamma$ for $\gamma<\omega^\mu$. Our induction hypothesis takes care of the case when μ is a limit ordinal or when $\gamma<\omega^{\mu_-}$, so we may assume that $\mathfrak{l} = \ell_\gamma$, where $\omega^{\mu_-}\leqslant\gamma<\omega^\mu$. By the inductive definitions of $L_\gamma(g\circ s)$ and $\ell_\gamma\circ g$, we may further reduce to the case when $\gamma = \omega^{\mu_-}$. Lemma 4.4.10 takes care of the case $\mu = 1$, so we may assume that $\mu > 1$. In summary, we thus need to show that $L_{\omega^{\mu_-}}(g\circ s) = (\ell_{\omega^{\mu_-}}\circ g)\circ s$, where $\mu > 1$.

Set $\mathfrak{a} := \mathfrak{d}_{\omega^{\mu_-}}(g)\in\mathfrak{L}_{<\alpha}$. We claim that $(\ell_{\omega^{\mu_-}}\circ\mathfrak{a})\circ s = L_{\omega^{\mu_-}}(\mathfrak{a}\circ s)$. We have $\mathfrak{a} = \ell_{\sigma+\omega^{\mu_{--}}k}$, where $\omega^{\mu_-}\leqslant_o\sigma<\alpha$, $k\in\mathbb{N}$, and $k = 0$ if μ_- is a limit ordinal. Since $\ell_{\sigma+\omega^{\mu_{--}}k} = \ell_{\omega^{\mu_{--}}k}\circ\ell_\sigma$, we have

$$\ell_{\omega^{\mu_-}}\circ\mathfrak{a} = \ell_{\omega^{\mu_-}}\circ(\ell_{\omega^{\mu_{--}}k}\circ\ell_\sigma) = (\ell_{\omega^{\mu_-}}\circ\ell_{\omega^{\mu_{--}}k})\circ\ell_\sigma = (\ell_{\omega^{\mu_-}} - k)\circ\ell_\sigma = \ell_{\sigma+\omega^{\mu_-}} - k.$$

This gives

$$\begin{aligned}(\ell_{\omega^{\mu_-}}\circ\mathfrak{a})\circ s &= (\ell_{\sigma+\omega^{\mu_-}} - k)\circ s = L_{\sigma+\omega^{\mu_-}}(s) - k = L_{\omega^{\mu_-}}(L_\sigma(s)) - k\\ &= L_{\omega^{\mu_-}}(L_{\omega^{\mu_{--}}k}(L_\sigma(s))) = L_{\omega^{\mu_-}}(L_{\sigma+\omega^{\mu_{--}}k}(s)) = L_{\omega^{\mu_-}}(\mathfrak{a}\circ s),\end{aligned}$$

where the first equality in the second line follows from Proposition 4.3.11.

Having proved our claim, let us now show that $(\ell_{\omega^{\mu_-}}\circ g)\circ s = L_{\omega^{\mu_-}}(g\circ s)$. Take $\gamma<\omega^{\mu_-}$ with $L_\gamma(g\circ s) - L_\gamma(\mathfrak{d}_{\omega^{\mu_-}}(g\circ s))\prec 1$, and $\varepsilon := \ell_\gamma\circ g - \ell_\gamma\circ\mathfrak{a}\prec 1$. We have

$$\ell_{\omega^{\mu_-}}\circ g = \ell_{\omega^{\mu_-}}^{\uparrow\gamma}\circ(\ell_\gamma\circ g) = \ell_{\omega^{\mu_-}}^{\uparrow\gamma}\circ(\ell_\gamma\circ\mathfrak{a}) + \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(\ell_\gamma\circ\mathfrak{a},\varepsilon) = \ell_{\omega^{\mu_-}}\circ\mathfrak{a} + \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(\ell_\gamma\circ\mathfrak{a},\varepsilon).$$

As $\ell_\gamma \in \mathbb{L}_{<\omega^{\mu_-}}$ and $(\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \in \mathbb{L}_{<\omega^{\mu_-}}$ for all $k>0$, by Lemma 4.1.2, our induction hypothesis applied to μ_- gives

$$((\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \circ (\ell_\gamma \circ \mathfrak{a})) \circ s = (\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \circ ((\ell_\gamma \circ \mathfrak{a}) \circ s) = (\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \circ L_\gamma(\mathfrak{a} \circ s)$$

for $k>0$. Along with **C1$_\nu$**, we thus have

$$\begin{aligned} \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(\ell_\gamma \circ \mathfrak{a}, \varepsilon) \circ s &= \left(\sum_{k \in \mathbb{N}^>} \frac{(\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \circ (\ell_\gamma \circ \mathfrak{a})}{k!} \varepsilon^k \right) \circ s = \sum_{k \in \mathbb{N}^>} \frac{((\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \circ (\ell_\gamma \circ \mathfrak{a})) \circ s}{k!} \varepsilon^k \circ s \\ &= \sum_{k \in \mathbb{N}^>} \frac{(\ell_{\omega^{\mu_-}}^{\uparrow\gamma})^{(k)} \circ L_\gamma(\mathfrak{a} \circ s)}{k!} (\varepsilon \circ s)^k = \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(L_\gamma(\mathfrak{a} \circ s), \varepsilon \circ s). \end{aligned}$$

Using also our claim that $(\ell_{\omega^{\mu_-}} \circ \mathfrak{a}) \circ s = L_{\omega^{\mu_-}}(\mathfrak{a} \circ s)$, we obtain

$$(\ell_{\omega^{\mu_-}} \circ g) \circ s = (\ell_{\omega^{\mu_-}} \circ \mathfrak{a}) \circ s + \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(\ell_\gamma \circ \mathfrak{a}, \varepsilon) \circ s = L_{\omega^{\mu_-}}(\mathfrak{a} \circ s) + \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(L_\gamma(\mathfrak{a} \circ s), \varepsilon \circ s).$$

It remains to show that $L_{\omega^{\mu_-}}(g \circ s) = L_{\omega^{\mu_-}}(\mathfrak{a} \circ s) + \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(L_\gamma(\mathfrak{a} \circ s), \varepsilon \circ s)$. Now

$$L_\gamma(g \circ s) - L_\gamma(\mathfrak{a} \circ s) = (\ell_\gamma \circ g) \circ s - (\ell_\gamma \circ \mathfrak{a}) \circ s = \varepsilon \circ s \prec 1,$$

so $\mathfrak{d}_{\omega^{\mu_-}}(\mathfrak{a} \circ s) = \mathfrak{d}_{\omega^{\mu_-}}(g \circ s)$ and $L_\gamma(\mathfrak{a} \circ s) - L_\gamma(\mathfrak{d}_{\omega^{\mu_-}}(\mathfrak{a} \circ s)) \prec 1$. By analyticity of $L_{\omega^{\mu_-}}$, we can conclude that $L_{\omega^{\mu_-}}(g \circ s) = L_{\omega^{\mu_-}}(\mathfrak{a} \circ s) + \mathcal{T}_{\ell_{\omega^{\mu_-}}^{\uparrow\gamma}}(L_\gamma(\mathfrak{a} \circ s), \varepsilon \circ s)$. □

Now that each $\ell'_\rho \circ s$ is defined for $\rho < \alpha$, the same arguments as in Proposition 4.4.12 yield:

Proposition 4.4.12. *For $\rho < \alpha$, the function L_ρ is analytic on $\mathbb{T}^{>,\succ}$ with*

$$\begin{aligned} \mathrm{Conv}(L_\rho)_s &\supseteq s + \mathbb{T}^{\prec s} \quad \textit{and} \\ L_\rho^{(k)}(s) &= \ell_\rho^{(k)} \circ s \end{aligned}$$

for all $s \in \mathbb{T}^{>,\succ}$ and $k \in \mathbb{N}$.

Proposition 4.4.13. *The function $\circ$ satisfies **C4$_\nu$**, i.e. for all $f \in \mathbb{L}_{<\alpha}$, all $t \in \mathbb{T}^{>,\succ}$ and all $\delta \in \mathbb{T}$ with $\delta \prec t$, we have*

$$f \circ (t + \delta) = f \circ t + \mathcal{T}_f(t, \delta).$$

Proof. Fix $t \in \mathbb{T}^{>,\succ}$ and $\delta \in \mathbb{T}$ with $\delta \prec t$. Let $T: \mathbb{L}_{<\alpha} \to \mathbb{T}$ be the map given by

$$T(f) := f \circ t + \mathcal{T}_f(t, \delta).$$

We need to show that $f \circ (t + \delta) = T(f)$ for all $f \in \mathbb{L}_{<\alpha}$. By Lemma 4.1.1, the map T is strongly linear, so it suffices to show that $\mathfrak{l} \circ (t + \delta) = T(\mathfrak{l})$ for all $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$. Since log is injective, it is enough to show that $\log(\mathfrak{l} \circ (t + \delta)) = \log T(\mathfrak{l})$. Now $\log(\mathfrak{l} \circ (t + \delta)) = (\log \mathfrak{l}) \circ (t + \delta)$ by Lemma 4.4.10 and $\log T(\mathfrak{l}) = T(\log \mathfrak{l})$ by [33, Lemma 8.3]. By Proposition 4.4.12 and strong linearity, we have

$$T(\log \mathfrak{l}) = T\left(\sum_{\gamma < \alpha} \mathfrak{l}_\gamma \ell_{\gamma+1} \right) = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma T(\ell_{\gamma+1}) = \sum_{\gamma < \alpha} \mathfrak{l}_\gamma L_{\gamma+1}(t + \delta) = (\log \mathfrak{l}) \circ (t + \delta).$$

We conclude that $\log(\mathfrak{l} \circ (t + \delta)) = (\log \mathfrak{l}) \circ (t + \delta) = T(\log \mathfrak{l}) = \log T(\mathfrak{l})$. □

To conclude our proof of Theorem 4.3.1, we prove the uniqueness of $\circ$.

Proposition 4.4.14. *The function $\circ$ is unique to satisfy **C1$_\nu$**, **C2$_\nu$**, **C3$_\nu$**, and **C4$_\nu$**.*

Proof. Let $\bullet$ be a composition satisfying conditions **C1$_\nu$**, **C2$_\nu$**, **C3$_\nu$**, and **C4$_\nu$** and let $s \in \mathbb{T}^{>,\succ}$. We first show that $\ell_1 \bullet s = \ell_1 \circ s$. Write $s = c\mathfrak{m} + \delta$, with $c \in \mathbb{R}^>$, $\mathfrak{m} := \mathfrak{d}_s$, and $\delta \prec s$. By **C4$_\nu$**, we have

$$\ell_1 \bullet s = \ell_1 \bullet (c\mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \bullet (c\mathfrak{m})}{k!} \delta.$$

For $k>0$, we have $\ell_1^{(k)}=(-1)^{k-1}(k-1)!\,\ell_0^{-k}$, so **C2$_\nu$** gives

$$\ell_1^{(k)}\bullet(c\,\mathfrak{m})=(-1)^{k-1}(k-1)!\,(c\,\mathfrak{m})^{-k}=\ell_1^{(k)}\circ(c\,\mathfrak{m}).$$

Thus, it remains to show that $\ell_1\bullet(c\mathfrak{m})=\ell_1\circ(c\mathfrak{m})$. Using **C2$_\nu$**, **C3$_\nu$**, and the identity $c\,\mathfrak{m}=(c\,\ell_0)\bullet\mathfrak{m}$, we see that

$$\ell_1\bullet(c\,\mathfrak{m})=\ell_1\bullet((c\,\ell_0)\bullet\mathfrak{m})=(\ell_1\circ(c\,\ell_0))\bullet\mathfrak{m}=(\ell_1+\log c)\bullet\mathfrak{m}=L_1(\mathfrak{m})+\log c.$$

Likewise $\ell_1\circ(c\,\mathfrak{m})=L_1(\mathfrak{m})+\log c$.

Now we turn to the task of showing that $f\bullet s=f\circ s$ for $f\in\mathbb{L}_{<\alpha}$. We make the inductive assumption that for $\mu<\nu$ and $f\in\mathbb{L}_{<\omega^\mu}$, we have $f\bullet s=f\circ s$ (if $\mu=0$, this is Proposition 4.4.1). By strong linearity, it suffices to verify that $\mathfrak{l}\bullet s=\mathfrak{l}\circ s$ for any monomial $\mathfrak{l}\in\mathfrak{L}_{<\alpha}$. As $(\mathfrak{l}\bullet s)^{-1}=\mathfrak{l}^{-1}\bullet s$ and likewise for $\mathfrak{l}\circ s$, it suffices to show this only for $\mathfrak{l}\in\mathfrak{L}_{<\alpha}^{\succ}$. Given $\mathfrak{l}=\prod_{\gamma<\alpha}\ell_\gamma^{\mathfrak{l}_\gamma}\in\mathfrak{L}_{<\alpha}$, we have by **C3$_\nu$** that

$$\begin{aligned}\ell_1\bullet(\mathfrak{l}\bullet s) &= (\ell_1\circ\mathfrak{l})\bullet s=\sum_{\gamma<\alpha}\mathfrak{l}_\gamma(\ell_{\gamma+1}\bullet s),\\ \ell_1\circ(\mathfrak{l}\circ s) &= (\ell_1\circ\mathfrak{l})\circ s=\sum_{\gamma<\alpha}\mathfrak{l}_\gamma(\ell_{\gamma+1}\circ s).\end{aligned}$$

Thus, it suffices to show that $\ell_\gamma\bullet s=\ell_\gamma\circ s$ for all $\gamma<\alpha$. By our induction hypothesis, we only need to handle the case that ν is a successor and $\gamma\geqslant\omega^{\nu_-}$. If $\gamma=\omega^{\nu_-}$, then by Proposition 4.3.5, there is an ordinal $\sigma<\omega^{\nu_-}$ with $\varepsilon:=\ell_\sigma\circ s-L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))\prec 1$. Our inductive hypothesis and Lemma 4.1.2 yield

$$\begin{aligned}\ell_\sigma\bullet s &= \ell_\sigma\circ s=L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))+\varepsilon,\\ (\ell_{\omega^{\nu_-}}^{\uparrow\sigma})^{(k)}\bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s)) &= (\ell_{\omega^{\nu_-}}^{\uparrow\sigma})^{(k)}\circ L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s)) \qquad (\text{for } k\in\mathbb{N}^>)\end{aligned}$$

Thus,

$$\begin{aligned}\ell_{\omega^{\nu_-}}\bullet s &= \ell_{\omega^{\nu_-}}^{\uparrow\sigma}\bullet(\ell_\sigma\bullet s)=\ell_{\omega^{\nu_-}}^{\uparrow\sigma}\bullet(L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))+\varepsilon) && (\text{by } \mathbf{C3}_\nu)\\ &= \ell_{\omega^{\nu_-}}^{\uparrow\sigma}\bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))+\sum_{k\in\mathbb{N}^>}\frac{(\ell_{\omega^{\nu_-}}^{\uparrow\sigma})^{(k)}\bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))}{k!}\varepsilon^k && (\text{by } \mathbf{C4}_\nu)\\ &= (\ell_{\omega^{\nu_-}}^{\uparrow\sigma}\circ\ell_\sigma)\bullet\mathfrak{d}_{\omega^{\nu_-}}(s)+\sum_{k\in\mathbb{N}^>}\frac{(\ell_{\omega^{\nu_-}}^{\uparrow\sigma})^{(k)}\bullet L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))}{k!}\varepsilon^k && (\text{by } \mathbf{C3}_\nu \text{ and } \mathbf{C2}_\nu)\\ &= L_{\omega^{\nu_-}}(\mathfrak{d}_{\omega^{\nu_-}}(s))+\sum_{k\in\mathbb{N}^>}\frac{(\ell_{\omega^{\nu_-}}^{\uparrow\sigma})^{(k)}\circ L_\sigma(\mathfrak{d}_{\omega^{\nu_-}}(s))}{k!}\varepsilon^k\\ &= \ell_{\omega^{\nu_-}}\circ s.\end{aligned}$$

Now suppose $\gamma>\omega^{\nu_-}$ and assume by induction that $\ell_\sigma\bullet s=\ell_\sigma\circ s$ for all $\sigma<\gamma$. Take $\sigma<\gamma$ with $\gamma=\omega^{\nu_-}+\sigma$. Then **C3$_\nu$** and our inductive assumption gives

$$\ell_\gamma\bullet s=(\ell_\sigma\circ\ell_{\omega^{\nu_-}})\bullet s=\ell_\sigma\bullet(\ell_{\omega^{\nu_-}}\bullet s)=\ell_\sigma\circ(\ell_{\omega^{\nu_-}}\circ s)=(\ell_\sigma\circ\ell_{\omega^{\nu_-}})\circ s=\ell_\gamma\circ s.$$

This concludes the proof. □

Chapter 5

Hyperexponentiation

We now tackle the problem of hyperexponentiation in a confluent hyperserial skeleton $\mathbb{T}$, equipped with its external composition law $\circ$ from Theorem 4.3.1. For the sake of the present discussion, we take $\mathbb{T}$ as having force **On**. Our treatment of exponentiation will take the form of an inductive proof which spans over Chapter 5 and Chapter 6. The reason for this is that we require each function

$$L_{\omega^\mu}^{\uparrow\gamma} \colon \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}; s \mapsto \ell_{\omega^\mu}^{\uparrow\gamma} \circ s$$

for $\mu \in \mathbf{On}$ and $\gamma < \omega^\mu$ to be strictly increasing (for now, we only know this to be true for $\gamma = 0$ by Lemma 4.3.12), in order to be able to study the properties of hyperexponentiation. Yet such strict monotonicity result seems to be difficult to obtain unless it is known already that $\mathbb{T}$ embeds into a field $\tilde{\mathbb{T}}$ where all hyperexponentials E_γ of all strength $\gamma < \omega^\mu$ are defined. In that case it follows that $L_{\omega^\mu}^{\uparrow\gamma} = L_{\omega^\mu} \circ E_\gamma$ is strictly increasing. Now constructing extensions of $\mathbb{T}$ where E_γ is defined for all $\gamma < \omega^\mu$ requires $L_{\omega^\eta}^{\uparrow\rho}$ to be strictly increasing for all $\eta < \mu$ and $\rho < \omega^\eta$ ...hence the inductive structure of the proof.

5.1 Inductive setting for Chapters 5 and 6

Our goal for Chapters 5 and 6 is to prove the Theorem 5.1.5 below. We first need a few definitions. They implicitely rely on the extension of partial hyperlogarithms into strictly increasing functions $\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ as a consequence of Section 4.3.

Definition 5.1.1. *Let $\mathbb{T}$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. We say that $\mathbb{T}$ has* **force** $(\boldsymbol{\nu}, \boldsymbol{\mu})$ *if for each $\eta < \boldsymbol{\mu}$, the function $L_{\omega^\eta} \colon \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ is bijective.*

Note that if $\mathbb{T}$ has force $(\boldsymbol{\nu}, \boldsymbol{\mu})$, then $L_\gamma \colon \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ is bijective for all $\gamma < \omega^{\boldsymbol{\mu}}$.

Remark 5.1.2. Every confluent hyperserial skeleton of force $\boldsymbol{\nu}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, 0)$. Given a set-sized field of transseries $\mathbb{T}$, we recall that the exponential function cannot be total [68]. Thus, any confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ with $\boldsymbol{\mu} > 0$ is necessarily a proper class.

Remark 5.1.3. Let $\mathbb{T}$ be a hyperserial skeleton of force **On**. Then $\mathbb{T}$ is hyperserial of force $(\mathbf{On}, \mu)$ if and only if $(\mathbb{T}, (L_{\omega^\eta})_{\eta < \nu})$ is hyperserial of force (ν, μ) for all $\nu \geqslant \mu$. Similarly, $\mathbb{T}$ is hyperserial of force $(\mathbf{On}, \mathbf{On})$ if and only if $\mathbb{T}$ is hyperserial of force $(\mathbf{On}, \mu)$ for all μ.

Definition 5.1.4. *Let $\mathbb{T}$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. A* **hyperexponential closure** *of $\mathbb{T}$* **of force** $\boldsymbol{\mu}$ *is a confluent extension $\mathbb{T}_{(<\boldsymbol{\mu})}$ of $\mathbb{T}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ with the following initial property: if $\mathbb{U}$ is another confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ and if $\Phi \colon \mathbb{T} \longrightarrow \mathbb{U}$ is an embedding of force $\boldsymbol{\nu}$, then there is a unique embedding $\Psi \colon \mathbb{T}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{U}$ of force $\boldsymbol{\nu}$ that extends Φ.*

$$\begin{array}{ccc} \mathbb{T} & \overset{\subseteq}{\longrightarrow} & \mathbb{T}_{(<\boldsymbol{\mu})} \\ & {\scriptstyle\Phi}\searrow & \downarrow \exists!\Lambda \\ & & \mathbb{U} \end{array}$$

A **hyperexponential closure** *of* $\mathbb{T}$ *is a hyperexponential closure of* $\mathbb{T}$ *of force* $\boldsymbol{\nu}$*.*

Note that a hyperexponential closure of force $\boldsymbol{\mu}$ if it exists is unique up to unique isomorphism. We will write $\mathbb{T}_{(<\boldsymbol{\mu})}$ for the hyperexponential closure of $\mathbb{T}$ of force $\boldsymbol{\mu}$ if it exists. We can now state Theorem 5.1.5.

Theorem 5.1.5. *Let* $\mathbb{T}$ *be a confluent hyperserial skeleton of force* $\boldsymbol{\nu} \leqslant \mathbf{On}$ *and let* $\boldsymbol{\pi} \leqslant \boldsymbol{\nu}$*. Then* $\mathbb{T}$ *has a hyperexponential closure of force* $\boldsymbol{\pi}$*.*

We will prove Theorem 5.1.5 by induction on $\boldsymbol{\pi}$ (for all $\mathbb{T}$). Note that it holds trivially if $\boldsymbol{\pi} = 0$. Consider a generalized ordinal $\boldsymbol{\pi} \leqslant \boldsymbol{\nu}$. Throughout Chapters 5 and 6, we make the induction hypothesis that Theorem 5.1.5 holds for all $\eta < \boldsymbol{\pi}$:

Induction hypothesis (Chapters 5 and 6). *Each confluent hyperserial skeleton of force* $\eta < \boldsymbol{\pi}$ *has a hyperexponential closure of force* $\eta < \boldsymbol{\pi}$*.*

We first treat the case when $\boldsymbol{\pi}$ is a limit. For each $\eta < \boldsymbol{\pi}$, and each confluent hyperserial skeleton $\mathbb{T}$ of force $\boldsymbol{\nu}$, we have an exponential closure $\mathbb{T}_{(<\eta)} = \mathbb{R}[[\mathfrak{M}_{(<\eta)}]]$ of $\mathbb{T}$ of force η.

Each ordinal $\gamma \in \mathbf{On}$ can be written uniquely as $\gamma = \boldsymbol{\pi}\, \chi(\gamma) + \rho(\gamma)$ where $\chi(\gamma) \in \mathbf{On}$ and $\rho(\gamma) < \boldsymbol{\pi}$ if we impose $\chi(\gamma) = 0$ in the case when $\boldsymbol{\pi} = \mathbf{On}$. Setting $\mathfrak{M}_{(0)} := \mathfrak{M}$, we define an extension $\mathbb{T}_{(\gamma)} := \mathbb{R}[[\mathfrak{M}_{(\gamma)}]]$ by induction on $\gamma \in \mathbf{On}$ as follows:

- $\mathfrak{M}_{(\gamma+1)} := (\mathfrak{M}_{(\gamma)})_{(<\rho(\gamma))}$.
- $\mathfrak{M}_{(\gamma)} := \bigcup_{\sigma<\gamma} \mathfrak{M}_{(\sigma)}$ if γ is a non-zero limit.

So $\mathbb{T}_{(0)} = \mathbb{T}$ and we have the force $\boldsymbol{\nu}$ inclusion $\mathbb{T}_{(\sigma)} \subseteq \mathbb{T}_{(\gamma)}$ whenever $\sigma < \gamma$. We set

$$\mathfrak{M}_{(<\boldsymbol{\pi})} := \bigcup_{\gamma \in \mathbf{On}} \mathfrak{M}_{(\gamma)}, \qquad \mathbb{T}_{(<\boldsymbol{\pi})} := \bigcup_{\gamma \in \mathbf{On}} \mathbb{T}_{(\gamma)}$$

Note that $\mathbb{T}_{(<\boldsymbol{\pi})} = \mathbb{R}[[\mathfrak{M}_{(<\boldsymbol{\pi})}]]$ by Lemma 1.1.9.

Proposition 5.1.6. *The hyperserial skeleton* $\mathbb{T}_{(<\boldsymbol{\pi})}$ *is a hyperexponential closure of* $\mathbb{T}$ *of force* $\boldsymbol{\pi}$*.*

Proof. We first prove that $\mathbb{T}_{(<\boldsymbol{\pi})}$ has force $(\boldsymbol{\nu}, \boldsymbol{\pi})$. Let $\eta < \boldsymbol{\pi}$ and $s \in \mathbb{T}^{>,\succ}_{(<\boldsymbol{\pi})}$. So $s \in \mathbb{T}_{(\gamma)}$ for a certain $\gamma \in \mathbf{On}$. We have $s \in \mathbb{T}_{(\gamma+\eta+2)}$ where $\mathbb{T}_{(\gamma+\eta+2)}$ has force $(\boldsymbol{\nu}, \gamma+\eta+1)$, hence force $(\boldsymbol{\nu}, \eta+1)$ thus $s \in L_{\omega^\eta}(\mathbb{T}^{>,\succ}_{(\gamma+\eta+2)}) \subseteq L_{\omega^\eta}(\mathbb{T}_{(<\boldsymbol{\pi})})$. So $\mathbb{T}_{(<\boldsymbol{\pi})}$ has force $(\boldsymbol{\nu}, \boldsymbol{\pi})$.

Let $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ be a hyperserial embedding of force $\boldsymbol{\nu}$ into a confluent hyperserial embedding $\mathbb{U}$ of force $(\boldsymbol{\nu}, \boldsymbol{\pi})$. We will show for each $\gamma \in \mathbf{On}$ and that there is a unique force $\boldsymbol{\nu}$ embedding Ψ_γ: $\mathbb{T}_{(\gamma)} \longrightarrow \mathbb{U}$ extending Φ. We have $\Psi_0 = \Phi$, so assume that we have defined this unique embedding Ψ_ρ when $\rho < \gamma$. If $\gamma = \eta + 1$ is a successor, then $\mathbb{T}_{(\gamma)} = (\mathbb{T}_{(\eta)})_{(<\rho(\eta))}$, so by 5.1.5, the embedding Ψ_η extends uniquely to an embedding $\Psi_\gamma: \mathbb{T}_{(\gamma)} \longrightarrow \mathbb{U}$. Since Ψ_γ uniquely extends Ψ_η and since Ψ_σ uniquely extends Φ, we see that Ψ_γ uniquely extends Φ. If γ is a limit, then we set $\Psi_\gamma := \bigcup_{\sigma<\gamma} \Psi_\sigma$. The map Ψ_γ is only defined on $\bigcup_{\sigma<\gamma} \mathbb{T}_{(\sigma)}$, which may not equal $\mathbb{T}_{(\sigma)}$, but Ψ_γ *is* defined on all of $\mathfrak{M}_{(\gamma)}$ and so Ψ_γ extends uniquely to a force $\boldsymbol{\nu}$ embedding $\mathbb{T}_{(\gamma)} \longrightarrow \mathbb{U}$, which we also denote by Ψ_γ. Since each $\Psi_\sigma, \sigma < \gamma$ uniquely extends Φ, we see that Ψ_γ uniquely extends Φ as well. Likewise, we define Ψ to be the unique force $\boldsymbol{\nu}$ embedding extending $\bigcup_{\gamma \in \mathbf{On}} \Psi_\gamma$. □

We now assume that $\boldsymbol{\pi} = \mu + 1$ is a successor. So Theorem 5.1.5 holds for μ by Hypothesis 5.1.5 and we have an initial extension $\mathbb{T} \subseteq \mathbb{T}_{(<\mu)}$ for each confluent hyperserial skeleton of force $\boldsymbol{\nu}$. In order to prove Theorem 5.1.5, we have to show how to define missing hyperexponentials of the form $E_{\omega^\mu}(s)$ for $s \in \mathbb{T}^{>,\succ}$. In Section 5.2, we start by giving a formula for hyperexponentials $E_{\omega^\mu}(s)$ that are already defined in $\mathbb{T}^{>,\succ}$ (showing in fact that they are analytic). In Section 5.3, we show that defining hyperexponentials reduces to defining them on specific series called truncated series. We will only prove Theorem 5.1.5 in the next chapter.

Before we continue, let us fix some notation. Set

$$\begin{aligned} \boldsymbol{\alpha} &:= \omega^{\boldsymbol{\nu}} \\ \beta &:= \omega^{\mu} \end{aligned}$$

Given $\gamma<\beta$, we set

$$\ell_{[\gamma,\beta)}:=\prod_{\gamma\leqslant\sigma<\beta}\ell_\sigma\in\mathfrak{L}_{[\gamma,\beta)},\qquad \ell_{(\gamma,\beta)}:=\prod_{\gamma<\sigma<\beta}\ell_\sigma,\qquad \ell_{<\beta}:=\ell_{[0,\beta)}.$$

Note that $\ell_\beta'=\ell_{<\beta}^{-1}$ and that $\ell_{[\gamma,\beta)}^{\uparrow\gamma}=\prod_{\gamma\leqslant\sigma<\beta}\ell_\sigma^{\uparrow\gamma}$. Given $s\in\mathbb{T}^{>,\succ}$, we set

$$L_{[\gamma,\beta)}(s):=\ell_{[\gamma,\beta)}\circ s,\qquad L_\beta^{\uparrow\gamma}(s):=\ell_\beta^{\uparrow\gamma}\circ s,\qquad L_{[\gamma,\beta)}^{\uparrow\gamma}(s):=\ell_{[\gamma,\beta)}^{\uparrow\gamma}\circ s,$$

and we view $L_{[\gamma,\beta)}$, $L_\beta^{\uparrow\gamma}$, and $L_{[\gamma,\beta)}^{\uparrow\gamma}$ as functions from $\mathbb{T}^{>,\succ}$ to $\mathbb{T}^{>,\succ}$. We define $L_{(\gamma,\beta)}$ and $L_{(\gamma,\beta)}^{\uparrow\gamma}$ analogously.

Given $\gamma<\boldsymbol{\alpha}$, we say that $E_\gamma(s)$ is defined if $s\in L_\gamma(\mathbb{T}^{>,\succ})$. If $\mathbb{T}$ is of force $(\boldsymbol{\nu},\mu)$, then $E_\gamma(s)$ is defined for all $\gamma<\omega^\mu$ and $s\in\mathbb{T}^{>,\succ}$. Lemma 4.3.12 tells us that L_γ is strictly increasing; in particular, it is injective. We let $E_\gamma\colon L_\gamma(\mathbb{T}^{>,\succ})\longrightarrow\mathbb{T}^{>,\succ}$ be its functional inverse, which is again strictly increasing. We may also consider E_γ as a partially defined function on $\mathbb{T}^{>,\succ}$.

Our induction hypothesis, that $\mathbb{T}_{(<\mu)}$ exists, has the following consequence:

Lemma 5.1.7. *For $\gamma<\beta$, the function $L_\beta^{\uparrow\gamma}$ is strictly increasing on $\mathbb{T}^{>,\succ}$.*

Proof. Let $s,t\in\mathbb{T}^{>,\succ}$ with $s<t$. By our inductive assumption, $E_\gamma(s)$ and $E_\gamma(t)$ both exist in $\mathbb{T}_{(<\mu)}$. As E_γ and L_β are strictly increasing on $\mathbb{T}_{(<\mu)}^{>,\succ}$ and $s<t$, we have $E_\gamma(s)<E_\gamma(t)$ and

$$L_\beta^{\uparrow\gamma}(s)=L_\beta(E_\gamma(s))<L_\beta(E_\gamma(t))=L_\beta^{\uparrow\gamma}(t).\qquad\square$$

5.2 Local hyperexponentiation

The situation in this section is the same as that which we encountered when studying the exponential in transserial fields. The function L_β being analytic, its definition around a point $s\in\mathbb{T}^{>,\succ}$ is given by a fixed power series, whose formal functional inverse if it exists provides a local inverse of L_β around $L_\beta(s)$.

There is a question as to whether such inversion can be done purely formally, i.e. in the general context of analytic functions on fields of well-based series. The problem here is that although a formal inverse of a power series is always defined, its convergence on a sufficiently large neighborhood of 0 may be problematic to establish, unless one has control over the way differentiation acts on analytic functions. Since analytic functions do not form a group of well-based series, one is left with few tools to tackle such problem

5.2.1 Local inversion of log

We first consider the case when $\beta=1$, so $\mu=0$. This is the well-known case of exponentiation in fields of well-based series, which we partially studied in Section 3.2.1. Indeed, recall that $(\mathbb{T},\log)$ is in particular a transserial field. For $s\in\mathbb{T}^>$, we showed that

$$\log s\in\mathbb{T}_\succ\Longleftrightarrow s\in\mathfrak{M},$$

and that

$$s\in\log\mathbb{T}^>\Longleftrightarrow s_\succ\in\log\mathbb{T}^>.\tag{5.2.1}$$

Furthermore, the function $\log\colon\mathbb{T}^>\longrightarrow\mathbb{T}$ is bijective if and only if $\log\mathfrak{M}=\mathbb{T}_\succ$.

Corollary 5.2.1. *The skeleton $\mathbb{T}$ has force $(1,1)$ if and only if $\mathbb{T}_\succ\subseteq\log\mathbb{T}^>$.*

Recall by (3.2.2) that for $s\in\log\mathbb{T}^>$, $r\in\mathbb{R}$ and $\varepsilon\prec 1$, we have

$$\exp(s+r+\varepsilon)=\exp(r)\exp(s)\left(\sum_{k\in\mathbb{N}}\frac{1}{k!}\varepsilon^k\right).$$

For $s\in\mathbb{T}^{>,\succ}$, we write

$$\mathcal{L}_1[s]:=\{t\in\mathbb{T}^{>,\succ}:s-t\preccurlyeq 1\}.$$

We note the following:

Lemma 5.2.2. *For $s \in \mathbb{T}^{>}$, we have $(\log s)_{\succ} = \log \mathfrak{d}_s$. Thus, $s \in \mathfrak{M}$ if and only if $\log s \in \mathbb{T}_{\succ}$. Moreover, L_1 is a bijection between $\mathcal{E}_1[\mathfrak{m}]$ and $\mathcal{L}_1[L_1(\mathfrak{m})]$ for each $\mathfrak{m} \in \mathfrak{M}^{\succ}$.*

Proof. Given $s \in \mathbb{T}^{>}$, write $s = r\, \mathfrak{d}_s\, (1+\varepsilon)$, where $r \in \mathbb{R}^{>}$ and $\varepsilon \prec 1$. We have

$$\log s = \log \mathfrak{d}_s + \log r + \tilde{L}(\varepsilon)$$

where $\log \mathfrak{m}$ is purely large. If $r \neq 1$, then $\operatorname{supp} \log c = \{1\}$ and if $\varepsilon \neq 0$, then $\tilde{L}(\varepsilon) \sim \varepsilon$, so $\operatorname{supp} \tilde{L}(\varepsilon) \prec 1$. Thus, $(\log s)_{\succ} = \log \mathfrak{d}_s$, as desired. Now assume that $s \succ 1$ and let $\mathfrak{m} \in \mathfrak{M}^{\succ}$. Then

$$s \in \mathcal{E}_1[\mathfrak{m}] \iff \mathfrak{d}_s = \mathfrak{m} \iff \sharp_1(L_1(s)) = L_1(\mathfrak{d}_s) = L_1(\mathfrak{m}) \iff L_1(s) \in \mathcal{L}_1[L_1(\mathfrak{m})],$$

so $L_1(\mathcal{E}_1[\mathfrak{m}]) = \mathcal{L}_1[L_1(\mathfrak{m})] \cap L_1(\mathbb{T}^{>,\succ})$. By (5.2.1), we have $\mathcal{L}_1[L_1(\mathfrak{m})] \cap L_1(\mathbb{T}^{>,\succ}) = \mathcal{L}_1[L_1(\mathfrak{m})]$, hence the result. □

5.2.2 Local inversion of the hyperlogarithms

In this subsection, we study the range of the functions $L_\beta^{\uparrow\gamma}$ for $\gamma < \beta$ and give a formula for their partial functional inverses. We fix $a \in \mathbb{T}^{>,\succ}$ and set $\varphi := L_\beta(a) \in \mathbb{T}^{>,\succ}$. We also fix $\lambda < \beta$. For $k \in \mathbb{N}$, we define series $t_k \in \mathbb{L}_{<\beta}$ inductively by

$$\begin{aligned} t_0 &:= \ell_\lambda \\ t_{k+1} &:= \ell_{<\beta}\, t_k' \end{aligned}$$

Intuitively speaking, the series $t_k \circ a$ is to be thought of as $(\ell_\lambda^{\uparrow\beta})^{(k)} \circ \varphi$, whereas the sum $\sum_{k\in\mathbb{N}} \frac{t_k \circ a}{k!} \varepsilon^k$ behaves like $L_\lambda(E_\beta(\varphi+\varepsilon))$ for $\varepsilon \prec L_{(\lambda,\beta)}(a)^{-1}$. The latter thereby provides a functional inverse of $L_\beta^{\uparrow\lambda}$ on a neighborhood of φ.

Proposition 5.2.3. *Let $\varepsilon \in \mathbb{T}$ with $\varepsilon \prec L_{(\lambda,\beta)}(a)^{-1}$. Then the family $((t_k \circ a)\, \varepsilon^k)_{k\in\mathbb{N}}$ is well-based and $t_0 \circ a \succ (t_k \circ a)\, \varepsilon^k$ for $k > 0$.*

Proof. Consider the derivative $\partial_{[\lambda,\beta)} := \ell_{[\lambda,\beta)}^{\uparrow\lambda}\, \partial$ on $\mathbb{L}_{<\beta}$. We claim that $t_k = \partial_{[\lambda,\beta)}^k(\ell_0) \circ \ell_\lambda$ for all $k \in \mathbb{N}$. This is clear for $k = 0$. Assuming that the claim holds for a given k, we have

$$\begin{aligned} t_{k+1} &= \ell_{<\beta}\, t_k' = \ell_{<\beta}\, (\partial_{[\lambda,\beta)}^k(\ell_0) \circ \ell_\lambda)' = \ell_{<\beta}\, (\partial_{[\lambda,\beta)}^k(\ell_0)' \circ \ell_\lambda)\, \ell_\lambda' \\ &= \ell_{[\lambda,\beta)}\, (\partial_{[\lambda,\beta)}^k(\ell_0)' \circ \ell_\lambda) = (\ell_{[\lambda,\beta)}^{\uparrow\lambda}\, \partial_{[\lambda,\beta)}^k(\ell_0)') \circ \ell_\lambda = \partial_{[\lambda,\beta)}^{k+1}(\ell_0) \circ \ell_\lambda. \end{aligned}$$

In light of this claim, we have $t_k \circ a = \partial_{[\lambda,\beta)}^k(\ell_0) \circ L_\lambda(a)$. Recall that ∂ has well-based operator support $\operatorname{supp}_* \partial = \{\ell_{\gamma+1}' : \gamma < \beta\} \preccurlyeq \ell_0^{-1}$ as an operator on $\mathbb{L}_{<\beta}$, so

$$\operatorname{supp}_* \partial_{[\lambda,\beta)} \preccurlyeq \ell_0^{-1}\, \ell_{[\lambda,\beta)}^{\uparrow\lambda} = \ell_0^{-1} \prod_{\lambda \leqslant \gamma < \beta} \ell_\gamma^{\uparrow\lambda} = \prod_{\lambda < \gamma < \beta} \ell_\gamma^{\uparrow\lambda} = \ell_{(\lambda,\beta)}^{\uparrow\lambda}.$$

Consider the strongly linear map

$$\begin{aligned} \Phi \colon \mathbb{L}_{<\beta} &\longrightarrow \mathbb{T} \\ f &\longmapsto f \circ L_\lambda(a) \end{aligned}$$

and set

$$\mathfrak{A} := \bigcup_{\mathfrak{m} \in \operatorname{supp}_* \partial_{[\lambda,\beta)}} \operatorname{supp} \Phi(\mathfrak{m}),$$

so $\mathfrak{A}$ is well-based and $\mathfrak{A} \preccurlyeq L_{(\lambda,\beta)}^{\uparrow\lambda}(L_\lambda(a)) = L_{(\lambda,\beta)}(a)$. For $k \in \mathbb{N}$, we have $t_k \circ a = \Phi(\partial_{[\lambda,\beta)}^k(\ell_0))$, so for $\mathfrak{m} \in \operatorname{supp}(t_k \circ a)$, there exist $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \operatorname{supp} \partial_{[\lambda,\beta)}$ with

$$\mathfrak{m} \in (\operatorname{supp} \Phi(\mathfrak{m}_1) \cdots \operatorname{supp} \Phi(\mathfrak{m}_k)) \cdot \operatorname{supp} \Phi(\ell_0).$$

This gives us

$$\operatorname{supp}(t_k \circ a) \subseteq \mathfrak{A}^k \cdot \operatorname{supp} \Phi(\ell_0)$$

and it follows that

$$\operatorname{supp}((t_k \circ a)\, \varepsilon^k) \subseteq (\mathfrak{A} \cdot \operatorname{supp} \varepsilon)^k \cdot \operatorname{supp} \Phi(\ell_0).$$

As $\varepsilon \prec L_{(\lambda,\beta)}(a)^{-1}$, we have $\mathfrak{A} \cdot \operatorname{supp} \varepsilon \prec 1$, so we deduce that $((t_k \circ a)\, \varepsilon^k)_{k \in \mathbb{N}}$ is well-based and that $t_0 \circ a \succ (t_k \circ a)\, \varepsilon^k$ for $k > 0$. □

For our next result, we need a combinatorial lemma for power series over a differential field. Let (K, ∂) be a differential field. Then the ring $K[[z]]$ is naturally equipped with two derivations:

$$\begin{aligned} \left(\sum_{n=0}^{\infty} a_n\, z^n\right)' &:= \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, z^n, \\ \partial\left(\sum_{n=0}^{\infty} a_n\, z^n\right) &:= \sum_{n=0}^{\infty} \partial(a_n)\, z^n. \end{aligned}$$

We also have a composition $\circ: K[[z]] \times z\, K[[z]] \longrightarrow K[[z]]$ given by

$$R \circ (z\, S) \longmapsto R(z\, S)$$

for $R, S \in K[[z]]$. This composition interacts with our derivations as follows:

$$\partial(R \circ (z\, S)) = (\partial R) \circ (z\, S) + (R' \circ (z\, S))\, z\, \partial S, \qquad (R \circ (z\, S))' = (R' \circ (z\, S))\, (z\, S)'$$

Lemma 5.2.4. *Let $S = \sum_{n \in \mathbb{N}} a_n\, z^n \in K[[z]]$ and $R = \sum_{m \in \mathbb{N}} b_m\, z^m \in K[[z]]$. Write $F := R \circ (z\, S)$ and assume that we have*

$$u\, a_0\, \partial b_0 = 1, \qquad (n+2)\, a_{n+1} = u\, a_0\, \partial a_n, \qquad (m+1)\, b_{m+1} = u\, \partial b_m$$

for each n and m, where $u \in K$. Then $F = b_0 + z$.

Proof. The last two assumptions give us the following identities

$$R' = u\, \partial R, \quad \text{and} \tag{5.2.2}$$

$$(z\, S)' = a_0\, (1 + u\, z\, \partial S). \tag{5.2.3}$$

We claim that $(\partial b_0)\, F' = \partial F$. Indeed, we have

$$\begin{aligned} \partial F &= \partial(R \circ (z\, S)) \\ &= (\partial R) \circ (z\, S) + (R' \circ (z\, S))\, z\, \partial S \\ &= (u^{-1} R') \circ (z\, S) + (R' \circ (z\, S))\, z\, \partial S && \text{(by (5.2.2))} \\ &= (u^{-1} + z\, \partial S)\, (R' \circ (z\, S)) \\ &= u^{-1}\, (1 + u\, z\, \partial S)\, (R' \circ (z\, S)) \\ &= u^{-1}\, a_0^{-1}\, (z\, S)'\, (R' \circ (z\, S)) && \text{(by (5.2.3))} \\ &= (\partial b_0)\, (z\, S)'\, (R' \circ (z\, S)) && \text{(since } u\, a_0\, \partial b_0 = 1) \\ &= (\partial b_0)\, (R \circ (z\, S))' \\ &= (\partial b_0)\, F'. \end{aligned}$$

Write $F = \sum_{k=0}^{\infty} F_k\, z^k$. The identity $(\partial b_0)\, F' = \partial F$ yields $F_{k+1} = \frac{1}{(k+1)\, \partial b_0}\, \partial F_k$ for each k. Since $F_0 = b_0$, we conclude that $F_1 = 1$ and $F_k = 0$ for $k > 1$. □

Lemma 5.2.5. *Let $\varepsilon \in \mathbb{T}$ with $\varepsilon \prec L_{(\lambda,\beta)}(a)^{-1}$. Then*

$$L_\beta^{\uparrow\lambda}\left(\sum_{n \in \mathbb{N}} \frac{t_n \circ a}{n!}\, \varepsilon^n\right) = \varphi + \varepsilon. \tag{5.2.4}$$

Proof. We have

$$L_\beta^{\uparrow\lambda}\Big(\sum_{n \in \mathbb{N}} \frac{t_n \circ a}{n!}\, \varepsilon^n\Big) = L_\beta^{\uparrow\lambda}\Big(t_0 \circ a + \sum_{n \geqslant 1} \frac{t_n \circ a}{n!}\, \varepsilon^n\Big) = \sum_{m \in \mathbb{N}} \frac{(\ell_\beta^{\uparrow\lambda})^{(m)} \circ (t_0 \circ a)}{m!} \Big(\sum_{n \geqslant 1} \frac{t_n \circ a}{n!}\, \varepsilon^n\Big)^m.$$

Consider the formal power series

$$F=\sum_{m\in\mathbb{N}}\frac{(\ell_\beta^{\uparrow\lambda})^{(m)}\circ t_0}{m!}\Big(\sum_{n\geqslant 1}\frac{t_n}{n!}z^n\Big)^m\in\mathbb{L}_{<\alpha}[[z]].$$

Writing $F=\sum_{k\in\mathbb{N}}F_k z^k$, we have

$$\sum_{k\in\mathbb{N}}(F_k\circ a)\,\varepsilon^k=L_\beta^{\uparrow\lambda}\Big(\sum_{n\in\mathbb{N}}\frac{t_n\circ a}{n!}\varepsilon^n\Big).$$

Thus, it suffices to show that $F=\ell_\beta+z$.

Let $a_n:=\frac{1}{(n+1)!}t_{n+1}$ and $b_m:=\frac{1}{m!}(\ell_\beta^{\uparrow\lambda})^{(m)}\circ t_0$. Then by factoring out z from the inner sum and re-indexing, we obtain

$$F=\sum_{m\in\mathbb{N}}b_m\left(z\sum_{n\in\mathbb{N}}a_n z^n\right)^m.$$

Note that the sequence $(a_n)_{n\in\mathbb{N}}$ satisfies the identities:

$$a_0=t_1=\ell_{<\beta}\,\ell_\lambda'=\ell_{[\lambda,\beta)},\qquad a_{n+1}=\frac{t_{n+2}}{(n+2)!}=\frac{\ell_{<\beta}\,t_{n+1}'}{(n+2)!}=\frac{\ell_{<\beta}\,a_n'}{n+2}.$$

Since $((\ell_\beta^{\uparrow\lambda})^{(m)}\circ t_0)'=((\ell_\beta^{\uparrow\lambda})^{(m+1)}\circ t_0)\,t_0'=((\ell_\beta^{\uparrow\lambda})^{(m+1)}\circ t_0)\,\ell_\lambda'$, the sequence (b_m) satisfies the identities

$$b_0=\ell_\beta^{\uparrow\lambda}\circ t_0=\ell_\beta,\qquad b_{m+1}=\frac{1}{(m+1)!}(\ell_\beta^{\uparrow\lambda})^{(m+1)}\circ t_0=\frac{b_m'}{(m+1)\,\ell_\lambda'}.$$

Setting $u:=\ell_{<\lambda}$, we have

$$u\,a_0\,b_0'=\ell_{<\beta}\,b_0'=1,\qquad (n+2)\,a_{n+1}=\ell_{<\beta}\,a_n'=u\,a_0\,a_n',\qquad (m+1)\,b_{m+1}=\frac{b_m'}{\ell_\lambda'}=u\,b_m'.$$

Using Lemma 5.2.4, we conclude that $F=b_0+z=\ell_\beta+z$. □

Proposition 5.2.6. *The map $s\longmapsto L_\beta^{\uparrow\lambda}(s)$ is a bijection from $L_\lambda(a)+\mathbb{T}^{\prec L_\lambda(a)}$ to $L_\beta(a)+\mathbb{T}^{\prec L_{(\lambda,\beta)}(a)^{-1}}$.*

Proof. Let $\delta\prec L_\lambda(a)$ and let $s:=L_\lambda(a)+\delta$. We have $L_\beta^{\uparrow\lambda}(s)=L_\beta(a)+\mathcal{T}_{\ell_\beta^{\uparrow\lambda}}(L_\lambda(a),\delta)$, so

$$L_\beta^{\uparrow\lambda}(s)-L_\beta(a)\sim((\ell_\beta^{\uparrow\lambda})'\circ L_\lambda(a))\,\delta\prec((\ell_\beta^{\uparrow\lambda})'\circ L_\lambda(a))\,L_\lambda(a).$$

Since $\ell_\beta'=(\ell_\beta^{\uparrow\lambda}\circ\ell_\lambda)'=((\ell_\beta^{\uparrow\lambda})'\circ\ell_\lambda)\,\ell_\lambda'$, we have

$$((\ell_\beta^{\uparrow\lambda})'\circ\ell_\lambda)\,\ell_\lambda=\frac{\ell_\beta'}{\ell_\lambda'}\,\ell_\lambda=\ell_{(\lambda,\beta)}^{-1},$$

so $L_\beta^{\uparrow\lambda}(s)-L_\beta(a)\prec L_{(\lambda,\beta)}(a)^{-1}$. This gives $L_\beta^{\uparrow\lambda}(s)\in L_\beta(a)+\mathbb{T}^{\prec L_{(\lambda,\beta)}(a)^{-1}}$.

Conversely, given $\varepsilon\prec L_{(\lambda,\beta)}(a)^{-1}$, Lemma 5.2.5 yields $L_\beta^{\uparrow\lambda}\big(\sum_{k\in\mathbb{N}}\frac{t_k\circ a}{k!}\varepsilon^k\big)=L_\lambda(a)+\varepsilon$. Let us show by induction on $k\geqslant 1$ that $t_k\preccurlyeq\ell_{(\lambda,\beta)}^k\,\ell_\lambda$. We have $t_1=\ell_{<\beta}\,\ell_\lambda'=\ell_{[\lambda,\beta)}=\ell_{(\lambda,\beta)}\,\ell_\lambda$. Assuming that $t_k\preccurlyeq\ell_{(\lambda,\beta)}^k\,\ell_\lambda$, we have

$$t_{k+1}=\ell_{<\beta}\,t_k'\preccurlyeq\ell_{<\beta}\,(\ell_{(\lambda,\beta)}^k\,\ell_\lambda)'=\ell_{<\beta}\,(k\,\ell_{(\lambda,\beta)}^{k-1}\,\ell_{(\lambda,\beta)}'\,\ell_\lambda+\ell_{(\lambda,\beta)}^k\,\ell_\lambda').$$

We have

$$\ell_{(\lambda,\beta)}'\,\ell_\lambda=\ell_\lambda\sum_{\lambda<\sigma<\beta}\ell_\sigma^{-1}\,\ell_{<\sigma}^{-1}\,\ell_{(\lambda,\beta)}\sim\ell_\lambda\,\ell_{\lambda+1}^{-1}\,\ell_{<(\lambda+1)}^{-1}\,\ell_{(\lambda,\beta)}\prec\ell_\lambda\,\ell_{<(\lambda+1)}^{-1}\,\ell_{(\lambda,\beta)}=\ell_{<\lambda}^{-1}\,\ell_{(\lambda,\beta)}=\ell_\lambda'\,\ell_{(\lambda,\beta)},$$

so $k\,\ell_{(\lambda,\beta)}^{k-1}\,\ell_{(\lambda,\beta)}'\,\ell_\lambda+\ell_{(\lambda,\beta)}^k\,\ell_\lambda'\sim\ell_{(\lambda,\beta)}^k\,\ell_\lambda'$. This gives

$$t_{k+1}\preccurlyeq\ell_{<\beta}\,\ell_{(\lambda,\beta)}^k\,\ell_\lambda'=\frac{\ell_{<\beta}\,\ell_{(\lambda,\beta)}^k}{\ell_{<\lambda}}=\ell_{[\lambda,\beta)}\,\ell_{(\lambda,\beta)}^k=\ell_{(\lambda,\beta)}^{k+1}\,\ell_\lambda.$$

It follows that $(t_k\circ a)\,\varepsilon^k\prec(t_k\circ a)\,L_{(\lambda,\beta)}(a)^{-k}\preccurlyeq L_\lambda(a)$ for each $k>0$, so $\sum_{k\in\mathbb{N}^>}\frac{t_k\circ a}{k!}\varepsilon^k\prec L_\lambda(a)$. Since $t_0\circ a=L_\lambda(a)$, we conclude that $\sum_{k\in\mathbb{N}}\frac{t_k\circ a}{k!}\in L_\lambda(a)+\mathbb{T}^{\prec L_\lambda(a)}$. □

5.3 Truncated series

The notion of α-truncation for $\alpha \in \omega^{\mathbf{On}}$ is a generalization of pure largeness (i.e. the fact of having only infinite monomials in one's support). Like the exponential in transserial fields is determined by its restriction to purely large series, the hyperexponential function of strength α is determined by its values for α-truncated series. This seemingly unpractical notion will turn out to have many regularities due to the relative simplicity of hyperexponential functions. It will play a crucial role in the sequel of the thesis.

5.3.1 β-Truncation

Definition 5.3.1. *For $0 < \eta \leqslant \mu$, we say that $\varphi \in \mathbb{T}^{>,\succ}$ is **ω^η-truncated** if $\varphi > L_{\omega^\eta}^{\uparrow\gamma}(\mathfrak{m}^{-1})$ for all $\mathfrak{m} \in (\operatorname{supp}\varphi)^{\prec}$ and all $\gamma < \omega^\eta$. We also say that a series $\varphi \in \mathbb{T}$ is **1-truncated** if it is purely infinite, i.e. if $\operatorname{supp}\varphi \subseteq \mathfrak{M}^{\succ}$. We write $\mathbb{T}_{\succ,\omega^\eta}$ for the class of ω^η-truncated series in $\mathbb{T}$. So $\mathbb{T}_{\succ,1} = \mathbb{T}_{\succ}$.*

In Subsection 4.2.3, we showed for $\eta < \boldsymbol{\nu}$ that the class $\mathbb{T}^{>,\succ}$ can be partitioned into convex subclasses $\mathcal{E}_{\omega^\eta}[s]$, $s \in \mathbb{T}^{>,\succ}$, each of which contains a unique $L_{<\omega^\eta}$-atomic element $\mathfrak{d}_{\omega^\eta}(s)$. In this section, we describe a different partition of $\mathbb{T}$ into convex subclasses, each of which will contain a unique ω^η-truncated series $\sharp_{\omega^\eta}(s)$. We will then show that L_β is bijective provided that $\mathbb{T}_{\succ,\beta} \subseteq L_\beta(\mathbb{T}^{>,\succ})$.

For the remainder of this section, we assume that $\mu > 0$.

Lemma 5.3.2. *We have $\mathbb{T}_{\succ,\beta} + \mathbb{R}^{\geqslant} \subseteq \mathbb{T}_{\succ,\beta}$. If μ is a successor, then $\mathbb{T}_{\succ,\beta} + \mathbb{R} = \mathbb{T}_{\succ,\beta}$.*

Proof. For $\varphi \in \mathbb{T}_{\succ,\beta}$ and $r \in \mathbb{R}^{\geqslant}$, we have $(\operatorname{supp}\varphi + r)^{\prec} = (\operatorname{supp}\varphi)^{\prec}$ and $\varphi + r \geqslant \varphi$ so $\varphi + r \in \mathbb{T}_{\succ,\beta}$. Assume now that μ is a successor and let $\varphi \in \mathbb{T}_{\succ,\beta}$ and $r \in \mathbb{R}$. Again, $(\operatorname{supp}\varphi + r)^{\prec} = (\operatorname{supp}\varphi)^{\prec}$. Take $n \in \mathbb{N}$ with $n > -r$. Then for all $\gamma < \beta$ and $\mathfrak{m} \in (\operatorname{supp}\varphi)^{\prec}$, we have

$$\varphi > L_\beta^{\uparrow\gamma+\omega^{\mu_-}n}(\mathfrak{m}^{-1}) = L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1}) + n > L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1}) - r,$$

so $\varphi + r > L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1})$. □

Lemma 5.3.3. *Let $a \in \mathbb{T}^{>,\succ}$ and let $\varphi := L_\beta(a) \in \mathbb{T}^{>,\succ}$. Then φ is β-truncated if and only if $\operatorname{supp}\varphi \succ L_\gamma(a)^{-1}$ for all $\gamma < \beta$.*

Proof. We have $(\operatorname{supp}\varphi)^{\succcurlyeq} \succ L_\gamma(a)^{-1}$ for all $\gamma < \beta$ since the series $L_\gamma(a)$ is infinite. Let $\mathfrak{m} \in (\operatorname{supp}\varphi)^{\prec}$ and let $\gamma < \beta$. By Lemma 5.1.7, the function $L_\beta^{\uparrow\gamma}$ is strictly increasing, so we have $\varphi = L_\beta^{\uparrow\gamma}(L_\gamma(a)) > L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1})$ if and only if $L_\gamma(a) > \mathfrak{m}^{-1}$, hence the result. □

By Lemma 5.3.3 and $\mathbf{R}_{\boldsymbol{\mu}}$, the series $L_\beta(\mathfrak{a})$ is β-truncated for all $\mathfrak{a} \in \mathfrak{M}_\beta$. The axiom $\mathbf{R_0}$ also gives that $L_1(\mathfrak{m})$ is 1-truncated for $\mathfrak{m} \in \mathfrak{M}_1$.

Lemma 5.3.4. *Let $s, t \in \mathbb{T}^{>,\succ}$ with $s \succcurlyeq t$ and let $\gamma < \beta$. Then $L_\beta^{\uparrow\gamma+1}(s) > L_\beta^{\uparrow\gamma}(t)$.*

Proof. Take $r \in \mathbb{R}^{>}$ with $r\,s > t$. Then Lemma 5.1.7 gives $L_\beta^{\uparrow\gamma}(r\,s) > L_\beta^{\uparrow\gamma}(t)$, so it is enough to prove that $L_\beta^{\uparrow\gamma+1}(s) > L_\beta^{\uparrow\gamma}(r\,s)$. For this, we may show that $\ell_\beta^{\uparrow\gamma+1} > \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_0)$ in $\mathbb{L}$. As the map $\mathbb{L} \longrightarrow \mathbb{L}; f \longmapsto f \circ \ell_1$ is order-preserving, it is enough to show that

$$\ell_\beta^{\uparrow\gamma} = \ell_\beta^{\uparrow\gamma+1} \circ \ell_1 > (\ell_\beta^{\uparrow\gamma} \circ (r\,\ell_0)) \circ \ell_1 = \ell_\beta^{\uparrow\gamma} \circ (r\,\ell_1).$$

This follows from Lemma 5.1.7 and the fact that $r\,\ell_1 < \ell_0$. □

Definition 5.3.5. *For $t \in \mathbb{T}^{>,\succ}$, we define*

$$\mathcal{L}_\beta[t] := \{s \in t + \mathbb{T}^{\prec} : s = t \text{ or } (s \neq t \text{ and } t < L_\beta^{\uparrow\gamma}(|s-t|^{-1}) \text{ for some } \gamma < \beta)\}.$$

Note that $|s-t|^{-1}$ is positive infinite whenever $s \in t + \mathbb{T}^{\prec}$ and $s \neq t$, so the expression $L_\beta^{\uparrow\gamma}(|s-t|^{-1})$ in the definition is warranted.

Proposition 5.3.6. *The classes $\mathcal{L}_\beta[t]$ form a partition of $\mathbb{T}^{>,\succ}$ into convex subclasses.*

Proof. Let $t\in\mathbb{T}^{>,\succ}$. The convexity of $\mathcal{L}_\beta[t]$ follows immediately from the definition of $\mathcal{L}_\beta[t]$ and Lemma 5.1.7. Let $s\in\mathcal{L}_\beta[t]$. We claim that $\mathcal{L}_\beta[t]\subseteq\mathcal{L}_\beta[s]$, from which it follows by symmetry that $\mathcal{L}_\beta[t]=\mathcal{L}_\beta[s]$. This clearly holds if $s=t$, so assume that $s\neq t$.

We first show that $t\in\mathcal{L}_\beta[s]$. Let $\varepsilon:=s-t\prec 1$ and let $\gamma<\beta$ with $t<L_\beta^{\uparrow\gamma}(|\varepsilon|^{-1})$ for some $\gamma<\beta$. Given σ with $\beta>\sigma>\gamma$, we have $\ell_\sigma^{\uparrow\gamma}\circ\ell_\gamma=\ell_\sigma<\ell_\gamma$, whence $\ell_\sigma^{\uparrow\gamma}<\ell_0$. Therefore,

$$L_\beta^{\uparrow\gamma}(|\varepsilon|^{-1})=L_\beta^{\uparrow\sigma}(L_\sigma^{\uparrow\gamma}(|\varepsilon|^{-1}))<L_\beta^{\uparrow\sigma}(|\varepsilon|^{-1})$$

by Lemma 5.1.7, so $t<L_\beta^{\uparrow\sigma}(|\varepsilon|^{-1})$ for all such σ.

If μ is a successor, take $n<\omega$ with $\gamma<\omega^{\mu_-}n$. Then $t<L_\beta^{\uparrow\omega^{\mu_-}n}(|\varepsilon|^{-1})$ and since $s-t=\varepsilon\prec 1$, we have

$$s=t+\varepsilon<L_\beta^{\uparrow\omega^{\mu_-}n}(|\varepsilon|^{-1})+\varepsilon<L_\beta(|\varepsilon|^{-1})+n+1=L_\beta^{\uparrow\omega^{\mu_-}(n+1)}(|\varepsilon|^{-1}).$$

If μ is a limit, take $\eta<\mu$ with $\gamma<\omega^\eta$, so that $t<L_\beta^{\uparrow\omega^\eta}(|\varepsilon|^{-1})$. Let us show that $s<L_\beta^{\uparrow\omega^{\eta+1}}(|\varepsilon|^{-1})$. Suppose for contradiction that $s\geqslant L_\beta^{\uparrow\omega^{\eta+1}}(|\varepsilon|^{-1})$. By (4.1.4), we have

$$\ell_\beta^{\uparrow\omega^\eta}-\ell_\beta\sim\frac{1}{\ell'_{\omega^{\eta+1}}}\ell'_\beta,\qquad \ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\sim\frac{1}{\ell'_{\omega^{\eta+2}}}\ell'_\beta.$$

Since $\ell'_{\omega^{\eta+2}}\prec\ell'_{\omega^{\eta+1}}$, we have $\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\succ\ell_\beta^{\uparrow\omega^\eta}-\ell_\beta$, so

$$\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta^{\uparrow\omega^\eta}=\big(\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\big)-(\ell_\beta^{\uparrow\omega^\eta}-\ell_\beta)\sim\ell_\beta^{\uparrow\omega^{\eta+1}}-\ell_\beta\sim\frac{1}{\ell'_{\omega^{\eta+2}}}\ell'_\beta=\ell^{-1}_{[\omega^{\eta+2},\beta)}.$$

Therefore,

$$\varepsilon=s-t\geqslant L_\beta^{\uparrow\omega^{\eta+1}}(|\varepsilon|^{-1})-L_\beta^{\uparrow\omega^\eta}(|\varepsilon|^{-1})\sim L_{[\omega^{\eta+2},\beta)}(|\varepsilon|^{-1})^{-1}.$$

This means that $|\varepsilon|^{-1}\preccurlyeq L_{[\omega^{\eta+2},\beta)}(|\varepsilon|^{-1})$: a contradiction since $\ell_{[\omega^{\eta+2},\beta)}\prec\ell_0$.

Now let $u\in\mathcal{L}_\beta[t]$ and let us show $u\in\mathcal{L}_\beta[s]$. This is clear if $u=s$ or if $u=t$, so we assume that u, s, and t are pairwise distinct. By our claim, we have $t\in\mathcal{L}_\beta[s]$ and $t\in\mathcal{L}_\beta[u]$, so take $\gamma<\beta$ with $s<L_\beta^{\uparrow\gamma}(|t-s|^{-1})$ and $u<L_\beta^{\uparrow\gamma}(|t-u|^{-1})$. Note that

$$|s-u|\leqslant|t-s|+|t-u|\leqslant 2\max(|t-s|,|t-u|),$$

thus, $|s-u|^{-1}\geqslant\frac{1}{2}\min(|t-s|^{-1},|t-u|^{-1})$. Lemmas 5.1.7 and 5.3.4 yield

$$L_\beta^{\uparrow\gamma+1}(|s-u|^{-1})>L_\beta^{\uparrow\gamma+1}(2\,|s-u|^{-1})>\min(L_\beta^{\uparrow\gamma}(|t-s|^{-1}),L_\beta^{\uparrow\gamma}(|t-u|^{-1})).$$

If $L_\beta^{\uparrow\gamma+1}(|s-u|^{-1})>L_\beta^{\uparrow\gamma}(|t-s|^{-1})>s$, then $u\in\mathcal{L}_\beta[s]$ by definition. If $L_\beta^{\uparrow\gamma+1}(|s-u|^{-1})>L_\beta^{\uparrow\gamma}(|t-u|^{-1})>u$, then $s\in\mathcal{L}_\beta[u]$, so $u\in\mathcal{L}_\beta[s]$ by our claim. □

Proposition 5.3.7. *Let $t\in\mathbb{T}^{>,\succ}$. Then the class $\mathcal{L}_\beta[t]$ contains exactly one β-truncated element.*

Proof. Let us first show that $\mathcal{L}_\beta[t]$ contains a β-truncated element. Suppose that t itself is not β-truncated, let $\mathfrak{m}\in(\operatorname{supp}t)^\prec$ be greatest such that $t\leqslant L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1})$ for some $\gamma<\beta$. Setting $\varphi:=t_{\succ\mathfrak{m}}$, we have $\varphi-t\asymp\mathfrak{m}$, so $L_\beta^{\uparrow\gamma+1}(|\varphi-t|^{-1})>L_\beta^{\uparrow\gamma}(\mathfrak{m}^{-1})$ by Lemma 5.3.4. Our assumption on $\mathfrak{m}$ therefore yields $L_\beta^{\uparrow\gamma+1}(|\varphi-t|^{-1})>t$, whence $\varphi\in\mathcal{L}_\beta[t]$.

We claim that φ is β-truncated. Fix $\mathfrak{n}\in(\operatorname{supp}\varphi)^\prec$. By definition of φ, we have $t>L_\beta^{\uparrow\gamma+1}(\mathfrak{n}^{-1})$ for all $\gamma<\beta$. Since $t-\varphi_{\succ\mathfrak{n}}\asymp\mathfrak{n}$, Lemma 5.3.4 gives $L_\beta^{\uparrow\gamma+1}(\mathfrak{n}^{-1})>L_\beta^{\uparrow\gamma}(|t-\varphi_{\succ\mathfrak{n}}|^{-1})$ for all $\gamma<\beta$, so $\varphi_{\succ\mathfrak{n}}\notin\mathcal{L}_\beta[t]=\mathcal{L}_\beta[\varphi]$. By definition, this means that $\varphi\geqslant L_\beta^{\uparrow\gamma+1}(|\varphi-\varphi_{\succ\mathfrak{n}}|^{-1})$ for all $\gamma<\beta$. Since $\varphi-\varphi_{\succ\mathfrak{n}}\asymp\mathfrak{n}$, we have $L_\beta^{\uparrow\gamma+1}(|\varphi-\varphi_{\succ\mathfrak{n}}|^{-1})>L_\beta^{\uparrow\gamma}(\mathfrak{n}^{-1})$, by Lemma 5.3.4. Thus, $\varphi>L_\beta^{\uparrow\gamma}(\mathfrak{n}^{-1})$, as claimed.

Now let $\varphi,\psi\in\mathbb{T}^{>,\succ}$ be β-truncated series with $\varphi\in\mathcal{L}_\beta[\psi]$. We need to show that $\varphi=\psi$. Take $\gamma<\beta$ with $\varphi<L_\beta^{\uparrow\gamma}(|\varphi-\psi|^{-1})$. For $\mathfrak{m}\in(\operatorname{supp}\varphi)^\prec$, we have $\varphi>L_\beta^{\uparrow\gamma+1}(\mathfrak{m}^{-1})$ since φ is β-truncated. Therefore,

$$L_\beta^{\uparrow\gamma}(|\varphi-\psi|^{-1})>\varphi>L_\beta^{\uparrow\gamma+1}(\mathfrak{m}^{-1}),$$

so $|\varphi-\psi|^{-1}\succ\mathfrak{m}^{-1}$ by Lemma 5.3.4. Thus $(\operatorname{supp}\varphi)^{\prec}\succ|\varphi-\psi|$. Since $|\varphi-\psi|\prec 1$, we deduce $\operatorname{supp}\varphi\succ|\varphi-\psi|$, so $\varphi\trianglelefteq\psi$. We also have $\psi\in\mathcal{L}_{\beta}[\varphi]$, so the same argument gives $\psi\trianglelefteq\varphi$ and we conclude that $\varphi=\psi$. □

For $t\in\mathbb{T}^{>,\succ}$, we define $\sharp_{\beta}(t)$ to be the unique β-truncated series in $\mathcal{L}_{\beta}[t]$. Note that this definition extends the previous definition of $\sharp_1$. It follows from the proof of Proposition 5.3.7 that $\sharp_{\beta}(t)\trianglelefteq s$ for all $s\in\mathcal{L}_{\beta}[t]$ and that

$$\mathcal{L}_{\beta}[t] \;=\; \{s\in\mathbb{T}^{>,\succ}:\sharp_{\beta}(s)=\sharp_{\beta}(t)\}.$$

Proposition 5.3.8. *For $a\in\mathbb{T}^{>,\succ}$ we have*

$$\mathcal{L}_{\beta}[L_{\beta}(a)]=\{s\in\mathbb{T}^{>,\succ}:s-L_{\beta}(a)\prec L_{[\gamma,\beta)}(a)^{-1} \text{ for some } \gamma<\beta\}.$$

Proof. We have $s\in\mathcal{L}_{\beta}[L_{\beta}(a)]\setminus\{L_{\beta}(a)\}$ if and only if $L_{\beta}^{\uparrow\rho}(|s-L_{\beta}(a)|^{-1})>L_{\beta}(a)$ for some $\rho<\beta$. Since $L_{\beta}(a)=L_{\beta}^{\uparrow\rho}(L_{\rho}(a))$ for each $\rho<\beta$, this is in turn equivalent to $|s-L_{\beta}(a)|^{-1}>L_{\rho}(a)$ by Lemma 5.1.7 . Thus, $s\in\mathcal{L}_{\beta}[L_{\beta}(a)]$ if and only if $|s-L_{\beta}(a)|<L_{\rho}(a)^{-1}$ for some $\rho<\beta$, and it remains to show that $|s-L_{\beta}(a)|<L_{\rho}(a)^{-1}$ for some $\rho<\beta$ if and only if $|s-L_{\beta}(a)|\prec L_{[\gamma,\beta)}(a)^{-1}$ for some $\gamma<\beta$. This follows from the fact that if $\rho<\gamma<\beta$, then $\ell_{\rho}\succ\ell_{[\gamma,\beta)}\succ\ell_{\gamma}$, so $L_{\rho}(a)^{-1}\prec L_{[\gamma,\beta)}(a)^{-1}\prec L_{\gamma}(a)^{-1}$. □

Proposition 5.3.9. *For each $a\in\mathbb{T}^{>,\succ}$ we have $L_{\beta}(\mathcal{E}_{\beta}[a])\subseteq\mathcal{L}_{\beta}[L_{\beta}(a)]$.*

Proof. Let $u\in\mathcal{E}_{\beta}[a]$. Then there is $\lambda=\omega^{\eta}n<\beta$ with $L_{\lambda}(u)-L_{\lambda}(a)\prec 1$. Thus, $L_{\lambda}(u)\in L_{\lambda}(a)+\mathbb{T}^{\prec}$ and so $L_{\beta}(u)=L_{\beta}^{\uparrow\lambda}(L_{\lambda}(u))\in L_{\beta}(a)+\mathbb{T}^{\prec L_{[\lambda,\beta)}(a)^{-1}}$ by Proposition 5.2.6. Therefore, $L_{\beta}(u)\in\mathcal{L}_{\beta}[L_{\beta}(a)]$ by Proposition 5.3.8. □

Corollary 5.3.10. *We have $\sharp_{\beta}\circ L_{\beta}=L_{\beta}\circ\mathfrak{d}_{\beta}$ on $\mathbb{T}^{>,\succ}$. Thus, for $s\in\mathbb{T}^{>,\succ}$, we have $s\in\mathfrak{M}_{\beta}$ if and only if $L_{\beta}(s)\in\mathbb{T}_{\succ,\beta}$.*

Proof. Let $s\in\mathbb{T}^{>,\succ}$. Then $L_{\beta}(\mathfrak{d}_{\beta}(s))\in\mathcal{L}_{\beta}[L_{\beta}(s)]$ by Proposition 5.3.9 and $L_{\beta}(\mathfrak{d}_{\beta}(s))$ is β-truncated by $\mathbf{R}_{\boldsymbol{\mu}}$ and Lemma 5.3.3. Thus $L_{\beta}(\mathfrak{d}_{\beta}(s))=\sharp_{\beta}(L_{\beta}(s))$. The fact that $s\in\mathfrak{M}_{\beta}$ if and only if $L_{\beta}(s)\in\mathbb{T}_{\succ,\beta}$ follows from this and the fact that L_{β} is injective. □

Proposition 5.3.11. *Assume that $\mathbb{T}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu},\mu)$. Then $L_{\beta}(\mathcal{E}_{\beta}[s])=\mathcal{L}_{\beta}[L_{\beta}(s)]$ for all $s\in\mathbb{T}^{>,\succ}$. In particular, if $E_{\beta}(t)$ is defined for $t\in\mathbb{T}^{>,\succ}$, then E_{β} is defined on $\mathcal{L}_{\beta}[t]$.*

Proof. We prove this by induction on μ. Let $s\in\mathbb{T}^{>,\succ}$. By Proposition 5.3.9, we need only prove that $L_{\beta}(\mathcal{E}_{\beta}[s])\supseteq\mathcal{L}_{\beta}[L_{\beta}(s)]$. Let $t\in\mathcal{L}_{\beta}[L_{\beta}(s)]$. By Proposition 5.3.8, there is a $\lambda=\omega^{\eta}n<\beta$ with $t\in L_{\beta}(s)+\mathbb{T}^{\prec L_{[\lambda,\beta)}^{-1}(s)}$. By Proposition 5.2.6, there is a $v\in L_{\lambda}(s)+\mathbb{T}^{\prec L_{\lambda}(s)}$ with $t=L_{\beta}^{\uparrow\lambda}(v)$. Since $\mathbb{T}$ is hyperserial of force $(\boldsymbol{\nu},\mu)$, the hyperexponential $E_{\lambda}(v)$ is defined and

$$E_{\beta}(t)=E_{\lambda}(v).$$

Finally, since $v\sim L_{\lambda}(s)$, Lemma 4.3.4 and Proposition 4.3.5 imply $E_{\lambda}(v)\in\mathcal{E}_{\beta}[s]$. □

Corollary 5.3.12. *Assume that $\mathbb{T}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu},\mu)$. Then we have $E_{\beta}\circ\sharp_{\beta}=\mathfrak{d}_{\beta}\circ E_{\beta}$ whenever one of the sides is defined.*

Corollary 5.3.13. *The following are equivalent:*

a) $\mathbb{T}$ *has force* $(\boldsymbol{\nu},\mu+1)$.

b) *For all $\eta\leqslant\mu$, the function $E_{\omega^{\eta}}$ is defined on $\mathbb{T}_{\succ,\omega^{\eta}}$.*

c) *For all $\eta\leqslant\mu$ and $s\in\mathbb{T}^{>,\succ}$, the hyperexponential $E_{\omega^{\eta}}(t)$ is defined for some $t\in\mathcal{L}_{\omega^{\eta}}[s]$.*

d) *For all $\eta\leqslant\mu$, we have $L_{\omega^{\eta}}(\mathfrak{M}_{\omega^{\eta}})=\mathbb{T}_{\succ,\omega^{\eta}}$.*

Proof. The equivalence between a) and b) follows from Proposition 5.3.11 and the fact that we have

$$\mathbb{T}^{>,\succ} = \bigsqcup_{\varphi \in \mathbb{T}_{\succ,\omega^\eta}} \mathcal{L}_{\omega^\eta}[\varphi]$$

for all $\eta \leqslant \mu$. The equivalence between b) and c) follows directly from Proposition 5.3.11. The equivalence between b) and d) follows from Corollary 5.3.10. □

Set $\beta := \omega^\mu$ and assume that $\mathbb{T}$ has force $(\boldsymbol{\nu}, \mu)$. Then by Lemma 5.3.1, for all $s \in \mathbb{T}^{>,\succ}$, there is a $\gamma < \beta$ with $\varepsilon := s - \sharp_\beta(a) \prec \frac{\ell_\beta'}{\ell_\gamma'} \circ E_\beta \sharp_\beta(s)$. For any such γ, there is a family $(t_{\gamma,k})_{k\in\mathbb{N}} \in \mathbb{L}_{<\beta}^{\mathbb{N}}$ with $t_0 = \ell_\gamma$ such that $((t_{\gamma,k} \circ E_\beta \sharp_\beta(s))\, \varepsilon^k)_{k\in\mathbb{N}}$ is well-based and

$$E_\beta\, a = E_\gamma\!\left(\sum_{k\in\mathbb{N}} \frac{t_{\gamma,k} \circ E_\beta \sharp_\beta(s)}{k!}\, \varepsilon^k\right). \tag{5.3.1}$$

5.3.2 Useful properties of truncation

Throughout this subsection, we let $0 < \mu < \boldsymbol{\nu}$ and we set $\beta := \omega^\mu$ and $\theta := \omega^{\mu_-}$. Given $s, t \in \mathbb{T}^{>,\succ}$, it will be convenient to introduce the following notations:

$$\begin{array}{lllll} s <_\beta t & \Longleftrightarrow & \mathcal{L}_\beta[s] < \mathcal{L}_\beta[t] & \Longleftrightarrow & \sharp_\beta(s) < \sharp_\beta(t) \\ s =_\beta t & \Longleftrightarrow & \mathcal{L}_\beta[s] = \mathcal{L}_\beta[t] & \Longleftrightarrow & \sharp_\beta(s) = \sharp_\beta(t) \end{array}$$

Lemma 5.3.14. *Let $s \in \mathbb{T}^{>,\succ}$, $\gamma < \beta$, and $r \in \mathbb{R}^>$. We have*

$$L_\beta^{\uparrow\gamma}(r\, L_\gamma(s)) =_\beta L_\beta(s)$$

Proof. We claim that if $\ell_\beta \neq \ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma)$, then $\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma) \prec 1$ and

$$\ell_\beta < \ell_\beta^{\uparrow\gamma+1} \circ |\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma)|^{-1}.$$

Assuming that $\ell_\beta \neq \ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma)$, we have

$$\ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma) = \ell_\beta^{\uparrow\gamma+1} \circ \log(r\, \ell_\gamma) = \ell_\beta^{\uparrow\gamma+1} \circ (\ell_{\gamma+1} + \log r) = \sum_{k\in\mathbb{N}} \frac{(\ell_\beta^{\uparrow\gamma+1})^{(k)} \circ \ell_{\gamma+1}}{k!} (\log r)^k,$$

whence $\ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma) - \ell_\beta \sim ((\ell_\beta^{\uparrow\gamma+1})' \circ \ell_{\gamma+1}) \log r$. Now

$$(\ell_\beta^{\uparrow\gamma+1})' \circ \ell_{\gamma+1} = \frac{\ell_\beta'}{\ell_{\gamma+1}'} = \ell_{[\gamma+1,\beta)}^{-1},$$

so $\ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma) - \ell_\beta \asymp \ell_{[\gamma+1,\beta)}^{-1} \prec 1$. Since $\ell_{[\gamma+1,\beta)} \succ \ell_{\gamma+1}$, we have $|\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma)|^{-1} > \ell_{\gamma+1}$, so Lemma 5.1.7 gives

$$\ell_\beta^{\uparrow\gamma+1} \circ |\ell_\beta - \ell_\beta^{\uparrow\gamma} \circ (r\, \ell_\gamma)|^{-1} > \ell_\beta^{\uparrow\gamma+1} \circ \ell_{\gamma+1} = \ell_\beta,$$

as desired. Composing with s gives that if $L_\beta(s) \neq L_\beta^{\uparrow\gamma}(r\, L_\gamma(s))$, then $L_\beta(s) - L_\beta^{\uparrow\gamma}(r\, L_\gamma(s)) \prec 1$ and

$$L_\beta(s) < L_\beta^{\uparrow\gamma+1}(|L_\beta(s) - L_\beta^{\uparrow\gamma}(r\, L_\gamma(s))|^{-1}),$$

From which it follows that $L_\beta^{\uparrow\gamma}(r\, L_\gamma(s)) \in \mathcal{L}_\beta[L_\beta(s)]$. □

Corollary 5.3.15. *Let $s, t \in \mathbb{T}^{>,\succ}$ with $t \preccurlyeq s$. Then $L_\beta(s\, t) =_\beta L_\beta(s)$.*

Proof. We have $L_\beta(s\, t) = L_\beta^{\uparrow 1}(L_1(s\, t)) = L_\beta^{\uparrow 1}(L_1(s) + L_1(t))$. Let $n > 0$ with $t < n\, s$. We have $0 < L_1(t) < L_1(s) + \log n < 2\, L_1(s)$, so

$$L_\beta^{\uparrow 1}(L_1(s)) < L_\beta^{\uparrow 1}(L_1(s) + L_1(t)) < L_\beta^{\uparrow 1}(3\, L_1(s))$$

by Lemma 5.1.7. Since $L_\beta(s) = L_\beta^{\uparrow 1}(L_1(s)) =_\beta L_\beta^{\uparrow 1}(3\,L_1(s))$ by Lemma 5.3.14 and $\mathcal{L}_\beta[s]$ is convex, we are done. □

Lemma 5.3.16. *For each $s \in \mathbb{T}^{>,\succ}$ and each $\gamma < \theta$, we have*

$$L_\beta^{\uparrow\gamma}(s) =_\beta L_\beta(L_\gamma(s)) =_\beta L_\beta(s).$$

Proof. Take $\lambda = \omega^\eta n$ with $\gamma \leqslant \lambda < \theta$. Since $\ell_\lambda^{\uparrow\gamma} \leqslant \ell_0$, we have $\ell_\beta^{\uparrow\gamma} = \ell_\beta^{\uparrow\lambda} \circ \ell_\lambda^{\uparrow\gamma} \leqslant \ell_\beta^{\uparrow\lambda} \circ \ell_0$ by Lemma 5.1.7. This gives

$$\ell_\beta^{\uparrow\gamma} \leqslant \ell_\beta^{\uparrow\lambda} = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ \ell_{\omega^{\eta+1}}^{\uparrow\lambda} = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (\ell_{\omega^{\eta+1}} + n) < \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (2\,\ell_{\omega^{\eta+1}}).$$

Thus, $L_\beta^{\uparrow\gamma}(s) < L_\beta^{\uparrow\omega^{\eta+1}}(2\,L_{\omega^{\eta+1}}(s))$. Likewise, since $\ell_\gamma \geqslant \ell_\lambda$, we have

$$\ell_\beta \circ \ell_\gamma \geqslant \ell_\beta \circ \ell_\lambda = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (\ell_{\omega^{\eta+1}} \circ \ell_\lambda) = \ell_\beta^{\uparrow\omega^{\eta+1}} \circ (\ell_{\omega^{\eta+1}} - n) > \ell_\beta^{\uparrow\omega^{\eta+1}} \circ \left(\frac{1}{2}\,\ell_{\omega^{\eta+1}}\right),$$

so $L_\beta(L_\gamma(s)) > L_\beta^{\uparrow\omega^{\eta+1}}\big(\frac{1}{2}\,L_{\omega^{\eta+1}}(s)\big)$. Lemma 5.1.7 gives $\ell_\beta^{\uparrow\gamma} = \ell_\beta^{\uparrow\gamma} \circ \ell_0 \geqslant \ell_\beta^{\uparrow\gamma} \circ \ell_\gamma = \ell_\beta \geqslant \ell_\beta \circ \ell_\gamma$, so we have

$$L_\beta^{\uparrow\omega^{\eta+1}}(2\,L_{\omega^{\eta+1}}(s)) > L_\beta^{\uparrow\gamma}(s) \geqslant L_\beta(L_\gamma(s)) > L_\beta^{\uparrow\omega^{\eta+1}}\left(\frac{1}{2}\,L_{\omega^{\eta+1}}(s)\right).$$

By Lemma 5.3.14, both $L_\beta^{\uparrow\omega^{\eta+1}}(2\,L_{\omega^{\eta+1}}(s))$ and $L_\beta^{\uparrow\omega^{\eta+1}}\big(\frac{1}{2}\,L_{\omega^{\eta+1}}(s)\big)$ are elements of $\mathcal{L}_\beta[L_\beta(s)]$. Since $\mathcal{L}_\beta[L_\beta(s)]$ is convex, this means that it also contains $L_\beta^{\uparrow\gamma}(s)$ and $L_\beta(L_\gamma(s))$. □

We have the following useful consequence:

Corollary 5.3.17. *Let $s, t \in \mathbb{T}^{>,\succ}$ be such that $L_\gamma(s) \asymp L_\sigma(t)$ for some $\gamma, \sigma < \theta$. Then*

$$L_\beta(s) =_\beta L_\beta(t).$$

Proof. Take $n \in \mathbb{N}^>$ with $\frac{1}{n}\,L_\gamma(s) < L_\sigma(t) < n\,L_\gamma(s)$. Then

$$L_\beta\left(\frac{1}{n}\,L_\gamma(s)\right) < L_\beta(L_\sigma(t)) < L_\beta(n\,L_\gamma(s)).$$

We have $L_\beta(n\,L_\gamma(s)) =_\beta L_\beta^{\uparrow\gamma}(n\,L_\gamma(s))$ by Lemma 5.3.16 and we have $L_\beta^{\uparrow\gamma}(n\,L_\gamma(s)) =_\beta L_\beta(s)$ by Lemma 5.3.14, so $L_\beta(n L_\gamma(s)) =_\beta L_\beta(s)$. Likewise, $L_\beta\big(\frac{1}{n}\,L_\gamma(s)\big) =_\beta L_\beta(s)$. Since $\mathcal{L}_\beta[L_\beta(s)]$ is convex, this yields $L_\beta(L_\sigma(t)) =_\beta L_\beta(s)$. Since $L_\beta(L_\sigma(t)) =_\beta L_\beta(t)$ by Lemma 5.3.16, we conclude that $L_\beta(t) =_\beta L_\beta(s)$. □

Corollary 5.3.18. *Let $s, t \in \mathbb{T}^{>,\succ}$ with $L_\beta(s) <_\beta L_\beta(t)$. Then $s^{-1} t \in \mathbb{T}^{>,\succ}$ and $L_\beta(s^{-1} t) =_\beta L_\beta(t)$.*

Proof. As L_β is strictly increasing, we have $s \leqslant t$, which gives $L_1(s) \leqslant L_1(t)$. We first claim that $L_1(s) \nsim L_1(t)$. If $\mu > 1$, then Corollary 5.3.17 gives that $L_1(s) \not\asymp L_1(t)$, so we may focus on the case when $\mu = 1$. Suppose toward contradiction that $L_\omega(s) <_\omega L_\omega(t)$ and that $L_1(t) = L_1(s) + \varepsilon$ for some $\varepsilon \prec L_1(s)$. Then

$$L_\omega(t) - L_\omega(s) = L_\omega^{\uparrow 1}(L_1(s) + \varepsilon) - L_\omega^{\uparrow 1}(L_1(s)) = \mathcal{T}_{\ell_\omega^{\uparrow 1}}(L_1(s), \varepsilon) \sim ((\ell_\omega^{\uparrow 1})' \circ L_1(s))\,\varepsilon.$$

Since $(\ell_\omega^{\uparrow 1}) = \ell_\omega + 1$, we have $(\ell_\omega^{\uparrow 1})' = \ell_\omega' = \ell_{[0,\omega)}^{-1}$, so $(\ell_\omega^{\uparrow 1})' \circ L_1(s) = \ell_{[0,\omega)}^{-1} \circ L_1(s) = L_{[1,\omega)}(s)^{-1}$. Since $\varepsilon \prec L_1(s)$, we have

$$L_\omega(t) - L_\omega(s) \sim ((\ell_\omega^{\uparrow 1})' \circ L_1(s))\,\varepsilon \prec L_{[2,\omega)}(s)^{-1},$$

so $L_\omega(s) =_\omega L_\omega(t)$ by Proposition 5.3.8, a contradiction.

From our claim, we get $0 < L_1(s^{-1}\,t) = L_1(t) - L_1(s) \asymp L_1(t)$. This yields $s^{-1}\,t \in \mathbb{T}^{>,\succ}$, as $L_1(s^{-1}\,t) \in \mathbb{T}^{>,\succ}$. Take $r \in \mathbb{R}^{>1}$ with $r^{-1}\,L_1(t) < L_1(s^{-1}\,t) < r\,L_1(t)$. Lemma 5.3.14 gives

$$L_\beta(t) = L_\beta^{\uparrow 1}(L_1(t)) =_\beta L_\beta^{\uparrow 1}(r^{-1}\,L_1(t)) =_\beta L_\beta^{\uparrow 1}(r\,L_1(t)),$$

so $L_\beta(t) =_\beta L_1(s^{-1}\,t)$ since $\mathcal{L}_\beta[L_\beta(t)]$ is convex and $L_\beta^{\uparrow 1}$ is strictly increasing. □

Chapter 6
Hyperexponential extensions

In this chapter, we continue and conclude the inductive proof we began in Chapter 5 (see Section 5.1) Recall that $\boldsymbol{\pi} = \mu + 1 \leqslant \boldsymbol{\nu}$ is a successor ordinal, and that 5.1.5 is assumed to hold for μ. We again set

$$\begin{array}{rcl} \boldsymbol{\alpha} & := & \omega^{\boldsymbol{\nu}} \\ \beta & := & \omega^{\mu} \\ \theta & := & \omega^{\mu_-}. \end{array}$$

Note that $\beta = \theta\,\omega$ if μ is a successor and $\beta = \theta$ if μ is a limit. We also fix a confluent hyperserial skeleton $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of force $(\boldsymbol{\nu}, \mu)$, and not just $\boldsymbol{\nu}$. The results in this chapter are contained, in this more general form, in [14, Sections 7 and 8].

In order to prove Theorem 5.1.5 for $\boldsymbol{\pi}$, we rely on a weaker extension theorem which we now describe. Given the class $\mathbf{T} \subseteq \mathbb{T}$ of β-truncated series φ whose hyperexponential $E_\beta(\varphi)$ is not defined, we construct a hyperserial skeleton $\mathbb{T}_{(\mu)} = \mathbb{R}[[\mathfrak{M}_{(\mu)}]]$ and a hyperserial embedding Ψ: $\mathbb{T} \longrightarrow \mathbb{T}_{(\mu)}$ with

$$\Psi(\mathbf{T}) \subseteq L_\beta(\mathbb{T}_{(\mu)}^{>,\succ}), \tag{6.0.1}$$

That is, seeing as $\mathbb{T}$ is naturally included in $\mathbb{T}_{(\mu)}$, the hyperexponentials E_β^φ of each element $\varphi \in \mathbf{T}$ are defined in $\mathbb{T}_{(\mu)}$. Furthermore, the extension $(\mathbb{T}_{(\mu)}, \Psi)$ is initial among extensions satisfying (6.0.1). More precisely, we will prove the following result.

Theorem 6.0.1. *Let $\mathbb{T}$ be a hyperserial skeleton of force $(\boldsymbol{\nu}, \mu)$ and let $\mathbf{T}$ denote the class of β-truncated series $\varphi \in \mathbb{T}$ with $\varphi \notin L_\beta(\mathbb{T}^{>,\succ})$. There is a hyperserial skeleton $\mathbb{T}_{(\mu)}$ of force $\boldsymbol{\nu}$ and a hyperserial embedding $\Psi \colon \mathbb{T} \longrightarrow \mathbb{T}_{(\mu)}$ of force $\boldsymbol{\nu}$ such that*

$$\Psi(\mathbf{T}) \subseteq L_\beta(\mathbb{T}_{(\mu)}^{>,\succ}).$$

Moreover, for any other such extension $(\mathbb{U}, \Phi)$, there is a unique hyperserial embedding $\Lambda \colon \mathbb{T}_{(\mu)} \longrightarrow \mathbb{U}$ of force $\boldsymbol{\nu}$ with $\Lambda \circ \Psi = \Phi$.

$$\begin{array}{ccl} \mathbb{T} & \overset{\Psi}{\longrightarrow} & \mathbb{T}_{(\mu)} \\ & {\scriptstyle \Phi}\searrow & \downarrow \exists!\Lambda \\ & & \mathbb{U} \end{array}$$

Section 6.1 is dedicated to the proof of the theorem for $\mu = 0$. In that case, a large part of the work has already been done in [92], but it contains a self-contained treatment for our setting. The case when $\mu > 0$, which is the longest argument of the thesis, is split into three sections. In Section 6.2, we define the structure of field of well-based series of $\mathbb{T}_{(\mu)}$, which involves defining its ordered group of monomials $\mathfrak{M}_{(\mu)}$. In Section 6.3, we define the hyperserial skeleton on $\mathbb{T}_{(\mu)}$ and conjunctively prove that it satisfies the axioms for hyperserial skeletons. In Section 6.4, we show that $\mathbb{T}_{(\mu)}$ is confluent, prove Theorem 6.0.1 and conclude our inductive proof of Theorem 5.1.5.

6.1 Exponential extensions

Let $\mathbf{T}$ be the class of all 1-truncated series $\varphi \in \mathbb{T}_{\succ,1}$ for which $\exp \varphi$ is not defined. Write $\langle \mathbf{T} \rangle$ be the $\mathbb{R}$-subspace of $\mathbb{T}_{\succ,1}$ generated by $\mathbf{T}$ and $\log \mathfrak{M}$. By Lemma 5.2.2, the class $\langle \mathbf{T} \rangle$ consists only of 1-truncated series. The reader may recall that we have already showed how to construct exponential extensions and closures (see Section 3.2). The difference here is that besides extending the logarithm throughout exponential extensions, we must also extend the whole hyperserial skeleton and show that the axioms are preserved.

6.1.1 Monomial group

We associate to each $\varphi \in \langle \mathbf{T} \rangle$ a formal symbol e^{φ} and we let $\mathfrak{M}_{(0)}$ denote the multiplicative $\mathbb{R}$-vector space of all such symbols, where $\mathrm{e}^{\varphi}\,\mathrm{e}^{\psi} = \mathrm{e}^{\varphi+\psi}$ and $(\mathrm{e}^{\varphi})^r = \mathrm{e}^{r\varphi}$. We use 1 in place of e^0. We order this space by setting $\mathrm{e}^{\varphi} \succ \mathrm{e}^{\psi} \Longleftrightarrow \varphi > \psi$. It is easy to see that $(\mathfrak{M}_{(0)}, \times, \prec, \mathbb{R})$ is an ordered $\mathbb{R}$-vector space which is isomorphic to $(\langle \mathbf{T} \rangle, +, <, \mathbb{R})$. We identify $\mathfrak{M}$ with the $\mathbb{R}$-subspace $\mathrm{e}^{\log \mathfrak{M}}$ of $\mathfrak{M}_{(\mu)}$ via the embedding $\mathfrak{m} \mapsto \mathrm{e}^{\log \mathfrak{m}}$. Let $\mathbb{T}_{(0)} := \mathbb{R}[[\mathfrak{M}_{(0)}]]$, so the identification $\mathfrak{M} \subseteq \mathfrak{M}_{(0)}$ induces an identification $\mathbb{T} \subseteq \mathbb{T}_{(0)}$.

6.1.2 Extending the logarithm and the first hyperlogarithm

For $\mathrm{e}^{\varphi} \in \mathfrak{M}_{(0)}$, we set

$$\log \mathrm{e}^{\varphi} := \varphi.$$

We let L_1 be the restriction of log to $\mathfrak{M}_{(0)}^{\succ}$. Note the following:

1. By construction, $(\mathbb{T}_{(0)}, L_1)$ satisfies $\mathbf{DD_0}$ and $\mathbf{FE_0}$. Moreover, $L_1(\mathfrak{m}) = L_1(\mathrm{e}^{L_1(\mathfrak{m})})$ for $\mathfrak{m} \in \mathfrak{M}^{\succ}$, so $(\mathbb{T}, L_1) \subseteq (\mathbb{T}_{(0)}, L_1)$
2. We claim that $(\mathbb{T}_{(0)}, L_1)$ satisfies $\mathbf{A_0}$. Suppose for contradiction that $\varphi = L_1(\mathrm{e}^{\varphi}) \succcurlyeq \mathrm{e}^{\varphi}$, where $\mathrm{e}^{\varphi} \in \mathfrak{M}_{(0)}^{\succ}$. Then $\mathfrak{d}_{\varphi} \succcurlyeq \mathrm{e}^{\varphi}$, so $L_1(\mathfrak{d}_{\varphi}) \geqslant \varphi$ by definition. This gives $L_1(\mathfrak{d}_{\varphi}) \succcurlyeq \mathfrak{d}_{\varphi}$, which contradicts the fact that $(\mathbb{T}, L_1)$ satisfies $\mathbf{A_0}$.
3. By definition, we have $\mathrm{e}^{\varphi} \in \mathfrak{M}_{(0)}^{\succ}$ if and only if $L_1(\mathrm{e}^{\varphi}) > 0$, so $(\mathbb{T}_{(0)}, L_1)$ satisfies $\mathbf{M_0}$.
4. Since $L_1(\mathrm{e}^{\varphi}) = \varphi \in \mathbb{T}_{\succ,1}$ for $\mathrm{e}^{\varphi} \in \mathfrak{M}_{(0)}^{\succ}$, the axiom $\mathbf{R_0}$ is satisfied.
5. As remarked in Remark 4.2.2, $\mathbf{P}_0$ follows from $\mathbf{FE_0}$.

Extending L_ω. For $\varphi \in \langle \mathbf{T} \rangle$ with $\mathrm{e}^{\varphi} \succ 1$, we have $L_1(\mathrm{e}^{\varphi}) \in \mathfrak{M}_{(0)}^{\succ}$ if and only if $\varphi \in \mathfrak{M}^{\succ}$, so $\mathrm{e}^{\varphi} \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\varphi \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\varphi \in \mathfrak{M}_{\omega}$. Accordingly, we set

$$\operatorname{dom} L_{\omega} := \{\mathrm{e}^{\varphi} : \varphi \in \langle \mathbf{T} \rangle \cap \mathfrak{M}_{\omega}\}, \qquad L_{\omega}(\mathrm{e}^{\varphi}) := L_{\omega}(\varphi) + 1.$$

This ensures that $\mathbf{DD}_1$ holds. Note that if $\mathfrak{a} \in \mathfrak{M}_{\omega}$, then $L_1(\mathfrak{a}) \in \langle \mathbf{T} \rangle \cap \mathfrak{M}_{\omega}$, so $\mathfrak{a} = \mathrm{e}^{L_1(\mathfrak{a})} \in \operatorname{dom} L_{\omega}$ and $L_{\omega}(\mathrm{e}^{L_1(\mathfrak{a})}) = L_{\omega}(L_1(\mathfrak{a})) + 1 = L_{\omega}(\mathfrak{a})$. Thus, $\mathfrak{M}_{\omega} \subseteq \operatorname{dom} L_{\omega}$ and $(\mathbb{T}, L_1, L_{\omega}) \subseteq (\mathbb{T}_{(0)}, L_1, L_{\omega})$. We also have the following:

1. For $\mathrm{e}^{\varphi} \in \operatorname{dom} L_{\omega}$, we have

   $$L_{\omega}(L_1(\mathrm{e}^{\varphi})) = L_{\omega}(\varphi) = L_{\omega}(\mathrm{e}^{\varphi}) - 1$$

   so $(\mathbb{T}_{(0)}, L_1, L_{\omega})$ satisfies $\mathbf{FE}_1$.
2. For $\mathrm{e}^{\varphi} \in \operatorname{dom} L_{\omega}$, we have $L_{\omega}(\varphi) + 1 = (\ell_{\omega} + 1) \circ \varphi \prec \ell_0 \circ \varphi = \varphi$, since $\ell_{\omega} + 1 \prec \ell_0$. Thus

   $$L_{\omega}(\mathrm{e}^{\varphi}) = L_{\omega}(\varphi) + 1 \prec \varphi = L_1(\mathrm{e}^{\varphi}),$$

   which proves $\mathbf{A}_1$.
3. $(\mathbb{T}_{(0)}, L_1, L_{\omega})$ satisfies $\mathbf{M}_1$. To see this, let $\mathrm{e}^{\varphi}, \mathrm{e}^{\psi} \in \operatorname{dom} L_{\omega}$ with $\mathrm{e}^{\varphi} \prec \mathrm{e}^{\psi}$ and let $n \in \mathbb{N}$. We want to show that $L_{\omega}(\mathrm{e}^{\varphi}) + L_n(\mathrm{e}^{\varphi})^{-1} < L_{\omega}(\mathrm{e}^{\psi}) - L_n(\mathrm{e}^{\psi})^{-1}$. Since $L_{n+1}(\mathrm{e}^{\varphi}) \prec L_n(\mathrm{e}^{\varphi})$ and $L_{n+1}(\mathrm{e}^{\psi}) \prec L_n(\mathrm{e}^{\psi})$ by $\mathbf{A_0}$, we may assume without loss of generality that $n > 0$. Now

   $$\begin{aligned} L_{\omega}(\mathrm{e}^{\varphi}) + L_n(\mathrm{e}^{\varphi})^{-1} &= L_{\omega}(\varphi) + 1 + L_{n-1}(\varphi)^{-1} \\ L_{\omega}(\mathrm{e}^{\psi}) - L_n(\mathrm{e}^{\psi})^{-1} &= L_{\omega}(\psi) + 1 - L_{n-1}(\psi)^{-1}. \end{aligned}$$

   Since $\varphi, \psi \in \mathfrak{M}_{\omega}$ and since $(\mathbb{T}, L_1, L_{\omega})$ satisfies $\mathbf{M}_1$, we have

   $$L_{\omega}(\varphi) + L_{n-1}^{-1}(\varphi) < L_{\omega}(\psi) - L_{n-1}^{-1}(\psi).$$

4. Let $\mathrm{e}^{\varphi} \in \operatorname{dom} L_{\omega}$. Since $\varphi \in \mathfrak{M}_{\omega}$ and $(\mathbb{T}, L_1, L_{\omega})$ satisfies $\mathbf{R}_1$, the hyperlogarithm $L_{\omega}(\varphi)$ is ω-truncated by Lemma 5.3.3. It follows from Lemma 5.3.2 that $L_{\omega}(\mathrm{e}^{\varphi}) = L_{\omega}(\varphi) + 1$ is also ω-truncated, so $(\mathbb{T}_{(0)}, L_1, L_{\omega})$ satisfies $\mathbf{R}_1$.

5. Let $\mathrm{e}^{\varphi} \in \operatorname{dom} L_{\omega}$ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. To show that $(\mathbb{T}_{(0)}, L_1, L_\omega)$ satisfies $\mathbf{P}_1$, we need to show that the sum $s = \sum_{n\in\mathbb{N}} r_n L_{n+1}(\mathrm{e}^{\varphi})$ is in $\log \mathfrak{M}_{(0)}$. We have

$$s = \sum_{n\in\mathbb{N}} r_n L_{n+1}(\mathrm{e}^{\varphi}) = r_0 \varphi + \sum_{n\in\mathbb{N}^>} r_n L_n(\varphi).$$

Since $\varphi \in \mathfrak{M}_\omega$ and since $(\mathbb{T}_{(0)}, L_1, L_\omega)$ satisfies $\mathbf{P}_1$, we have $\sum_{n\in\mathbb{N}^>} r_n L_n(\varphi) \in \log \mathfrak{M}$. It remains to note that $r_0 \varphi = r_0 L_1(\mathrm{e}^{\varphi}) = \log \mathrm{e}^{r_0 \varphi} \in \log \mathfrak{M}_{(0)}$ and that $\log \mathfrak{M}_{(0)}$ is closed under finite sums.

6.1.3 Extending L_{ω^η} for $1 < \eta < \nu$

Let $1 < \eta < \boldsymbol{\nu}$ and set $\operatorname{dom} L_{\omega^\eta} := \mathfrak{M}_{\omega^\eta}$. We need to show that $\mathbf{DD}_\eta$ holds for each η, and for this, it suffices to show that $\mathbf{DD}_2$ holds. Let $\mathrm{e}^{\varphi} \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_\omega^{\circ n}$ and take n with $L_{\omega(n+1)}(\varphi) \asymp L_{\omega(n+1)}(\mathfrak{d}_{\omega^2}(\varphi))$. Since $L_\omega(\varphi) + 1 \asymp L_\omega(\varphi)$, Lemma 4.3.4 yields

$$L_{\omega(n+1)}(\mathrm{e}^{\varphi}) = L_{\omega n}(L_\omega(\varphi) + 1) \asymp L_{\omega n}(L_\omega(\varphi)) = L_{\omega(n+1)}(\varphi) \asymp L_{\omega(n+1)}(\mathfrak{d}_{\omega^2}(\varphi)).$$

Since $L_{\omega(n+1)}(\mathrm{e}^{\varphi})$ and $L_{\omega(n+1)}(\mathfrak{d}_{\omega^2}(\varphi))$ are both monomials, they must be equal. The axiom $\mathbf{M}_1$ gives $\mathrm{e}^{\varphi} = \mathfrak{d}_{\omega^2}(\varphi) \in \mathfrak{M}_{\omega^2} = \operatorname{dom} L_{\omega^2}$.

Now $\mathbf{FE}_\eta$, $\mathbf{A}_\eta$, $\mathbf{M}_\eta$, $\mathbf{R}_\eta$, and $\mathbf{P}_\eta$ hold for each $1 < \eta < \boldsymbol{\nu}$, since they hold in $\mathbb{T}$. Furthermore, $\mathbf{P}_{\boldsymbol{\nu}}$ holds if $\boldsymbol{\nu} \in \mathbf{On}$; this is clear since $\boldsymbol{\nu} > 1$. Thus, $(\mathbb{T}_{(0)}, (L_{\omega^\mu})_{\eta<\boldsymbol{\nu}})$ is a hyperserial skeleton of force $\boldsymbol{\nu}$ which extends $(\mathbb{T}, (L_{\omega^\mu})_{\eta<\boldsymbol{\nu}})$.

Proposition 6.1.1. *Then $\mathbb{T}_{(0)}$ is $\boldsymbol{\nu}$-confluent.*

Proof. Clearly, $\mathbb{T}_{(0)}$ is 0-confluent. Let $s \in \mathbb{T}_{(0)}^{>,\succ}$ and take $\varphi \in \langle \mathbf{T} \rangle$ with $\mathfrak{d}_s = \mathrm{e}^{\varphi} \in \mathfrak{M}_{(0)}^{\succ}$. We have $L_1(\mathfrak{d}_1(s)) = L_1(\mathrm{e}^{\varphi}) = \varphi \in \mathbb{T}$. Let $\mathfrak{a} := \mathfrak{d}_\omega(\varphi)$ and take n with $(L_1 \circ \mathfrak{d}_1)^{\circ n}(\varphi) \asymp (L_1 \circ \mathfrak{d}_1)^{\circ n}(\mathfrak{a})$. We have $L_1(\mathfrak{d}_1(s)) = \varphi$. By definition of $\langle \mathbf{T} \rangle$, we either have $\mathfrak{a} \in \log \mathfrak{M}$ and $\mathfrak{a} \in \langle \mathbf{T} \rangle$ or $\mathfrak{a} \notin \log \mathfrak{M}$ and $\mathfrak{a} \in \mathbf{T}$, whence $\mathfrak{a} \in \langle \mathbf{T} \rangle$. So

$$(L_1 \circ \mathfrak{d}_1)^{\circ(n+1)}(s) \asymp (L_1 \circ \mathfrak{d}_1)^{\circ n}(\mathfrak{a}) = (L_1 \circ \mathfrak{d}_1)^{\circ(n+1)}(\mathrm{e}^{\mathfrak{a}}).$$

The fact that $\mathfrak{a} \in \mathfrak{M}_\omega$ implies that $\mathrm{e}^{\mathfrak{a}} \in \operatorname{dom} L_\omega$, so $\mathfrak{d}_\omega(s) = \mathrm{e}^{\mathfrak{a}}$. We have

$$L_\omega(\mathfrak{d}_\omega(s)) = L_\omega(\mathrm{e}^{\mathfrak{a}}) = L_\omega(\mathfrak{a}) + 1 \asymp L_\omega(\mathfrak{a}) = L_\omega(\mathfrak{d}_\omega(\mathfrak{a})),$$

so $\mathfrak{d}_{\omega^2}(s) = \mathfrak{d}_{\omega^2}(\mathfrak{a})$ and, more generally, $\mathfrak{d}_{\omega^\eta}(s) = \mathfrak{d}_{\omega^\eta}(\mathfrak{a})$ for $2 \leqslant \eta \leqslant \boldsymbol{\nu}$. Thus, the skeleton $\mathbb{T}_{(\mu)}$ is $\boldsymbol{\nu}$-confluent. □

Let us summarize:

Proposition 6.1.2. *The field $\mathbb{T}_{(0)}$ is a confluent hyperserial skeleton of force $\boldsymbol{\nu}$. It is an extension of $\mathbb{T}$ of force $\boldsymbol{\nu}$ with $\langle \mathbf{T} \rangle \subseteq L_1(\mathfrak{M}_{(0)})$.*

Using the composition from Theorem 4.3.1, we can check whether an embedding Φ of confluent hyperserial skeletons is of force $\boldsymbol{\nu}$ without having to verify that $\Phi(\mathfrak{M}_{\omega^\eta}) \subseteq \mathfrak{N}_{\omega^\eta}$ for all η.

Lemma 6.1.3. *Let $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu}$ with the external composition $\circ: \mathbb{L}_{<\boldsymbol{\alpha}} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$ from Theorem 4.3.1 and let $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ be a strongly linear field embedding. Suppose that $\Phi(\mathfrak{M}) \subseteq \mathfrak{N}$, that $\Phi(\mathfrak{m}^r) = \Phi(\mathfrak{m})^r$ for all $\mathfrak{m} \in \mathfrak{M}$ and all $r \in \mathbb{R}$, and that $\Phi(L_{\omega^\eta}(\mathfrak{a})) = L_{\omega^\eta}(\Phi(\mathfrak{a}))$ for all $\eta < \boldsymbol{\nu}$ and all $\mathfrak{a} \in \mathfrak{M}_{\omega^\eta}$. Then Φ is an embedding of force $\boldsymbol{\nu}$.*

Proof. We will show by induction on $\eta < \boldsymbol{\nu}$ that $\Phi(\mathfrak{M}_{\omega^\eta}) \subseteq \mathfrak{N}_{\omega^\eta}$. For $\eta = 0$, this holds since Φ is order-preserving. Let $\eta > 0$ and assume that $\Phi(\mathfrak{M}_{\omega^\iota}) \subseteq \mathfrak{N}_{\omega^\iota}$ for all $\iota < \eta$. If η is a limit, then by $\mathbf{DD}_\eta$, we have

$$\Phi(\mathfrak{M}_{\omega^\eta}) = \Phi\left(\bigcap_{\iota<\eta} \mathfrak{M}_{\omega^\iota}\right) = \bigcap_{\iota<\eta} \Phi(\mathfrak{M}_{\omega^\iota}) \subseteq \bigcap_{\iota<\eta} \mathfrak{N}_{\omega^\iota} = \mathfrak{N}_{\omega^\eta}.$$

Suppose η is a successor and let $\mathfrak{a} \in \mathfrak{M}_{\omega^\eta}$. We have $L_{\omega^{\eta-}n}(\mathfrak{a}) \in \mathfrak{M}_{\omega^{\eta-}}$ for all $n \in \mathbb{N}$ by $\mathbf{DD}_\eta$. Our induction hypothesis gives $L_{\omega^{\eta-}n}(\Phi(\mathfrak{a})) = \Phi(L_{\omega^{\eta-}n}(\mathfrak{a})) \in \mathfrak{N}_{\omega^{\eta-}}$ for all $n \in \mathbb{N}$. Applying $\mathbf{DD}_\eta$ again gives $\Phi(\mathfrak{a}) \in \mathfrak{N}_{\omega^\eta}$, so $\Phi(\mathfrak{M}_{\omega^\iota}) \subseteq \mathfrak{N}_{\omega^\iota}$. □

Proposition 6.1.4. *Let $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu}$ and let $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ be an embedding of force $\boldsymbol{\nu}$. If $\Phi(\mathbf{T}) \subseteq \log(\mathbb{U}^>)$, then there is a unique embedding*

$$\Psi: \mathbb{T}_{(0)} \longrightarrow \mathbb{U}$$

of force $\boldsymbol{\nu}$ that extends Φ.

Proof. As $\mathbb{U}$ is hyperserial of force $\boldsymbol{\nu}$, we have an external composition $\circ: \mathbb{L}_{<\boldsymbol{\alpha}} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$. Since $\Phi(\mathbf{T}) \subseteq \log \mathbb{U}^>$, Φ is $\mathbb{R}$-linear, and $\log \mathbb{U}^>$ is an $\mathbb{R}$-subspace of $\mathbb{U}$ containing $\Phi(\log \mathfrak{M})$, we have $\Phi(\langle \mathbf{T} \rangle) \subseteq \log \mathbb{U}^>$.

Since $\Phi(\mathfrak{M}^\succ) \subseteq \mathfrak{N}^\succ$, we have $\Phi(\mathbb{T}_{\succ,1}) \subseteq \mathbb{U}_{\succ,1}$ so $\Phi(\langle \mathbf{T} \rangle) \subseteq \log \mathbb{U}^> \cap \mathbb{U}_{\succ,1}$. Thus, $\exp(\Phi(\varphi))$ is a monomial for $\varphi \in \langle \mathbf{T} \rangle$ by Lemma 5.2.2. We define a map $\Psi: \mathfrak{M}_{(0)} \longrightarrow \mathfrak{N}$ by setting

$$\Psi(\mathrm{e}^\varphi) := \exp(\Phi(\varphi)).$$

It is routine to check that $\Psi: \mathfrak{M}_{(0)} \longrightarrow \mathfrak{N}$ is an embedding of ordered monomial groups with $\mathbb{R}$-powers. By Proposition 1.3.2, this embedding Ψ uniquely extends into a strongly linear field embedding of $\mathbb{T}_{(0)}$ into $\mathbb{U}$, which we will still denote by Ψ. Note that if $\mathfrak{m} \in \mathfrak{M}$, then $\Psi(\mathrm{e}^{\log(\mathfrak{m})}) = \exp(\Phi(\log \mathfrak{m})) = \exp(\log(\Phi(\mathfrak{m}))) = \Phi(\mathfrak{m})$, so Ψ extends Φ.

We now prove that Ψ is a force $\boldsymbol{\nu}$ embedding. By Lemma 6.1.3, we need only show that Ψ commutes with logarithms and hyperlogarithms. Given $\mathrm{e}^\varphi \in \mathfrak{M}_{(\mu)}$, we have

$$\Psi(\log(\mathrm{e}^\varphi)) = \Psi(\varphi) = \Phi(\varphi) = \log(\exp(\Phi(\varphi))) = \log(\Psi(\mathrm{e}^\varphi)).$$

Now let $\mu < \boldsymbol{\nu}$ with $\mu > 0$ and let $\mathrm{e}^\varphi \in (\mathfrak{M}_{(\mu)})_{\omega^\mu}$. If $\mu > 1$, then $\mathrm{e}^\varphi \in \mathfrak{M}_{\omega^\mu}$, so we automatically have $L_{\omega^\mu}(\Psi(\mathrm{e}^\varphi)) = \Psi(L_{\omega^\mu}(\mathrm{e}^\varphi))$, since Ψ extends Φ. If $\mu = 1$, then $\varphi \in \langle \mathbf{T} \rangle \cap \mathfrak{M}_\omega$, so

$$L_\omega(\Psi(\mathrm{e}^\varphi)) = L_\omega(\exp(\Phi(\varphi))) = L_\omega(\Phi(\varphi)) + 1 = \Phi(L_\omega(\varphi) + 1) = \Phi(L_\omega(\mathrm{e}^\varphi)) = \Psi(L_\omega(\mathrm{e}^\varphi)).$$

Let us finally assume that $\Lambda: \mathbb{T}_{(0)} \longrightarrow \mathbb{U}$ is another embedding of force $\boldsymbol{\nu}$ that extends Φ. To see that $\Lambda = \Psi$, it suffices to show that $\Lambda(\mathrm{e}^\varphi) = \Psi(\mathrm{e}^\varphi)$ for $\varphi \in \langle \mathbf{T} \rangle$. Now

$$\log(\Lambda(\mathrm{e}^\varphi)) = \Lambda(\log(\mathrm{e}^\varphi)) = \Lambda(\varphi) = \Phi(\varphi),$$

so $\Lambda(\mathrm{e}^\varphi) = \exp(\Phi(\varphi)) = \Psi(\mathrm{e}^\varphi)$. □

We next turn to the case of $\mu > 0$.

6.2 Structure of field of well-based series

We first define $\mathbb{T}_{(\mu)}$ as a field of well-based series by defining its monomial group $\mathfrak{M}_{(\mu)}$ as a linearly ordered Abelian group extension of $\mathfrak{M}$.

6.2.1 Selecting truncated series

Let $\mathbf{T}$ be the class of all β-truncated series $\varphi \in \mathbb{T}_{\succ,\beta}$ for which $E_\beta(\varphi)$ is not defined. Consider $\varphi \in \mathbf{T}$ and $s \in \mathbb{T}^{>,\succ}$. We have $\sharp_\beta(L_\beta(s)) = L_\beta(\mathfrak{d}_\beta(s)) \in L_\beta(\mathbb{T}^{>,\succ})$ by Corollary 5.3.10. Since $\mathcal{L}_\beta[L_\beta(s)]$ contains a unique β-truncated element, φ is β-truncated and $\varphi \notin L_\beta(\mathbb{T}^{>,\succ})$, it follows that $\varphi \notin \mathcal{L}_\beta[L_\beta(s)]$. Thus, we have

$$\begin{aligned} \varphi \leqslant L_\beta(s) &\iff \varphi <_\beta L_\beta(s) \\ \varphi \geqslant L_\beta(s) &\iff \varphi >_\beta L_\beta(s). \end{aligned}$$

If μ is a successor, then since $\mathbb{T}$ has force $(\boldsymbol{\nu}, \mu)$, we have

$$\mathbf{T} = \mathbf{T} + \mathbb{Z}. \tag{6.2.1}$$

6.2.2 The group of monomials

We associate to each $\mathfrak{l} \in \mathfrak{L}_{<\theta}$ and each $\varphi \in \mathbf{T}$ a formal symbol $\mathfrak{l}[\mathrm{e}_\beta^\varphi]$. This should be thought of as $\mathfrak{l} \circ \mathrm{e}_\beta^\varphi$ if e_β^φ is an element in a hyperserial extension of $\mathbb{T}$. Accordingly, we write e_β^φ in place of $\ell_0[\mathrm{e}_\beta^\varphi]$ and 1 in place of $1[\mathrm{e}_\beta^\varphi]$.

Remark 6.2.1. We will now construct the smallest subgroup $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ containing $\mathrm{e}_\beta^{\mathbf{T}}$ such that a composition law $\mathbb{L}_{<\boldsymbol{\alpha}} \times \mathbb{T}_{(\mu)}^{>,\succ} \longrightarrow \mathbb{T}_{(\mu)}$ may be defined for $\mathbb{T}_{(\mu)} = \mathbb{R}[[\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}] \times \mathfrak{M}]]$. For $\varphi \in \mathbf{T}$,and $\mathfrak{l} \in \mathfrak{L}_{<\beta}$, we should hacve a monomial $\mathfrak{l} \circ \mathrm{e}_\beta^\varphi$ in $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$. Since adding all these as formal symbols would create ambiguities generated by identities

$$\ell_\theta \circ \mathrm{e}_\beta^\varphi = \mathrm{e}_\beta^{\varphi-1}$$

when $\beta = \theta\,\omega$ (i.e. when μ is a successor). We choose to systemacitally write expressions $\ell_\theta \circ \mathrm{e}_\beta^\varphi$ as $\mathrm{e}_\beta^{\varphi-1}$ and thus only restrict our logarithmic hypermonomials t lie $\mathfrak{L}_{<\theta}$, compensating by allowing for certain transfinite products involving more than one $\varphi \in \mathbf{T}$.

We define the group $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ as follows. If μ is a limit, then $\theta = \beta$, and $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ is the group generated by the elements $\mathfrak{l}[\mathrm{e}_\beta^\varphi]$ with $\mathfrak{l} \in \mathfrak{L}_{<\theta}$ and satisfying the relations $\mathfrak{l}_1[\mathrm{e}_\beta^\varphi]\,\mathfrak{l}_2[\mathrm{e}_\beta^\varphi] = (\mathfrak{l}_1\,\mathfrak{l}_2)[\mathrm{e}_\beta^\varphi]$. Hence $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ is the group of products

$$\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi], \qquad \mathfrak{t}_\varphi \in \mathfrak{L}_{<\theta},$$

for which the *hypersupport*

$$\operatorname{hsupp} \mathfrak{t} := \{\varphi \in \mathbf{T} : \mathfrak{t}_\varphi \neq 1\}$$

of $\mathfrak{t}$ is finite. If μ is a successor, then let $\frown$ be the equivalence relation on $\mathbb{T}$ defined by

$$s \frown t \iff t - s \in \mathbb{Z}.$$

We let $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ be the group of formal products

$$\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi], \qquad \mathfrak{t}_\varphi \in \mathfrak{L}_{<\theta},$$

for which the hypersupport $\operatorname{hsupp} \mathfrak{t}$ is well-based and $\operatorname{hsupp} \mathfrak{t} / \frown$ is finite. Given $\mathfrak{s}, \mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$, we note that $\operatorname{hsupp} \mathfrak{s}^{-1}\mathfrak{t} \subseteq \operatorname{hsupp} \mathfrak{s} \cup \operatorname{hsupp} \mathfrak{t}$, whence $\mathfrak{s}^{-1}\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$. Hence $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ is indeed a group.

For $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]^{\neq 1}$, we define $\varphi_{\mathfrak{t}} := \max \operatorname{hsupp} \mathfrak{t}$ and $\gamma_{\mathfrak{t}} := \min\{\gamma < \theta : (\mathfrak{t}_{\varphi_{\mathfrak{t}}})_\gamma \neq 0\}$. We set $\mathfrak{t} \succ 1$ if $\mathfrak{t}_{\varphi_{\mathfrak{t}}} \succ 1$, which happens if and only if $(\mathfrak{t}_{\varphi_{\mathfrak{t}}})_{\gamma_{\mathfrak{t}}} > 0$. The following facts will be used frequently, where $\mathfrak{t}, \mathfrak{u}$ range over $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$:

- $\varphi_{\mathfrak{t}^{-1}} = \varphi_{\mathfrak{t}}$ for $\mathfrak{t} \neq 1$,
- $\varphi_{\mathfrak{t}\mathfrak{u}} \leqslant \max(\varphi_{\mathfrak{t}}, \varphi_{\mathfrak{u}})$, and if $\varphi_{\mathfrak{t}} \neq \varphi_{\mathfrak{u}}$ then $\varphi_{\mathfrak{t}\mathfrak{u}} = \max(\varphi_{\mathfrak{t}}, \varphi_{\mathfrak{u}})$
- If $1 \prec \mathfrak{t} \preccurlyeq \mathfrak{u}$ or $\mathfrak{u} \preccurlyeq \mathfrak{t} \prec 1$, then $\varphi_{\mathfrak{t}} \leqslant \varphi_{\mathfrak{u}}$,
- If $\mathfrak{t} \succ 1$ and $\mathfrak{u} \succcurlyeq 1$ or if $\mathfrak{t} \prec 1$ and $\mathfrak{u} \preccurlyeq 1$ then $\varphi_{\mathfrak{t}\mathfrak{u}} = \max(\varphi_{\mathfrak{t}}, \varphi_{\mathfrak{u}})$.

Let $\mathfrak{M}_{(\mu)}$ denote the direct product $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}] \times \mathfrak{M}$. We denote by $\mathfrak{t}\,\mathfrak{m}$ a general element $(\mathfrak{t}, \mathfrak{m})$ of this group, where we implicitly understand that $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ and $\mathfrak{m} \in \mathfrak{M}$; we also identify $(\mathfrak{t}, 1)$ and $(1, \mathfrak{m})$ with $\mathfrak{t}$ and $\mathfrak{m}$, respectively. We set $\mathbb{T}_{(\mu)} := \mathbb{R}[[\mathfrak{M}_{(\mu)}]]$.

Remark 6.2.2. Assume that μ is a successor and consider $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$ as above. The advantage of the representation of $\mathfrak{t}$ as an infinite product of terms of the form $\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi]$ with $\mathfrak{t}_\varphi \in \mathfrak{L}_{<\theta}$ is that such a representation is unique. Alternatively, it is possible to represent $\mathfrak{t}$ as a finite product of terms of the form $\mathfrak{l}[\mathrm{e}_\beta^\varphi]$ with $\mathfrak{l} \in \mathfrak{L}_{<\beta}$, but uniqueness is lost, since $\ell_0[\mathrm{e}_\beta^\varphi] = \ell_\theta[\mathrm{e}_\beta^{\varphi+1}]$.

Nevertheless, we may construct a privileged representation as a finite product as follows. Since $\operatorname{hsupp} \mathfrak{t} / \frown$ is finite, there exist $\varphi_1 > \cdots > \varphi_n \in \mathbf{T}$ with $\varphi_i \not\frown \varphi_j$ for $i \neq j$ and $\operatorname{hsupp} \mathfrak{t} / \frown = \{\varphi_1, \ldots, \varphi_n\} / \frown$. Since $\operatorname{hsupp} \mathfrak{t}$ is well-based, we may also take $\varphi_i = \max\{\varphi \in \operatorname{hsupp} \mathfrak{t} : \varphi \frown \varphi_i\}$ for all i. Then

$$\mathfrak{t} = \prod_{1 \leqslant i \leqslant n} \prod_{m \in \mathbb{N}} \mathfrak{t}_{\varphi_i - m}[\mathrm{e}_\beta^{\varphi_i - m}].$$

Fix $i\in\{1,\ldots,n\}$. For each $m\in\mathbb{N}$, we have $\log\mathfrak{t}_{\varphi_i-m}=\sum_{\gamma<\theta}(\mathfrak{t}_{\varphi_i-m})_\gamma\ell_{\gamma+1}$, so

$$\begin{aligned}\sum_{m\in\mathbb{N}}\log(\mathfrak{t}_{\varphi_i-m}\circ\ell_{\theta m}) &= \sum_{m\in\mathbb{N}}(\log\mathfrak{t}_{\varphi_i-m})\circ\ell_{\theta m}=\sum_{m\in\mathbb{N}}\left(\sum_{\gamma<\theta}(\mathfrak{t}_{\varphi_i-m})_\gamma\ell_{\gamma+1}\right)\circ\ell_{\theta m}\\ &= \sum_{m\in\mathbb{N}}\sum_{\gamma<\theta}(\mathfrak{t}_{\varphi_i-m})_\gamma\ell_{\theta m+\gamma+1}=\log\left(\prod_{m\in\mathbb{N}}\prod_{\gamma<\theta}\ell_{\theta m+\gamma}^{(\mathfrak{t}_{\varphi_i-m})_\gamma}\right).\end{aligned}$$

Set

$$\mathfrak{t}_{\varphi_i}^*:=\prod_{m\in\mathbb{N}}\prod_{\gamma<\theta}\ell_{\theta m+\gamma}^{(\mathfrak{t}_{\varphi_i-m})_\gamma}\in\mathfrak{L}_{<\beta}.$$

This gives us the finite representation

$$\mathfrak{t}=\prod_{1\leqslant i\leqslant n}\mathfrak{t}_{\varphi_i}^*[\mathrm{e}_\beta^{\varphi_i}].$$

Note that $\mathfrak{t}\succ 1\Longleftrightarrow\mathfrak{t}_{\varphi_1}\succ 1\Longleftrightarrow\mathfrak{t}_{\varphi_1}^*\succ 1$.

6.2.3 The ordering

Let $\mathfrak{M}_{(\mu)}^{\succ}$ be the set of all elements $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{(\mu)}$ that satisfy one of the following conditions:

$$\mathfrak{t}\succ 1,\ \mathfrak{m}\prec 1,\text{ and }\varphi_{\mathfrak{t}}>L_\beta(\mathfrak{m}^{-1}) \tag{I}$$

$$\mathfrak{t}\prec 1,\ \mathfrak{m}\succ 1,\text{ and }\varphi_{\mathfrak{t}}<L_\beta(\mathfrak{m}) \tag{II}$$

$$\mathfrak{t}\succcurlyeq 1\text{ and }\mathfrak{m}\succ 1 \tag{III}$$

$$\mathfrak{t}\succ 1\text{ and }\mathfrak{m}\succcurlyeq 1 \tag{IV}$$

We define the relation $\prec$ on $\mathfrak{M}_{(\mu)}$ by $\mathfrak{t}\,\mathfrak{m}\prec\mathfrak{u}\,\mathfrak{n}$ if and only if $(\mathfrak{u}\,\mathfrak{t}^{-1})\,(\mathfrak{n}\,\mathfrak{m}^{-1})\in\mathfrak{M}_{(\mu)}^{\succ}$.

Proposition 6.2.3. *The relation $\prec$ is an order on $\mathfrak{M}_{(\mu)}$ that extends the orderings on both $\mathfrak{M}$ and $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbb{T}}]$.*

Proof. By definition, the relation $\prec$ extends the orderings on $\mathfrak{M}$ and $\mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbb{T}}]$. In order to show that $\prec$ is an order, it suffices to check that $\mathfrak{M}_{(\mu)}^{\succ}$ is a total positive cone on $\mathfrak{M}_{(\mu)}$.

Let $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{(\mu)}\setminus\{1\}$. By the definition of $\mathfrak{M}_{(\mu)}^{\succ}$ and the fact that $\varphi_{\mathfrak{t}^{-1}}=\varphi_{\mathfrak{t}}$, it is clear that $\mathfrak{t}\,\mathfrak{m}$ and $(\mathfrak{t}\,\mathfrak{m})^{-1}$ cannot both be in $\mathfrak{M}_{(\mu)}^{\succ}$. Let us show that either $\mathfrak{t}\,\mathfrak{m}\in\mathfrak{M}_{(\mu)}^{\succ}$ or $(\mathfrak{t}\,\mathfrak{m})^{-1}\in\mathfrak{M}_{(\mu)}^{\succ}$. Assume that $\mathfrak{t}\,\mathfrak{m}\notin\mathfrak{M}_{(\mu)}^{\succ}$. If $\mathfrak{t}\prec 1$ and $\mathfrak{m}\preccurlyeq 1$ or $\mathfrak{t}\preccurlyeq 1$ and $\mathfrak{m}\prec 1$, then $(\mathfrak{t}\,\mathfrak{m})^{-1}$ satisfies (III) or (IV). Suppose that $\mathfrak{t}\succ 1$, $\mathfrak{m}\prec 1$, and $\varphi_{\mathfrak{t}}\leqslant L_\beta(\mathfrak{m}^{-1})$. Then $\varphi_{\mathfrak{t}}<L_\beta(\mathfrak{m}^{-1})$ since $\varphi_{\mathfrak{t}}\notin L_\beta(\mathbb{T}^{>,\succ})$, so $\varphi_{\mathfrak{t}^{-1}}=\varphi_{\mathfrak{t}}<L_\beta(\mathfrak{m}^{-1})$. Since $\mathfrak{t}^{-1}\prec 1$ and $\mathfrak{m}^{-1}\succ 1$, we conclude that $(\mathfrak{t}\,\mathfrak{m})^{-1}$ satisfies (II). If $\mathfrak{t}\prec 1$, $\mathfrak{m}\succ 1$, and $\varphi_{\mathfrak{t}}\geqslant L_\beta(\mathfrak{m})$ then $(\mathfrak{t}\,\mathfrak{m})^{-1}$ satisfies (I), for similar reasons.

Now let $\mathfrak{t}\,\mathfrak{m},\mathfrak{u}\,\mathfrak{n}\in\mathfrak{M}_{(\mu)}^{\succ}$. We will show that $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})\in\mathfrak{M}_{(\mu)}^{\succ}$. If both $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{s}\,\mathfrak{n}$ satisfy one of the last two rules, then this is clear. Thus, we may assume without loss of generality that $\mathfrak{t}\,\mathfrak{m}$ satisfies either rule (I) or rule (II). We consider the following cases:

Case 1: $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{u}\,\mathfrak{n}$ both satisfy (I) or they both satisfy (II). Suppose that they both satisfy (I). Then $\mathfrak{t}\,\mathfrak{u}\succ 1$ and $\mathfrak{m}\,\mathfrak{n}\prec 1$, so we need to verify that $\varphi_{\mathfrak{t}\mathfrak{u}}>L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1})$. By Corollary 5.3.15, we have $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1})=_\beta\max(L_\beta(\mathfrak{m}^{-1}),L_\beta(\mathfrak{n}^{-1}))$. Since $\mathfrak{t},\mathfrak{u}\succ 1$, we also have $\varphi_{\mathfrak{t}\mathfrak{u}}=\max(\varphi_{\mathfrak{t}},\varphi_{\mathfrak{u}})$, whence $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1})<_\beta\varphi_{\mathfrak{t}\mathfrak{u}}$. The case when $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{u}\,\mathfrak{n}$ both satisfy (II) is similar.

Case 2: $\mathfrak{t}\,\mathfrak{m}$ satisfies (I) and $\mathfrak{u}\,\mathfrak{n}$ satisfies (III) or (IV). We have $\mathfrak{t}\,\mathfrak{u}\succ 1$, so if $\mathfrak{m}\,\mathfrak{n}\succcurlyeq 1$, then $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (IV). Suppose that $\mathfrak{m}\,\mathfrak{n}\prec 1$. If $\mathfrak{n}=1$, then $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1})=L_\beta(\mathfrak{m}^{-1})$ and if $\mathfrak{n}\succ 1$, then $(\mathfrak{m}\,\mathfrak{n})^{-1}\prec\mathfrak{m}^{-1}$, so $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1})<L_\beta(\mathfrak{m}^{-1})$ as L_β is strictly increasing. Since $\mathfrak{t}\,\mathfrak{m}$ satisfies rule (I) and $\mathfrak{u}\succcurlyeq 1$, we have

$$\varphi_{\mathfrak{t}\mathfrak{u}}=\max(\varphi_{\mathfrak{t}},\varphi_{\mathfrak{u}})\geqslant\varphi_{\mathfrak{t}}>L_\beta(\mathfrak{m}^{-1})>L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}),$$

so $(\mathfrak{t}\,\mathfrak{u})(\mathfrak{m}\,\mathfrak{n})$ satisfies (I).

Case 3: $\mathfrak{t}\,\mathfrak{m}$ satisfies (II) and $\mathfrak{u}\,\mathfrak{n}$ satisfies (III) or (IV). We have $\mathfrak{m}\,\mathfrak{n}\succcurlyeq\mathfrak{m}\succ 1$, so if $\mathfrak{t}\,\mathfrak{u}\succcurlyeq 1$, then $(\mathfrak{t}\,\mathfrak{u})(\mathfrak{m}\,\mathfrak{n})$ satisfies (IV). Suppose that $\mathfrak{t}\,\mathfrak{u}\prec 1$. If $\mathfrak{u}\succ 1$, then $1\prec\mathfrak{u}\prec\mathfrak{t}^{-1}$, so $\varphi_{\mathfrak{u}}\leqslant\varphi_{\mathfrak{t}^{-1}}=\varphi_{\mathfrak{t}}$ and $\varphi_{\mathfrak{t}\mathfrak{u}}\leqslant\max(\varphi_{\mathfrak{t}},\varphi_{\mathfrak{u}})=\varphi_{\mathfrak{t}}$. Since $\mathfrak{t}\,\mathfrak{m}$ satisfies rule (II), we have $\varphi_{\mathfrak{t}}<_\beta L_\beta(\mathfrak{m})$, so

$$\varphi_{\mathfrak{t}\mathfrak{u}}\leqslant\varphi_{\mathfrak{t}}<_\beta L_\beta(\mathfrak{m})\leqslant L_\beta(\mathfrak{m}\,\mathfrak{n}).$$

Hence $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (II).

Case 4: One of the monomials $\mathfrak{t}\,\mathfrak{m}$ and $\mathfrak{u}\,\mathfrak{n}$ satisfies (I) and the other one satisfies (II). Without loss of generality, we may assume that $\mathfrak{t}\,\mathfrak{m}$ satisfies (I) and $\mathfrak{u}\,\mathfrak{n}$ satisfies (II). Let us first consider the case when $\mathfrak{t}\,\mathfrak{u} \prec 1$. Then $1 \prec \mathfrak{t} \prec \mathfrak{u}^{-1}$, so $\varphi_{\mathfrak{t}} \leqslant \varphi_{\mathfrak{u}^{-1}} = \varphi_{\mathfrak{u}}$ and $\varphi_{\mathfrak{t}\mathfrak{u}} \leqslant \varphi_{\mathfrak{u}}$. Since $\varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1})$ and $\varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n})$, we deduce that $L_\beta(\mathfrak{m}^{-1}) <_\beta L_\beta(\mathfrak{n})$, so $L_\beta(\mathfrak{m}\,\mathfrak{n}) =_\beta L_\beta(\mathfrak{n})$ by Corollary 5.3.18. Since $\mathfrak{u}\,\mathfrak{n}$ satisfies (II), we have $\varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n})$, so

$$\varphi_{\mathfrak{t}\mathfrak{u}} \leqslant \varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n}) =_\beta L_\beta(\mathfrak{m}\,\mathfrak{n})$$

and $(\mathfrak{t}\,\mathfrak{s})(\mathfrak{m}\,\mathfrak{n})$ satisfies (II).

Let us now consider the case when $\mathfrak{t}\,\mathfrak{u} \succcurlyeq 1$. If $\mathfrak{m}\,\mathfrak{n} \succ 1$, then $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (III). If $\mathfrak{m}\,\mathfrak{n} = 1$ and $\mathfrak{t}\,\mathfrak{u} \succ 1$, then $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (IV). If $\mathfrak{m}\,\mathfrak{n} = \mathfrak{t}\,\mathfrak{u} = 1$, then $\mathfrak{m}\,\mathfrak{n} = (\mathfrak{t}\,\mathfrak{u})^{-1}$, so $\mathfrak{t}\,\mathfrak{m} = (\mathfrak{u}\,\mathfrak{n})^{-1}$, contradicting that $\mathfrak{t}\mathfrak{m}, \mathfrak{u}\,\mathfrak{n} \in \mathfrak{M}_{(\mu)}^{\succ}$. It remains to consider the case that $\mathfrak{m}\,\mathfrak{n} \prec 1$. Then $\mathfrak{m}^{-1} \succ \mathfrak{n} \succ 1$, so $L_\beta(\mathfrak{m}^{-1}) > L_\beta(\mathfrak{n})$ as L_β is strictly increasing. Since $\varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1})$ and $\varphi_{\mathfrak{u}} < L_\beta(\mathfrak{n})$, we deduce that $\varphi_{\mathfrak{t}} > \varphi_{\mathfrak{u}}$, so $\varphi_{\mathfrak{t}\mathfrak{u}} = \varphi_{\mathfrak{t}}$. Since $\mathfrak{n}^{-1} \prec 1$, we have $(\mathfrak{m}\,\mathfrak{n})^{-1} \prec \mathfrak{m}^{-1}$, so $L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}) < L_\beta(\mathfrak{m}^{-1})$. This gives

$$\varphi_{\mathfrak{t}\mathfrak{u}} = \varphi_{\mathfrak{t}} > L_\beta(\mathfrak{m}^{-1}) > L_\beta((\mathfrak{m}\,\mathfrak{n})^{-1}),$$

so $(\mathfrak{t}\,\mathfrak{u})\,(\mathfrak{m}\,\mathfrak{n})$ satisfies (I). □

Remark 6.2.4. Given $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]^{\succ}$, we have

$$\mathfrak{m} \prec \mathfrak{t} \Longleftrightarrow \mathfrak{m}^{-1}\,\mathfrak{t} \succ 1 \Longleftrightarrow L_\beta(\mathfrak{m}) < \varphi_{\mathfrak{t}}.$$

Since $\mathfrak{m} \not\asymp \mathfrak{t}$, we also have $\mathfrak{m} \succ \mathfrak{t} \Longleftrightarrow L_\beta(\mathfrak{m}) > \varphi_{\mathfrak{t}}$. More generally, for $s \in \mathbb{T}^{>,\succ}$, we have

$$s \prec \mathfrak{t} \Longleftrightarrow L_\beta(s) < \varphi_{\mathfrak{t}}, \qquad s \succ \mathfrak{t} \Longleftrightarrow L_\beta(s) > \varphi_{\mathfrak{t}}.$$

This is because $L_\beta(s) =_\beta L_\beta(\mathfrak{d}_s)$ by Corollary 5.3.17 with $\sigma = \gamma = 0$.

6.3 Extending the hyperlogarithmic structure

In this subsection, we extend the hyperlogarithms L_{ω^η} from $\mathbb{T}$ to $\mathbb{T}_{(\mu)}$, while verifying that they satisfy the axioms for hyperserial skeletons. We separate various cases as a function of η, including the case of the ordinary logarithm when $\eta = 0$ and starting with the real power operation.

In each case, we start with the definition of the domain $\operatorname{dom} L_{\omega^\eta}$ of the extended hyperlogarithm L_{ω^η} on $\mathbb{T}_{(\mu)}$ and then define L_{ω^η} on the elements of $\operatorname{dom} L_{\omega^\eta}$ which do not already lie in $\mathfrak{M}_{\omega^\eta}$. We next check that $(\mathbb{T}, (L_{\omega^\eta})_{\iota \leqslant \eta})$ satisfies the domain definition axioms $\mathbf{DD}_\eta$, as well as the other axioms for hyperserial skeletons.

6.3.1 Extending the real power operation

For $r \in \mathbb{R}$ and $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}$, define $(\mathfrak{t}\,\mathfrak{m})^r := \mathfrak{t}^r\,\mathfrak{m}^r$ where $\mathfrak{m}^r$ is as defined in $\mathfrak{M}$, and

$$\mathfrak{t}^r := \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi^r[\mathrm{e}_\beta^\varphi] \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}].$$

It is easy to check that this defines a real power operation on $\mathfrak{M}_{(\mu)}$. Note that $\varphi_{\mathfrak{t}^r} = \varphi_{\mathfrak{t}}$ for each non-zero $r \in \mathbb{R}$.

Now that we have defined an ordering and a real power operation on $\mathfrak{M}_{(\mu)}$, we let $\mathbb{T}_{(\mu)} := \mathbb{R}[[\mathfrak{M}_{(\mu)}]]$. Then $\mathbb{T}_{(\mu)}$ is a field of well-based series extending $\mathbb{T}$.

6.3.2 Extending the logarithm when $\mu = 1$

Suppose that $\mu = 1$, so $\beta = \omega$ and $\theta = 1$. For $\ell_0^r \in \mathfrak{L}_{<1}$ and $\varphi \in \mathbf{T}$, we define

$$\log(\ell_0^r[\mathrm{e}_\omega^\varphi]) := r\,\mathrm{e}_\omega^{\varphi - 1}$$

We extend log to $\mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$ by setting

$$\log \mathfrak{t} := \sum_{\varphi \in \mathbf{T}} \log(\ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi])$$

for $\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi] \in \mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$. Note that $\log(\ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi]) \neq 0$ if and only if $\varphi \in \operatorname{hsupp} \mathfrak{t}$. We claim that $\log \mathfrak{t}$ is well-defined. Let $\varphi_1 > \cdots > \varphi_n \in \mathbf{T}$. As in Remark 6.2.2, we have $\mathfrak{t} = \prod_{i=1}^n \prod_{m \in \mathbb{N}} \ell_0^{r_{\varphi_i - m}}[\mathrm{e}_\omega^{\varphi_i - m}]$ and

$$\log \mathfrak{t} = \sum_{i=1}^n \sum_{m \in \mathbb{N}} \log(\ell_0^{r_{\varphi_i - m}}[\mathrm{e}_\omega^{\varphi_i - m}]) = \sum_{i=1}^n \sum_{m \in \mathbb{N}} r_{\varphi_i - m}\, \mathrm{e}_\omega^{\varphi_i - m - 1}.$$

Each sum $\sum_{m \in \mathbb{N}} r_{\varphi_i - m}\, \mathrm{e}_\omega^{\varphi_i - m - 1}$ exists in $\mathbb{T}_{(\mu)}$, since the support $(\mathrm{e}_\omega^{\varphi_i - m - 1})_{m \in A_i}$ is a strictly decreasing sequence in $\mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$. Thus, $\log \mathfrak{t}$ is well-defined. If $\mathfrak{t} \neq 1$, then we note that

$$\log \mathfrak{t} \sim \log(\ell_0^{r_{\varphi_\mathfrak{t}}}[\mathrm{e}_\omega^{\varphi_\mathfrak{t}}]) \sim r_{\varphi_\mathfrak{t}}\, \mathrm{e}_\omega^{\varphi_\mathfrak{t} - 1}$$

Finally, we extend log to all of $\mathfrak{M}_{(\mu)}$ by setting $\log(\mathfrak{t}\,\mathfrak{m}) := \log \mathfrak{t} + \log \mathfrak{m}$. We let L_1 be the restriction of log to the class $\mathfrak{M}_{(\mu)}^{\succ}$, so $(\mathbb{T}_{(\mu)}, L_1)$ satisfies $\mathbf{DD_0}$.

Using the definition of real powers, it is straightforward to check that $(\mathbb{T}_{(\mu)}, L_1)$ satisfies $\mathbf{FE_0}$. For $\varphi \in \mathbf{T}$, we have $\varphi - 1 \in \mathbf{T}$ by (6.2.1), whence $\mathrm{e}_\omega^{\varphi - 1} \in \mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]^{\succ}$. Therefore $\log(\ell_0^r[\mathrm{e}_\omega^\varphi]) \succ 1$ for all $r \in \mathbb{R}$. It follows that

$$\operatorname{supp} L_1(\mathfrak{t}\,\mathfrak{m}) \subseteq \operatorname{supp} \log \mathfrak{t} \cup \operatorname{supp} \log \mathfrak{m} \succ 1$$

for $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^{\succ}$ and $\mathbf{R_0}$ is satisfied. The axiom $\mathbf{P}_0$ follows from $\mathbf{FE_0}$, so it remains to be shown that $(\mathbb{T}_{(\mu)}, L_1)$ satisfies $\mathbf{A_0}$ and $\mathbf{M_0}$.

Lemma 6.3.1. *$(\mathbb{T}_{(\mu)}, L_1)$ satisfies $\mathbf{A_0}$.*

Proof. Given $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^{\succ}$, we must show that $L_1(\mathfrak{t}\,\mathfrak{m}) \prec \mathfrak{t}\,\mathfrak{m}$. We proceed by case distinction:

1. If $\mathfrak{t} = 1$, then $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{m}) \prec \mathfrak{m} = \mathfrak{t}\,\mathfrak{m}$ since $(\mathbb{T}, L_1)$ satisfies $\mathbf{A_0}$.
2. If $\mathfrak{m} = 1$, then $\mathfrak{t} \succ 1$ and
$$\mathfrak{d}_{L_1(\mathfrak{t})} = \mathrm{e}_\omega^{\varphi_\mathfrak{t} - 1}$$
We have $\mathfrak{d}_{L_1(\mathfrak{t})} \in \mathfrak{L}_{<1}[\mathrm{e}_\omega^{\mathbf{T}}]$ and $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}} = \varphi_\mathfrak{t} - 1$. Thus $\mathfrak{d}_{L_1(\mathfrak{t})} \prec \mathfrak{t}$ since $\varphi_\mathfrak{t} - 1 < \varphi_\mathfrak{t}$. Thus $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) \prec \mathfrak{t} = \mathfrak{t}\,\mathfrak{m}$.
3. Suppose $\mathfrak{t} \succ 1$, $\mathfrak{m} \prec 1$, and $\varphi_\mathfrak{t} > L_\omega(\mathfrak{m}^{-1})$. We have $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) - L_1(\mathfrak{m}^{-1})$, so it is enough to show that $L_1(\mathfrak{t}) \prec \mathfrak{t}\,\mathfrak{m}$ and $L_1(\mathfrak{m}^{-1}) \prec \mathfrak{t}\,\mathfrak{m}$. We have
$$L_\omega(\mathfrak{m}^{-2}) = L_\omega^{\uparrow 1}(2\, L_1(\mathfrak{m}^{-1})) =_\omega L_\omega(\mathfrak{m}^{-1})$$
by Lemma 5.3.14, so $\varphi_\mathfrak{t} > L_\omega(\mathfrak{m}^{-2})$, whence $\mathfrak{t}\,\mathfrak{m}^2 \succ 1$ and $\mathfrak{t}\,\mathfrak{m} \succ \mathfrak{m}^{-1} \succ L_1(\mathfrak{m}^{-1})$. Since $\varphi_{\mathfrak{t}^{1/2}} = \varphi_\mathfrak{t} > L_\omega(\mathfrak{m}^{-1})$, we also have $\mathfrak{t}^{1/2}\,\mathfrak{m} \succ 1$, so
$$\mathfrak{t}\,\mathfrak{m} \succ \mathfrak{t}^{1/2} \succ L_1(\mathfrak{t}^{1/2}) \asymp L_1(\mathfrak{t}).$$
4. Suppose $\mathfrak{t} \prec 1$, $\mathfrak{m} \succ 1$, and $\varphi_\mathfrak{t} < L_\omega(\mathfrak{m})$. This time, we need to show that $L_1(\mathfrak{t}^{-1}) \prec \mathfrak{t}\,\mathfrak{m}$ and $L_1(\mathfrak{m}) \prec \mathfrak{t}\,\mathfrak{m}$. Using that $\varphi_{\mathfrak{t}^2} = \varphi_\mathfrak{t}$ and that $L_\omega(\mathfrak{m}^{1/2}) =_\omega L_\omega(\mathfrak{m})$, we have $\mathfrak{t}^2\,\mathfrak{m}, \mathfrak{t}\,\mathfrak{m}^{1/2} \succ 1$, so
$$\mathfrak{t}\,\mathfrak{m} \succ \mathfrak{t}^{-1} \succ L_1(\mathfrak{t}^{-1}), \qquad \mathfrak{t}\,\mathfrak{m} \succ \mathfrak{m}^{1/2} \succ L_1(\mathfrak{m}^{1/2}) \asymp L_1(\mathfrak{m}).$$
5. If $\mathfrak{t} \succ 1$ and $\mathfrak{m} \succ 1$, then $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) + L_1(\mathfrak{m})$. So the result follows from the fact that $L_1(\mathfrak{t}) \prec \mathfrak{t} \prec \mathfrak{t}\,\mathfrak{m}$ and $L_1(\mathfrak{m}) \prec \mathfrak{m} \prec \mathfrak{t}\,\mathfrak{m}$. □

Lemma 6.3.2. *$(\mathbb{T}_{(\mu)}, L_1)$ satisfies $\mathbf{M_0}$.*

Proof. Given $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^{\succ}$, we need to show that $L_1(\mathfrak{t}\,\mathfrak{m}) > 0$. If $\mathfrak{t} = 1$, then $\mathfrak{m} \succ 1$ so $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{m}) > 0$ since $(\mathbb{T}, L_1)$ satisfies $\mathbf{M_0}$. If $\mathfrak{m} = 1$, then $\mathfrak{t} \succ 1$, so $r_{\varphi_\mathfrak{t}} > 0$. Since

$$L_1(\mathfrak{t}) \sim r_{\varphi_\mathfrak{t}}\, \mathrm{e}_\omega^{\varphi_\mathfrak{t} - 1}$$

we have $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) > 0$. If $\mathfrak{t}, \mathfrak{m} \succ 1$, then $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) + L_1(\mathfrak{m}) > 0$.

Consider now the case that $\mathfrak{t} \prec 1$, $\mathfrak{m} \succ 1$, and $\varphi_\mathfrak{t} < L_\omega(\mathfrak{m})$. Since $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{m}) - L_1(\mathfrak{t}^{-1})$, we need to show that $L_1(\mathfrak{t}^{-1}) < L_1(\mathfrak{m})$. For each $r \in \mathbb{R}^{>}$, we have

$$L_\omega(\mathfrak{m}) =_\omega L_\omega^{\uparrow 1}(r\, L_1(\mathfrak{m})) = L_\omega(r\, L_1(\mathfrak{m})) + 1$$

by Lemma 5.3.14. Since $\varphi_{\mathfrak{t}} <_{\omega} L_{\omega}(\mathfrak{m})$, this gives $\varphi_{\mathfrak{t}} - 1 < L_{\omega}(r\, L_1(\mathfrak{m}))$. We have

$$L_1(\mathfrak{t}^{-1}) \asymp \mathrm{e}_{\omega}^{\varphi_{\mathfrak{t}} - 1} \prec r\, L_1(\mathfrak{m}) \asymp L_1(\mathfrak{m})$$

by Remark 6.2.4. This gives $L_1(\mathfrak{t}^{-1}) \asymp E_{\omega}(\varphi_{\mathfrak{t}} - 1) \prec L_1(\mathfrak{m})$.

Finally, suppose that $\mathfrak{t} \succ 1$, $\mathfrak{m} \prec 1$, and $\varphi_{\mathfrak{t}} > L_{\omega}(\mathfrak{m}^{-1})$. The same argument as above gives $\varphi_{\mathfrak{t}} - 1 > L_{\omega}(r\, L_1(\mathfrak{m}^{-1}))$ for $r \in \mathbb{R}^{>}$, so $L_1(\mathfrak{t}) \succ L_1(\mathfrak{m}^{-1})$ and $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) - L_1(\mathfrak{m}^{-1}) > 0$. □

6.3.3 Extending the logarithm when $\mu > 1$

For $\mathfrak{l} = \prod_{\gamma < \theta} \ell_{\gamma}^{\mathfrak{l}_{\gamma}} \in \mathfrak{L}_{<\theta}$ and $\varphi \in \mathbf{T}$, we define

$$\log(\mathfrak{l}[\mathrm{e}_{\beta}^{\varphi}]) := \sum_{\gamma < \theta} \mathfrak{l}_{\gamma}\, \ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}].$$

This sum is well-defined, as $\ell_{\sigma+1}[\mathrm{e}_{\beta}^{\varphi}] \prec \ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}]$ for $\gamma < \sigma < \theta$. For $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_{\beta}^{\mathbf{T}}]$, we set

$$\log \mathfrak{t} := \sum_{\varphi \in \operatorname{hsupp} \mathfrak{t}} \log(\mathfrak{t}_{\varphi}[\mathrm{e}_{\beta}^{\varphi}]) = \sum_{\varphi \in \operatorname{hsupp} \mathfrak{t}} \sum_{\gamma < \theta} (\mathfrak{t}_{\varphi})_{\gamma}\, \ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}].$$

This sum is also well-defined, as $\operatorname{hsupp} \mathfrak{t}$ is well-based and $\ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}] \prec \ell_{\sigma+1}[\mathrm{e}_{\beta}^{\psi}]$ for all $\gamma, \sigma < \theta$, and $\varphi, \psi \in \mathbf{T}$ with $\varphi < \psi$. If $\mathfrak{t} \neq 1$, then note that $\log \mathfrak{t} \sim (\mathfrak{t}_{\varphi_{\mathfrak{t}}})_{\gamma_{\mathfrak{t}}}\, \ell_{\gamma_{\mathfrak{t}}+1}[\mathrm{e}_{\beta}^{\varphi}]$, so

$$\mathfrak{d}_{\log \mathfrak{t}} = \ell_{\gamma_{\mathfrak{t}}+1}[\mathrm{e}_{\beta}^{\varphi_{\mathfrak{t}}}] = \mathfrak{d}_{\log \mathfrak{t}_{\varphi_{\mathfrak{t}}}}[\mathrm{e}_{\beta}^{\varphi_{\mathfrak{t}}}]$$

and $\log \mathfrak{t} > 0$ whenever $\mathfrak{t} \succ 1$. Finally, we extend log to all of $\mathfrak{M}_{(\mu)}$ by setting

$$\log(\mathfrak{t}\,\mathfrak{m}) := \log \mathfrak{t} + \log \mathfrak{m}.$$

for $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}$. As before, we let L_1 be the restriction of log to $\mathfrak{M}_{(\mu)}^{\succ}$, so $(\mathbb{T}_{(\mu)}, L_1)$ satisfies $\mathbf{DD_0}$.

The axiom $\mathbf{FE}_0$ (and thus $\mathbf{P}_0$) follow easily from the definition of L_1 and the axiom $\mathbf{R_0}$ holds since $\ell_{\gamma+1}[\mathrm{e}_{\beta}^{\varphi}] \succ 1$ for each γ. Let us prove that $\mathbf{A_0}$ holds for $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_{\beta}^{\mathbf{T}}]^{\succ}$. Given $\mathfrak{t} \succ 1$, we need to show that $\mathfrak{t}\,\mathfrak{d}_{L_1(\mathfrak{t})}^{-1} \succ 1$. Since $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}} = \varphi_{\mathfrak{t}}$, it suffices to show that $(\mathfrak{t}\,\mathfrak{d}_{L_1(\mathfrak{t})}^{-1})_{\varphi_{\mathfrak{t}}} = \mathfrak{t}_{\varphi_{\mathfrak{t}}} (\mathfrak{d}_{L_1(\mathfrak{t})}^{-1})_{\varphi_{\mathfrak{t}}} \succ 1$. Since $(\mathfrak{d}_{L_1(\mathfrak{t})}^{-1})_{\varphi_{\mathfrak{t}}} = \mathfrak{d}_{L_1(\mathfrak{t}_{\varphi_{\mathfrak{t}}})}^{-1}$, this further reduces to showing that $\mathfrak{t}_{\varphi_{\mathfrak{t}}} \succ L_1(\mathfrak{t}_{\varphi_{\mathfrak{t}}})$. But this follows from the fact that $\mathbf{A_0}$ holds for $\mathbb{L}_{<\theta}$. The proof that $\mathbf{A_0}$ holds for a general element $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^{\succ}$ is identical to cases 3–5 of Lemma 6.3.1. Let us now show that $(\mathbb{T}_{(\mu)}, L_1)$ also satisfies $\mathbf{M_0}$.

Lemma 6.3.3. $(\mathbb{T}_{(\mu)}, L_1)$ *satisfies* $\mathbf{M_0}$.

Proof. We have $L_1(\mathfrak{t}) > 0$ for $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_{\beta}^{\mathbf{T}}]^{\succ}$ and $L_1(\mathfrak{m}) > 0$ for $\mathfrak{m} \in \mathfrak{M}^{\succ}$. It follows that $L_1(\mathfrak{t}\,\mathfrak{m}) > 0$ for $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^{\succ}$ so long as $\mathfrak{t}, \mathfrak{m} \succcurlyeq 1$. Suppose that $\mathfrak{t} \succ 1$, $\mathfrak{m} \prec 1$, and $\varphi_{\mathfrak{t}} > L_{\beta}(\mathfrak{m}^{-1})$. Then $L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{t}) - L_1(\mathfrak{m}^{-1})$, so it is enough to show that $L_1(\mathfrak{t}) \succ L_1(\mathfrak{m}^{-1})$. As shown in the proof that $\mathbf{A_0}$ holds, we have $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}} = \varphi_{\mathfrak{t}}$. By Lemma 5.3.16, we also have $L_{\beta}(\mathfrak{m}^{-1}) =_{\beta} L_{\beta}(L_1(\mathfrak{m}^{-1}))$. Thus, $\varphi_{\mathfrak{d}_{L_1(\mathfrak{t})}} > L_{\beta}(L_1(\mathfrak{m}^{-1}))$, so $L_1(\mathfrak{t}) \asymp \mathfrak{d}_{L_1(\mathfrak{t})} \succ L_1(\mathfrak{m}^{-1})$; see Remark 6.2.4. The case that $\mathfrak{t} \prec 1$, $\mathfrak{m} \succ 1$, and $\varphi_{\mathfrak{t}} < L_{\beta}(\mathfrak{m})$ is similar. □

6.3.4 Extending $L_{\omega^{\eta}}$ when $0 < \eta < \mu_-$

Given $0 < \eta < \mu_-$, we set

$$\operatorname{dom} L_{\omega^{\eta}} := \mathfrak{M}_{\omega^{\eta}} \cup \{\ell_{\gamma}[\mathrm{e}_{\beta}^{\varphi}] : \varphi \in \mathbf{T} \text{ and } \omega^{\eta_-} \leqslant_o \gamma < \theta\}.$$

Given γ with $\omega^{\eta_-} \leqslant_o \gamma < \theta$, we decompose $\gamma = \gamma_{\geqslant \omega^{\eta}} + \omega^{\eta_-} n$, and define

$$L_{\omega^{\eta}}(\ell_{\gamma}[\mathrm{e}_{\beta}^{\varphi}]) := \ell_{\gamma_{\geqslant \omega^{\eta}} + \omega^{\eta}}[\mathrm{e}_{\beta}^{\varphi}] - n.$$

Note that $n = 0$ and $L_{\omega^{\eta}}(\ell_{\gamma}[\mathrm{e}_{\beta}^{\varphi}]) = \ell_{\gamma + \omega^{\eta}}[\mathrm{e}_{\beta}^{\varphi}]$ whenever η is a limit ordinal. More generally, we have

$$L_{\omega^{\iota}}(\ell_{\gamma}[\mathrm{e}_{\beta}^{\varphi}]) = \ell_{\gamma + \omega^{\iota}}[\mathrm{e}_{\beta}^{\varphi}]$$

whenever $\iota \leqslant \eta_-$ (including the case when $\iota = 0$).

Lemma 6.3.4. *$(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{DD}_\eta$ for each $\eta < \mu_-$.*

Proof. We prove this by induction on $\eta < \mu_-$, beginning with $\eta = 1$. Let $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^{\succ}$, so

$$L_1(\mathfrak{t}\,\mathfrak{m}) = \log \mathfrak{m} + \sum_{\varphi \in \operatorname{hsupp} \mathfrak{t}} \sum_{\gamma < \theta} (\mathfrak{t}_\varphi)_\gamma \,\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi].$$

If $L_1(\mathfrak{t}\,\mathfrak{m}) \in \mathfrak{M}_{(\mu)}^{\succ}$, then either $\mathfrak{t} = 1$ or $\mathfrak{m} = 1$. If $\mathfrak{t} = 1$, then $\mathfrak{m} \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\mathfrak{m} \in \mathfrak{M}_\omega$. If $\mathfrak{m} = 1$, then $L_1(\mathfrak{t}) \in \mathfrak{M}_{(\mu)}^{\succ}$ if and only if $\mathfrak{t} = \ell_\gamma[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_\omega$. It remains to note that $L_n(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma+n}[\mathrm{e}_\beta^\varphi] \in \mathfrak{M}_{(\mu)}^{\succ}$ for all n.

Now assume that $\eta > 1$ and that $\mathbf{DD}_\iota$ holds for all $\iota < \eta$. Since $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{DD}_\eta$ for each $\eta < \mu_-$, we may focus on elements of the form $\ell_\gamma[\mathrm{e}_\beta^\varphi]$ where $\gamma < \theta$ and $\varphi \in \mathbf{T}$. For the remainder of the proof, we fix such an element. If η is a successor, then we need to show that $\ell_\gamma[\mathrm{e}_\beta^\varphi] \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\eta_-}}^{\circ n}$ if and only if $\gamma \geqslant_o \omega^{\eta_-}$. One direction is clear: if $\gamma \geqslant_o \omega^{\eta_-}$ then $L_{\omega^{\eta_-} n}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma+\omega^{\eta_-} n}[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_{\omega^{\eta_-}}$ for each n. For the other direction, if $\ell_\gamma[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_{\omega^{\eta_-}}$, then $\gamma \geqslant_o \omega^{\eta_{--}}$, so write $\gamma = \gamma_{\geqslant\omega^{\eta_-}} + \omega^{\eta_{--}} m$ and note that $L_{\omega^{\eta_-}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant\omega^{\eta_-}} + \omega^{\eta_-}}[\mathrm{e}_\beta^\varphi] - m$ is a monomial if and only if $m = 0$. If η is a limit, then $\gamma \geqslant_o \omega^{\iota_-}$ for all $\iota < \eta$ if and only if $\gamma \geqslant_o \omega^{\eta_-} = \omega^\eta$, so we have $\ell_\gamma[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_{\omega^\eta}$ if and only if $\ell_\gamma[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_{\omega^\iota}$ for all $\iota < \eta$. □

Lemma 6.3.5. *$(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{A}_\eta$ for each $\eta < \mu_-$.*

Proof. Let $\varphi \in \mathbf{T}$ and $\eta, \iota, \gamma \in \mathbf{On}$ with $0 \leqslant \iota < \eta < \mu_-$ and $\omega^{\eta_-} \leqslant_o \gamma < \theta$. Since $(\mathbb{T}, (L_{\omega^\sigma})_{\sigma<\mu_-})$ satisfies $\mathbf{A}_\sigma$ for each $\sigma < \mu_-$, it suffices to show that $L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) < L_{\omega^\iota}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$. Decomposing $\gamma = \gamma_{\geqslant\omega^\eta} + \omega^{\eta_-} n$, we have $\gamma_{\geqslant\omega^\eta} + \omega^\eta > \gamma + \omega^\iota$, so

$$L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] - n \leqslant \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] < \ell_{\gamma+\omega^\iota}[\mathrm{e}_\beta^\varphi] = L_{\omega^\iota}(\ell_\gamma[\mathrm{e}_\beta^\varphi]). \qquad \square$$

Let $0 < \eta < \mu_-$, let $\omega^{\eta_-} \leqslant_o \gamma < \theta$, and let $\varphi \in \mathbf{T}$. We note that $L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$ has no infinitesimal terms in its support, so $\mathbf{R}_\eta$ is satisfied since it holds in $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu_-})$. To see that $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{FE}_\eta$, suppose that η is a successor and write $\gamma = \gamma_{\geqslant\omega^\eta} + \omega^{\eta_-} n$. Then

$$L_{\omega^\eta}(L_{\omega^{\eta_-}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^\eta}(\ell_{\gamma_{\geqslant\omega^\eta}+\omega^{\eta_-}(n+1)}[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] - (n+1) = L_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) - 1.$$

Lemma 6.3.6. *$(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{M}_\eta$ for each $\eta < \mu_-$ with $\eta > 0$.*

Proof. Let $\eta < \mu_-$ with $\eta > 0$, let $\mathfrak{a}, \mathfrak{b} \in (\mathfrak{M}_{(\mu)})_{\omega^\eta}$ with $\mathfrak{a} \prec \mathfrak{b}$, and let $\omega^\iota n < \omega^\eta$. We want to show that

$$L_{\omega^\eta}(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_{\omega^\eta}(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_{\omega^\eta}$, then this holds because $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{M}_\eta$. Consider the following cases:

1. If $\mathfrak{a} = \ell_\gamma[\mathrm{e}_\beta^\varphi]$ and $\mathfrak{b} = \ell_\sigma[\mathrm{e}_\beta^\psi]$, then write $\gamma = \gamma_{\geqslant\omega^\eta} + \omega^{\eta_-} m$ and $\sigma = \sigma_{\geqslant\omega^\eta} + \omega^{\eta_-} k$. We have

$$\begin{aligned} L_{\omega^\eta}(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} &= \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] - m + \ell_{\gamma+\omega^\eta n}^{-1}[\mathrm{e}_\beta^\varphi] \\ L_{\omega^\eta}(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1} &= \ell_{\sigma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\psi] - k - \ell_{\sigma+\omega^\eta n}^{-1}[\mathrm{e}_\beta^\psi]. \end{aligned}$$

   Since $\mathfrak{a} \prec \mathfrak{b}$, we have $\varphi \leqslant \psi$. If $\varphi < \psi$, then $\ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] \prec \ell_{\sigma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\psi]$. If $\varphi = \psi$, then $\gamma > \sigma$, so either $\gamma_{\geqslant\omega^\eta} > \sigma_{\geqslant\omega^\eta}$ or $\gamma_{\geqslant\omega^\eta} = \sigma_{\geqslant\omega^\eta}$ and $m > k$. In both cases, we have $L_{\omega^\eta}(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_{\omega^\eta}(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$.

2. If $\mathfrak{a} = \ell_\gamma[\mathrm{e}_\beta^\varphi]$ and $\mathfrak{b} \in \mathfrak{M}_{\omega^\eta}$, then we must have $\varphi < L_\beta(\mathfrak{b})$ by Remark 6.2.4. Writing $\gamma = \gamma_{\geqslant\omega^\eta} + \omega^{\eta_-} m$, we have $L_{\omega^\eta}(\mathfrak{a}) = \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] - m$, so $\mathfrak{d}_{L_{\omega^\eta}(\mathfrak{a})} = \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]$. By Corollary 5.3.17, we have $L_\beta(L_{\omega^\eta}(\mathfrak{b})) =_\beta L_\beta(\mathfrak{b}) > \varphi$, so

$$L_{\omega^\eta}(\mathfrak{b}) \succ \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi] \asymp L_{\omega^\eta}(\mathfrak{a}),$$

   again by Remark 6.2.4. In particular, $L_{\omega^\eta}(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_{\omega^\eta}(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$.

3. If $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$ and $\mathfrak{b}=\ell_\gamma[\mathrm{e}_\beta^\varphi]$, then $\varphi>L_\beta(\mathfrak{a})$. Arguing as in the previous case, we have $L_{\omega^\eta}(\mathfrak{b})\asymp \ell_{\gamma_{\geqslant\omega^\eta}+\omega^\eta}[\mathrm{e}_\beta^\varphi]\succ L_{\omega^\eta}(\mathfrak{a})$. □

Lemma 6.3.7. $(\mathbb{T}_{(\mu)},(L_{\omega^\eta})_{\eta<\mu_-})$ *satisfies* $\mathbf{P}_\eta$ *for each* $0<\eta<\mu_-$.

Proof. Let $\mathfrak{a}\in(\mathfrak{M}_{(\mu)})_{\omega^\eta}$ and let $(r_\gamma)_{\gamma<\omega^\eta}$ be a sequence of real numbers. Consider the sum $s:=\sum_{\gamma<\omega^\eta} r_\gamma L_{\gamma+1}(\mathfrak{a})$. We need to show that $s\in\log\mathfrak{M}_{(\mu)}$. If $\mathfrak{a}\in\mathfrak{M}_{\omega^\eta}$, then $s\in\log\mathfrak{M}$. Suppose $\mathfrak{a}=\ell_\sigma[\mathrm{e}_\beta^\varphi]$ with $\omega^{\eta_-}\leqslant_o\sigma<\theta$. Then $L_\gamma(\mathfrak{a})=\ell_{\sigma+\gamma}[\mathrm{e}_\beta^\varphi]$ for all $\gamma<\omega^\eta$, so

$$s=\sum_{\gamma<\omega^\eta} r_\gamma L_{\gamma+1}(\ell_\sigma[\mathrm{e}_\beta^\varphi])=\sum_{\gamma<\omega^\eta} r_\gamma \ell_{\sigma+\gamma+1}[\mathrm{e}_\beta^\varphi]=\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi])$$

where $\mathfrak{l}:=\prod_{\gamma<\omega^\eta}\ell_{\sigma+\gamma}^{r_\gamma}\in\mathfrak{L}_{<\theta}$. □

6.3.5 Extending L_θ if $\mu>1$ is a successor

Assume that $\mu>1$ is a successor and let $\xi:=\omega^{\mu_{--}}$. We take

$$\operatorname{dom} L_\theta:=\mathfrak{M}_\theta\cup\{\ell_\gamma[\mathrm{e}_\beta^\varphi]:\varphi\in\mathbf{T}\text{ and }\xi\leqslant_o\gamma<\theta\}.$$

Note that $\xi\leqslant_o\gamma<\theta$ implies $\gamma=\xi n$ for some $n\in\mathbb{N}$. Moreover, if μ_- is a limit, then $n=0$. In other words,

$$\operatorname{dom} L_\theta \;=\; \begin{cases}\mathfrak{M}_\theta\cup\{\ell_{\xi n}[\mathrm{e}_\beta^\varphi]:\varphi\in\mathbf{T}\text{ and }n\in\mathbb{N}\} & \text{if } \mu_- \text{ is a successor.}\\ \mathfrak{M}_\theta\cup\{\mathrm{e}_\beta^\varphi:\varphi\in\mathbf{T}\} & \text{if } \mu_- \text{ is a limit.}\end{cases}$$

We define

$$L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]):=\mathrm{e}_\beta^{\varphi-1}-n.$$

The proofs of Lemmas 6.3.4 and 6.3.7 can be amended to show that $(\mathbb{T}_{(\mu)},(L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{DD}_{\mu_-}$ and $\mathbf{P}_{\mu_-}$; just replace η with μ_-. Since $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{R}_{\mu_-}$, $\mathbf{FE}_{\mu_-}$, and $\mathbf{A}_{\mu_-}$, it suffices to check these axioms for elements of the form $\ell_{\xi n}[\mathrm{e}_\beta^\varphi]$, where $\varphi\in\mathbf{T}$ and $\xi n<\theta$. We have $\mathrm{e}_\beta^{\varphi-1}\in\mathfrak{M}_{(\mu)}^\succ$ so $\operatorname{supp} L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi])\succcurlyeq 1$ and $(\mathbb{T}_{(\mu)},(L_{\omega^\eta})_{\eta<\mu_-})$ satisfies $\mathbf{R}_{\mu_-}$. As for $\mathbf{FE}_{\mu_-}$, suppose that μ_- is a successor. We have

$$L_\theta(L_\xi(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]))=L_\theta(\ell_{\xi(n+1)}[\mathrm{e}_\beta^\varphi])=\mathrm{e}_\beta^{\varphi-1}-(n+1)=L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi])-1.$$

Lemma 6.3.8. $(\mathbb{T}_{(\mu)},(L_{\omega^\eta})_{\eta<\mu})$ *satisfies* $\mathbf{A}_{\mu_-}$.

Proof. Let $\varphi\in\mathbf{T}$, $\xi n<\theta$, and $\iota<\mu_-$. We have

$$L_\theta(\ell_{\xi n}[\mathrm{e}_\beta^\varphi])=\mathrm{e}_\beta^{\varphi-1}-n\prec\ell_{\xi n+\omega^\iota}[\mathrm{e}_\beta^\varphi]=L_{\omega^\iota}(\ell_{\xi n}[\mathrm{e}_\beta^\varphi]).$$ □

Lemma 6.3.9. $(\mathbb{T}_{(\mu)},(L_{\omega^\eta})_{\eta<\mu})$ *satisfies* $\mathbf{M}_{\mu_-}$.

Proof. Let $\mathfrak{a},\mathfrak{b}\in(\mathfrak{M}_{(\mu)})_\theta$ with $\mathfrak{a}\prec\mathfrak{b}$ and let $\omega^\iota n<\theta$. We need to show that

$$L_\theta(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1}<L_\theta(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

We proceed by case distinction.

1. If $\mathfrak{a},\mathfrak{b}\in\mathfrak{M}_\theta$, then this holds because $(\mathbb{T},(L_{\omega^\eta})_{\eta<\mu})$ satisfies $\mathbf{M}_{\mu_-}$.
2. Suppose $\mathfrak{a}=\ell_{\xi m}[\mathrm{e}_\beta^\varphi]$ and $\mathfrak{b}=\ell_{\xi k}[\mathrm{e}_\beta^\psi]$ for some $\xi m,\xi k<\theta$ and some $\varphi,\psi\in\mathbf{T}$. Then

$$\begin{aligned}L_\theta(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1} &= \mathrm{e}_\beta^{\varphi-1}-m+\ell_{\xi m+\omega^\iota n}[\mathrm{e}_\beta^\varphi]^{-1}, \quad\text{and}\\ L_\theta(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1} &= \mathrm{e}_\beta^{\psi-1}-k-\ell_{\xi k+\omega^\iota n}[\mathrm{e}_\beta^\psi]^{-1}.\end{aligned}$$

   Recall that $\varphi-1,\psi-1\in\mathbf{T}$ by (6.2.1). So either $\varphi<\psi$ or $\varphi=\psi$ and $m>k$. In either case, we have $L_\theta(\mathfrak{a})+L_{\omega^\iota n}(\mathfrak{a})^{-1}<L_\theta(\mathfrak{b})-L_{\omega^\iota n}(\mathfrak{b})^{-1}$.

3. Suppose $\mathfrak{a} = \ell_{\xi m}[\mathrm{e}_\beta^\varphi]$ for some $\xi m < \theta$ and some $\varphi \in \mathbf{T}$ and $\mathfrak{b} \in \mathfrak{M}_\theta$. Then $\varphi < L_\beta(\mathfrak{b})$ since $\mathfrak{a} \prec \mathfrak{b}$. For each $r \in \mathbb{R}^>$, we have $L_\beta(\mathfrak{b}) =_\beta L_\beta^{\uparrow\theta}(r L_\theta(\mathfrak{b})) = L_\beta(r L_\theta(\mathfrak{b})) + 1$ by Lemma 5.3.14, so $\varphi - 1 < L_\beta(r L_\theta(\mathfrak{b}))$. Now $L_\theta(\mathfrak{a}) \asymp \mathrm{e}_\beta^{\varphi-1} \prec r L_\theta(\mathfrak{b}) \asymp L_\theta(\mathfrak{b})$ by Remark 6.2.4. Hence,

$$L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} \asymp \mathrm{e}_\beta^{\varphi-1} \prec L_\theta(\mathfrak{b}) \asymp L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}.$$

4. Suppose $\mathfrak{a} \in \mathfrak{M}_\theta$ and $\mathfrak{b} = \ell_{\xi m}[\mathrm{e}_\beta^\varphi]$ for some $\xi m < \theta$ and some $\varphi \in \mathbf{T}$. Then $\varphi > L_\beta(\mathfrak{a})$, so similar arguments as above give $\varphi - 1 > L_\beta(r L_\theta(\mathfrak{a}))$ for each $r \in \mathbb{R}^>$. Again, we conclude that $L_\theta(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} \prec L_\theta(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$. □

6.3.6 Extending L_β

We define

$$\begin{aligned} \operatorname{dom} L_\beta &:= \mathfrak{M}_\beta \cup \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T}\} \\ L_\beta(\mathrm{e}_\beta^\varphi) &:= \varphi. \end{aligned}$$

Lemma 6.3.10. $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta < \mu+1})$ *satisfies* $\mathbf{DD}_\mu$.

Proof. If $\mu = 1$, let $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^\succ$, with $\mathfrak{t} = \prod_{\varphi \in \mathbf{T}} \ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi]$. We have

$$L_1(\mathfrak{t}\,\mathfrak{m}) = L_1(\mathfrak{m}) + \sum_{\varphi \in \mathbf{T}} L_1(\ell_0^{r_\varphi}[\mathrm{e}_\omega^\varphi]) = L_1(\mathfrak{m}) + \sum_{\varphi - 1 \in \mathbf{T}} r_\varphi\, \mathrm{e}_\omega^{\varphi-1} + \sum_{\varphi - 1 \notin \mathbf{T}} r_\varphi\, E_\omega(\varphi - 1).$$

If $L_1(\mathfrak{t}\,\mathfrak{m}) \in \mathfrak{M}_{(\mu)}^\succ$, then either $\mathfrak{t} = 1$ or $\mathfrak{m} = 1$. If $\mathfrak{t} = 1$, then $\mathfrak{m} \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} L_1^{\circ n}$ if and only if $\mathfrak{m} \in \mathfrak{M}_\omega$. If $\mathfrak{m} = 1$, then $L_1(\mathfrak{t}) \in \mathfrak{M}_{(\mu)}^\succ$ if and only if $\mathfrak{t} = \mathrm{e}_\omega^\varphi \in \operatorname{dom} L_\omega$. For $n \in \mathbb{N}$, we have

$$L_n(\mathrm{e}_\omega^\varphi) = \mathrm{e}_\omega^{\varphi - n}.$$

If $\mu > 1$ is a successor, then let $\varphi \in \mathbf{T}$ and $\xi m < \theta$. We need to show that $\ell_{\xi m}[\mathrm{e}_\beta^\varphi] \in \bigcap \operatorname{dom} L_\theta^{\circ n}$ if and only if $m = 0$. This holds since

$$L_\theta(\ell_{\xi m}[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^{\varphi-1} - m.$$

Finally, if μ is a non-zero limit, then we have

$$\bigcap_{\eta < \mu} \{\ell_\gamma[\mathrm{e}_\beta^\varphi] : \varphi \in \mathbf{T} \text{ and } \omega^{\eta_-} \leqslant_o \gamma < \theta\} = \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T}\}.$$ □

To see that $\mathbf{R}_\mu$ is satisfied, let $\varphi \in \mathbf{T}$ and let $\omega^\eta n < \beta$. We have

$$L_{\omega^\eta n}(\mathrm{e}_\beta^\varphi)^{-1} = \begin{cases} \ell_{\omega^\eta n}[\mathrm{e}_\beta^\varphi]^{-1} & \text{if } \eta < \mu_- \\ (\mathrm{e}_\beta^{\varphi - n})^{-1} & \text{if } \eta = \mu_-. \end{cases}$$

Let $\mathfrak{m} \in (\operatorname{supp} \varphi)^\prec$. Since φ is β-truncated, we have $\varphi > L_\beta(\mathfrak{m}^{-1})$. This gives $\ell_{\omega^\eta n}[\mathrm{e}_\beta^\varphi]^{-1} \prec \mathfrak{m}$ for $\eta < \mu_-$. If $\eta = \mu_-$, then $\varphi - n$ is also β-truncated by Lemma 5.3.2, so $\varphi - n > L_\beta(\mathfrak{m}^{-1})$ since $(\operatorname{supp} \varphi)^\prec = (\operatorname{supp}(\varphi - n))^\prec$. This yields $(\mathrm{e}_\beta^{\varphi-n}) \succ \mathfrak{m}^{-1}$, i.e. $L_{\omega^\eta n}(\mathrm{e}_\beta^\varphi)^{-1} \prec \mathfrak{m}$, so

$$\operatorname{supp} L_\beta(\mathrm{e}_\beta^\varphi) = \operatorname{supp} \varphi \succ L_{\omega^\eta n}(\mathrm{e}_\beta^\varphi)^{-1},$$

as desired.

If μ is a successor, then $L_\theta(\mathrm{e}_\beta^\varphi) = \mathrm{e}_\beta^{\varphi-1}$, so

$$L_\beta(L_\theta(\mathrm{e}_\beta^\varphi)) = \varphi - 1 = L_\beta(\mathrm{e}_\beta^\varphi) - 1,$$

so $\mathbf{FE}_\mu$ is satisfied. As for $\mathbf{A}_\mu$, let $\iota < \mu$. Since $\ell_0 > \ell_\beta$, we have $\varphi > L_\beta(\varphi)$, so Remark 6.2.4 with $\mathfrak{t} = \ell_{\omega^\iota}[\mathrm{e}_\beta^\varphi]$ and $s = \varphi$ gives

$$L_{\omega^\iota}(\mathrm{e}_\beta^\varphi) = \ell_{\omega^\iota}[\mathrm{e}_\beta^\varphi] \succ \varphi = L_\beta(\mathrm{e}_\beta^\varphi).$$

Lemma 6.3.11. $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+1})$ *satisfies* $\mathbf{M}_{\boldsymbol{\mu}}$.

Proof. Let $\mathfrak{a} \prec \mathfrak{b} \in \operatorname{dom} L_\beta$ and let $\omega^\iota n < \beta$. We want to show that

$$L_{\omega^\iota n}(\mathfrak{a})^{-1} + L_{\omega^\iota n}(\mathfrak{b})^{-1} < L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a}).$$

Note that $L_\beta(\mathfrak{a}), L_\beta(\mathfrak{b}) \in \mathbb{T}_{\succ,\beta}$. We claim that $L_\beta(\mathfrak{a}) < L_\beta(\mathfrak{b})$. If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$, then this follows from the fact that $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{M}_{\boldsymbol{\mu}}$. If $\mathfrak{a} = \mathrm{e}^\varphi_\beta$ and $\mathfrak{b} = \mathrm{e}^\psi_\beta$, then we have $L_\beta(\mathfrak{a}) = \varphi < \psi = L_\beta(\mathfrak{b})$. If $\mathfrak{a} = \mathrm{e}^\varphi_\beta$ and $\mathfrak{b} \in \mathfrak{M}_\beta$, then $L_\beta(\mathfrak{a}) = \varphi < L_\beta(\mathfrak{b})$ by Remark 6.2.4 and likewise, if $\mathfrak{a} \in \mathfrak{M}_\beta$ and $\mathfrak{b} = \mathrm{e}^\psi_\beta$, then $L_\beta(\mathfrak{a}) < \psi = L_\beta(\mathfrak{b})$.

Now suppose toward contradiction that $L_{\omega^\iota n}(\mathfrak{b})^{-1} + L_{\omega^\iota n}(\mathfrak{a})^{-1} \geqslant L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})$. We will show that $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$. As $L_\beta(\mathfrak{a})$ is the unique β-truncated element in $\mathcal{L}_\beta[L_\beta(\mathfrak{a})]$ and $L_\beta(\mathfrak{b})$ is β-truncated, this is a contradiction.

Since $L_{\omega^\iota n}(\mathfrak{a})^{-1} > L_{\omega^\iota n}(\mathfrak{b})^{-1}$ by $\mathbf{M}_\iota$, we have $2\, L_{\omega^\iota n}(\mathfrak{a})^{-1} > L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})$, so

$$\frac{1}{2}\, L_{\omega^\iota n}(\mathfrak{a}) < |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}.$$

By $\mathbf{A_0}$, we have $L_1(L_{\omega^\iota n}(\mathfrak{a})) \prec L_{\omega^\iota n}(\mathfrak{a}) \asymp \frac{1}{2}\, L_{\omega^\iota n}(\mathfrak{a})$, so

$$L_{\omega^\iota n+1}(\mathfrak{a}) \prec |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}.$$

If $L_{\omega^\iota n+1}(\mathfrak{a}) \in \mathbb{T}^{>,\succ}$, then Lemma 5.1.7 gives

$$L_\beta(\mathfrak{a}) = L_\beta^{\uparrow \omega^\iota n+1}(L_{\omega^\iota n+1}(\mathfrak{a})) < L_\beta^{\uparrow \omega^\iota n+1}(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}),$$

so $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$. Suppose $L_{\omega^\iota n+1}(\mathfrak{a}) \notin \mathbb{T}^{>,\succ}$ and let $\varphi \in \mathbf{T}$ with $\mathfrak{a} = \mathrm{e}^\varphi_\beta$. If $\iota < \mu_-$, then

$$L_{\omega^\iota n+1}(\mathfrak{a}) = \ell_{\omega^\iota n+1}[\mathrm{e}^\varphi_\beta] \prec |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1},$$

so $\varphi < L_\beta(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1})$ by Remark 6.2.4. As $\varphi = L_\beta(\mathfrak{a})$, this too gives $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$. Finally, if $\iota = \mu_- < \mu$, then

$$L_{\omega^\iota n+1}(\mathfrak{a}) = \ell_1[\mathrm{e}^{\varphi-n}_\beta] \prec |L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1},$$

so $\varphi - n < L_\beta(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1})$ by Remark 6.2.4. As $\ell_\beta + n = \ell_\beta^{\uparrow \theta n}$, we have

$$\varphi < L_\beta(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}) + n = L_\beta^{\uparrow \theta n}(|L_\beta(\mathfrak{b}) - L_\beta(\mathfrak{a})|^{-1}),$$

so $L_\beta(\mathfrak{b}) \in \mathcal{L}_\beta[L_\beta(\mathfrak{a})]$ once again. □

Lemma 6.3.12. $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+1})$ *satisfies* $\mathbf{P}_{\boldsymbol{\mu}}$.

Proof. Let $\mathfrak{a} \in \operatorname{dom} L_\beta$ and let $(r_\gamma)_{\gamma<\beta}$ be a sequence of real numbers. Consider the sum $s := \sum_{\gamma<\beta} r_\gamma L_{\gamma+1}(\mathfrak{a})$. If $\mathfrak{a} \in \mathfrak{M}_\beta$, then $s \in \log \mathfrak{M}$ since $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{P}_{\boldsymbol{\mu}}$. Assume therefore that $\mathfrak{a} = \mathrm{e}^\varphi_\beta$ for some $\varphi \in \mathbf{T}$. If μ is a limit, then $\beta = \theta$ and

$$s = \sum_{\gamma<\theta} r_\gamma L_{\gamma+1}(\mathrm{e}^\varphi_\beta) = \sum_{\gamma<\theta} r_\gamma \ell_{\gamma+1}[\mathrm{e}^\varphi_\beta] = \log(\mathfrak{l}[\mathrm{e}^\varphi_\beta])$$

where $\mathfrak{l} := \prod_{\gamma<\theta} \ell_\gamma^{r_\gamma} \in \mathfrak{L}^\succ_{<\theta}$. If μ is a successor, then we may write

$$s = \sum_{n\in\mathbb{N}} \sum_{\gamma<\theta} r_{\theta n+\gamma} L_{\theta n+\gamma+1}(\mathrm{e}^\varphi_\beta) = \sum_{n\in\mathbb{N}} \sum_{\gamma<\theta} r_{\theta n+\gamma} L_{\gamma+1}(L_{\theta n}(\mathrm{e}^\varphi_\beta)).$$

Since $\varphi - n \in \mathbf{T}$ for all n, we have

$$\sum_{n\in\mathbb{N}} \sum_{\gamma<\theta} r_{\theta n+\gamma} L_{\gamma+1}(L_{\theta n}(\mathrm{e}^\varphi_\beta)) = \sum_{n\in\mathbb{N}} \sum_{\gamma<\theta} r_{\theta n+\gamma} L_{\gamma+1}(\mathrm{e}^{\varphi-n}_\beta) = \log\left(\prod_{n\in\mathbb{N}} \mathfrak{l}_n[\mathrm{e}^{\varphi-n}_\beta]\right)$$

where $\mathfrak{l}_n := \prod_{\gamma<\theta} \ell_\gamma^{r_{\theta n+\gamma}} \in \mathfrak{L}^\succ_{<\theta}$. □

6.3.7 Extending $L_{\omega^{\mu+1}}$

Suppose $\boldsymbol{\nu} > \mu + 1$. We define

$$\begin{aligned} \operatorname{dom} L_{\omega^{\mu+1}} &:= \mathfrak{M}_{\omega^{\mu+1}} \cup \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}\} \\ L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) &:= L_{\omega^{\mu+1}}(\varphi) + 1. \end{aligned}$$

For $\varphi \in \mathbf{T}$, we have $\mathrm{e}_\beta^\varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_\beta^{\circ n}$ if and only if $\varphi \in \mathfrak{M}_{\omega^{\mu+1}}$ since $\varphi = L_\beta(\mathrm{e}_\beta^\varphi)$. This proves that $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{DD}_{\mu+1}$. Let $\varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}$. We have

$$L_{\omega^{\mu+1}}(L_\beta(\mathrm{e}_\beta^\varphi)) = L_{\omega^{\mu+1}}(\varphi) = L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) - 1,$$

so $\mathbf{FE}_{\mu+1}$ is satisfied. As for $\mathbf{A}_{\mu+1}$, it suffices to show that $L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) < L_\beta(\mathrm{e}_\beta^\varphi)$ since $L_\beta(\mathrm{e}_\beta^\varphi) < L_{\omega^\imath}(\mathrm{e}_\beta^\varphi)$ for all $\imath < \mu$ by $\mathbf{A}_\mu$. Since $\ell_{\omega^{\mu+1}} + 1 \prec \ell_0$, we have

$$L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) = L_{\omega^{\mu+1}}(\varphi) + 1 = (\ell_{\omega^{\mu+1}} + 1) \circ \varphi \prec \varphi = L_\beta(\mathrm{e}_\beta^\varphi).$$

Now for $\mathbf{R}_{\mu+1}$, let $\omega^\iota\, n < \omega^{\mu+1}$. Since $L_{\beta(n+1)}(\mathrm{e}_\beta^\varphi) \leqslant L_{\omega^\iota n}(\mathrm{e}_\beta^\varphi)$ by $\mathbf{A}_\mu$, it suffices to show that $\operatorname{supp} L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) \succ L_{\beta(n+1)}(\mathrm{e}_\beta^\varphi)^{-1}$. Since

$$\operatorname{supp} L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) = \operatorname{supp} L_{\omega^{\mu+1}}(\varphi) \cup \{1\}, \qquad L_{\beta(n+1)}(\mathrm{e}_\beta^\varphi)^{-1} = L_{\beta n}(\varphi)^{-1},$$

it is enough to show that $\operatorname{supp} L_{\omega^{\mu+1}}(\varphi) \succ L_{\beta n}(\varphi)^{-1}$. This holds because $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{R}_{\mu+1}$ and $\varphi \in \mathfrak{M}_{\omega^{\mu+1}}$.

Lemma 6.3.13. $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+2})$ *satisfies* $\mathbf{M}_{\mu+1}$.

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \operatorname{dom} L_{\omega^{\mu+1}}$ with $\mathfrak{a} \prec \mathfrak{b}$ and let $\omega^\iota\, n < \omega^{\mu+1}$. We want to show that $L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\omega^\iota n}(\mathfrak{a})^{-1} < L_{\omega^{\mu+1}}(\mathfrak{b}) - L_{\omega^\iota n}(\mathfrak{b})^{-1}$. Since $L_{\beta(n+1)}(\mathfrak{a}) \leqslant L_{\omega^\iota n}(\mathfrak{a})$ and likewise for $\mathfrak{b}$, it is enough to show that

$$L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\beta(n+1)}(\mathfrak{a})^{-1} < L_{\omega^{\mu+1}}(\mathfrak{b}) - L_{\beta(n+1)}(\mathfrak{b})^{-1}.$$

We proceed by case distinction:

1. If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$, then the result follows from $\mathbf{M}_{\mu+1}$ for $\mathbb{T}$.
2. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ and $\mathfrak{b} = \mathrm{e}_\beta^\psi$, then

$$\begin{aligned} L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\beta(n+1)}(\mathfrak{a})^{-1} &= L_{\omega^{\mu+1}}(\varphi) + 1 + L_{\beta n}(\varphi)^{-1} \\ L_{\omega^{\mu+1}}(\mathfrak{b}) - L_{\beta(n+1)}(\mathfrak{b})^{-1} &= L_{\omega^{\mu+1}}(\psi) + 1 - L_{\beta n}(\psi)^{-1}. \end{aligned}$$

   Since $\varphi, \psi \in \mathfrak{M}_{\omega^{\mu+1}}$ and $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{M}_{\mu+1}$, we have

$$L_{\omega^{\mu+1}}(\varphi) + L_{\beta n}(\varphi)^{-1} < L_{\omega^{\mu+1}}(\psi) - L_{\beta n}(\psi)^{-1}.$$

3. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ and $\mathfrak{b} \in \mathfrak{M}_{\omega^{\mu+1}}$, then $\varphi < L_\beta(\mathfrak{b})$. Since $\varphi, L_\beta(\mathfrak{b}) \in \mathfrak{M}_{\omega^{\mu+1}}$ and $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{M}_{\mu+1}$, we have

$$L_{\omega^{\mu+1}}(\varphi) + L_{\beta n}(\varphi)^{-1} < L_{\omega^{\mu+1}}(L_\beta(\mathfrak{b})) - L_{\beta n}(L_\beta(\mathfrak{b}))^{-1} = L_{\omega^{\mu+1}}(\mathfrak{b}) - 1 + L_{\beta(n+1)}(\mathfrak{b})^{-1}.$$

   Thus,

$$L_{\omega^{\mu+1}}(\mathfrak{a}) + L_{\beta(n+1)}(\mathfrak{a})^{-1} = L_{\omega^{\mu+1}}(\varphi) + 1 + L_{\beta n}(\varphi)^{-1} < L_{\omega^{\mu+1}}(\mathfrak{b}) + L_{\beta(n+1)}(\mathfrak{b})^{-1}.$$

4. If $\mathfrak{a} \in \mathfrak{M}_\beta$ and $\mathfrak{b} = \mathrm{e}_\beta^\psi$, then the argument is similar to the previous case. □

Lemma 6.3.14. $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+2})$ *satisfies* $\mathbf{P}_{\mu+1}$.

Proof. Let $\mathfrak{a} \in (\mathfrak{M}_{(\mu)})_{\omega^{\mu+1}}$ and let $(r_\gamma)_{\gamma<\omega^{\mu+1}}$ be a sequence of real numbers. We need to show that the sum $s = \sum_{\gamma<\omega^{\mu+1}} r_\gamma L_{\gamma+1}(\mathfrak{a})$ is in $\log \mathfrak{M}_{(\mu)}$. If $\mathfrak{a} \in \mathfrak{M}_{\omega^\eta}$, then $s \in \log \mathfrak{M}$. If $\mathfrak{a} = \mathrm{e}_\beta^\varphi$ for some $\varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}$, then

$$s = \sum_{n\in\mathbb{N}} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\beta n+\gamma+1}(\mathrm{e}_\beta^\varphi) = \sum_{\gamma<\beta} r_\gamma L_{\gamma+1}(\mathrm{e}_\beta^\varphi) + \sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\gamma+1}(L_{\beta n}(\mathrm{e}_\beta^\varphi)).$$

We have $\sum_{\gamma<\beta} r_\gamma L_{\gamma+1}(\mathrm{e}_\beta^\varphi) \in \log \mathfrak{M}_{(\mu)}$, since $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\mu+1})$ satisfies $\mathbf{P}_{\boldsymbol{\mu}}$. We also have

$$\sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\gamma+1}(L_{\beta n}(\mathrm{e}_\beta^\varphi)) = \sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\gamma+1}(L_{\beta(n-1)}(\varphi)) = \sum_{n\in\mathbb{N}^>} \sum_{\gamma<\beta} r_{\beta n+\gamma} L_{\beta(n-1)+\gamma+1}(\varphi) \in \log \mathfrak{M},$$

since $\varphi \in \mathfrak{M}_{\omega^{\mu+1}}$ and $(\mathbb{T}, (L_{\omega^\eta})_{\eta<\mu+2})$ satisfies $\mathbf{P}_{\mu+1}$. We conclude by noting that $\log \mathfrak{M}_{(\mu)}$ is closed under addition. □

Remark 6.3.15. In the case that $\boldsymbol{\nu} = \mu+1$, the argument that $\mathbf{DD}_{\mu+1}$ is satisfied gives

$$(\mathfrak{M}_{(\mu)})_{\omega^\nu} = \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_\beta^{\circ n} = \mathfrak{M}_{\omega^\nu} \cup \{\mathrm{e}_\beta^\varphi : \varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^\nu}\}$$

and the proof of Lemma 6.3.14 also tells us that $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\boldsymbol{\nu}})$ satisfies $\mathbf{P}_{\boldsymbol{\nu}}$.

6.3.8 Extending L_{ω^η} when $\mu+1<\eta<\nu$

If $\boldsymbol{\nu} > \mu+1$, then we will not extend the hyperlogarithms L_{ω^η} with $\eta > \mu+1$. So for $\eta < \boldsymbol{\nu}$ with $\eta > \mu+1$, we simply set

$$\operatorname{dom} L_{\omega^\eta} := \mathfrak{M}_{\omega^\eta}.$$

Lemma 6.3.16. *$(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\boldsymbol{\nu}})$ satisfies $\mathbf{DD}_\eta$ for all $\eta<\boldsymbol{\nu}$.*

Proof. It suffices to show that $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\boldsymbol{\nu}})$ satisfies $\mathbf{DD}_{\mu+2}$. Suppose toward contradiction that there is some $\varphi \in \mathbf{T} \cap \mathfrak{M}_{\omega^{\mu+1}}$ with $\mathrm{e}_\beta^\varphi \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} L_{\omega^{\mu+1}}^{\circ n}$. Take $n>0$ with $L_{\omega^{\mu+1}n}(\varphi) \asymp L_{\omega^{\mu+1}n}(\mathfrak{d}_{\omega^{\mu+2}}(\varphi))$. Since $L_{\omega^{\mu+1}}(\mathrm{e}_\beta^\varphi) = L_{\omega^{\mu+1}}(\varphi)+1 \asymp L_{\omega^{\mu+1}}(\varphi)$, Lemma 4.3.4 yields

$$L_{\omega^{\mu+1}n}(\mathrm{e}_\beta^\varphi) = L_{\omega^{\mu+1}(n-1)}(L_{\omega^{\mu+1}}(\varphi)+1) \asymp L_{\omega^{\mu+1}(n-1)}(L_{\omega^{\mu+1}}(\varphi)) \asymp L_{\omega^{\mu+1}n}(\mathfrak{d}_{\omega^{\mu+2}}(\varphi)).$$

Since $L_{\omega^{\mu+1}n}(\mathrm{e}_\beta^\varphi)$ and $L_{\omega^{\mu+1}n}(\mathfrak{d}_{\omega^{\mu+2}}(\varphi))$ are both monomials, they must be equal. The axiom $\mathbf{M}_{\mu+1}$ gives $\mathrm{e}_\beta^\varphi = \mathfrak{d}_{\omega^{\mu+2}}(\varphi) \in \mathbb{T}$, a contradiction. □

For all $\eta<\boldsymbol{\nu}$ with $\eta>\mu+1$, the axioms $\mathbf{FE}_\eta$, $\mathbf{A}_\eta$, $\mathbf{M}_\eta$, $\mathbf{R}_\eta$ and $\mathbf{P}_\eta$ automatically hold in $\mathbb{T}_{(\mu)}$ since they hold in $\mathbb{T}$, as does the axiom $\mathbf{P}_{\boldsymbol{\nu}}$ if $\boldsymbol{\nu}>\mu+1$ is an ordinal.

6.4 End of the proof of Theorem 4.2

We have completed the proof of the following:

Proposition 6.4.1. *$(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta<\boldsymbol{\nu}})$ is a hyperserial skeleton of force $\boldsymbol{\nu}$.*

Let us finally examine the confluence and universality of $\mathbb{T}_{(\mu)}$.

6.4.1 The extended hyperserial skeleton

Proposition 6.4.2. *We have $\mathfrak{M}_{\omega^{\mu+1}} \subseteq \mathbf{T} \cup L_\beta(\mathbb{T}^{>,\succ})$, and $\mathbb{T}_{(\mu)}$ is $\boldsymbol{\nu}$-confluent. In particular, $\mathbb{T}_{(\mu)}$ is $\boldsymbol{\nu}$-confluent.*

Proof. Let $\mathfrak{a} \in \mathfrak{M}_{\omega^{\mu+1}}$. If $\mathfrak{a} \notin L_\beta(\mathbb{T}^{>,\succ})$, then $\mathfrak{a} \in \mathbf{T}$ by definition, the first part of the statement is true. We turn to the second one.

Clearly, $\mathbb{T}_{(\mu)}$ is 0-confluent. Consider $s \in \mathbb{T}_{(\mu)}^{>,\succ}$ and write $\mathfrak{d}_1(s) = \mathfrak{d}_s = \mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}^\succ$. By our definition of L_1, we either have have $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \mathfrak{d}_1(L_1(\mathfrak{m}))$ or $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \ell_{\gamma_\mathfrak{t}+1}[\mathrm{e}_\beta^{\varphi_\mathfrak{t}}]$. If $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \mathfrak{d}_1(L_1(\mathfrak{m}))$, then $\mathfrak{d}_\omega(s) = \mathfrak{d}_\omega(\mathfrak{m})$ and, more generally, $\mathfrak{d}_{\omega^\eta}(s) = \mathfrak{d}_{\omega^\eta}(\mathfrak{m}) \in \mathfrak{M}_{\omega^\eta}$ for all $\eta \in \mathbf{On}$ with $1 \leqslant \eta \leqslant \boldsymbol{\nu}$, since $\mathcal{E}_\omega[\mathfrak{d}_\omega(\mathfrak{m})] = \mathcal{E}_\omega[\mathfrak{m}] \subseteq \mathcal{E}_{\omega^\eta}[\mathfrak{m}]$ by Lemma 4.2.7. Assume from now on that $\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \ell_{\gamma_\mathfrak{t}+1}[\mathrm{e}_\beta^{\varphi_\mathfrak{t}}]$.

We set $\gamma := \gamma_{\mathfrak{t}}$ and $\varphi := \varphi_{\mathfrak{t}}$. For $1 \leqslant \eta \leqslant \mu$, let us first show by induction that

$$\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant \omega^{\eta_-}}}[\mathrm{e}_\beta^\varphi] \in \mathfrak{M}_{\omega^\eta}.$$

If $\eta = 1$, then $\gamma = \gamma_{\geqslant 1}$ and

$$\mathfrak{d}_1(L_1(\mathfrak{d}_s)) = \ell_{\gamma+1}[\mathrm{e}_\beta^\varphi] = L_1(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_1(L_1(\ell_\gamma[\mathrm{e}_\beta^\varphi])),$$

so we indeed have $\mathfrak{d}_\omega(s) = \ell_{\gamma_{\geqslant 1}}[\mathrm{e}_\beta^\varphi]$. Let $1 < \eta \leqslant \mu$ and suppose that $\mathfrak{d}_{\omega^\sigma}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant \omega^{\sigma_-}}}[\mathrm{e}_\beta^\varphi]$ for $1 \leqslant \sigma < \eta$. If $1 < \eta < \mu$ and η is a successor, then our induction hypothesis yields

$$L_{\omega^{\eta_-}}(\mathfrak{d}_{\omega^{\eta_-}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^{\eta_-}}(\ell_{\gamma_{\geqslant \omega^{\eta_{--}}}}[\mathrm{e}_\beta^\varphi]).$$

Writing $\gamma_{\geqslant \omega^{\eta_{--}}} = \gamma_{\geqslant \omega^{\eta_-}} + \omega^{\eta_{--}} n$, we have

$$L_{\omega^{\eta_-}}(\ell_{\gamma_{\geqslant \omega^{\eta_{--}}}}[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant \omega^{\eta_-}} + \omega^{\eta_-}}[\mathrm{e}_\beta^\varphi] - n \asymp \ell_{\gamma_{\geqslant \omega^{\eta_-}} + \omega^{\eta_-}}[\mathrm{e}_\beta^\varphi] = L_{\omega^{\eta_-}}(\ell_{\gamma_{\geqslant \omega^{\eta_-}}}[\mathrm{e}_\beta^\varphi]),$$

so

$$\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^\eta}(\mathfrak{d}_{\omega^{\eta_-}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = \ell_{\gamma_{\geqslant \omega^{\eta_-}}}[\mathrm{e}_\beta^\varphi].$$

If $1 < \eta \leqslant \mu$ and $\eta = \eta_-$ is a limit, then there is $\sigma < \eta$ such that $\gamma_{\geqslant \omega^{\eta_-}} = \gamma_{\geqslant \omega^{\sigma_-}}$. For this σ, we have

$$L_{\omega^\sigma}(\mathfrak{d}_{\omega^\sigma}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^\sigma}(\ell_{\gamma_{\geqslant \omega^{\sigma_-}}}[\mathrm{e}_\beta^\varphi]) = L_{\omega^\sigma}(\ell_{\gamma_{\geqslant \omega^{\eta_-}}}[\mathrm{e}_\beta^\varphi]),$$

so $\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \ell_{\gamma_{\geqslant \omega^{\eta_-}}}[\mathrm{e}_\beta^\varphi] \in \mathfrak{M}_{\omega^\eta}$. Finally, if $\eta = \mu$ and μ is a successor, then $\gamma_{\geqslant \omega^{\mu_{--}}} = \omega^{\mu_{--}} n$, where $n = 0$ if μ_- is a limit. This gives

$$L_\theta(\mathfrak{d}_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_\theta(\ell_{\omega^{\mu_{--}} n}[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^{\varphi - 1} - n.$$

We thus have $L_\theta(\mathfrak{d}_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) \asymp L_\theta(\mathrm{e}_\beta^\varphi)$. Since $\gamma_{\geqslant \theta} = 0$, we deduce that $\mathfrak{d}_\beta(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^\varphi = \ell_{\gamma_{\geqslant \theta}}[\mathrm{e}_\beta^\varphi]$,

Let us now show that $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$ exists. Let $\mathfrak{a} := \mathfrak{d}_{\omega^{\mu+1}}(\varphi)$, so $\mathfrak{a} \in \mathbf{T} \cup L_\beta(\mathbb{T}^{>,\succ})$ by the first part of the statement. Take n with $(L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\varphi) \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{a})$. We have $L_\beta(\mathfrak{d}_\beta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_\beta(\mathrm{e}_\beta^\varphi) = \varphi$, so

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{a}) = \begin{cases} (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathrm{e}_\beta^{\mathfrak{a}}) & \text{if } \mathfrak{a} \in \mathbf{T} \\ (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(E_\beta(\mathfrak{a})) & \text{otherwise.} \end{cases}$$

Since $\mathfrak{a}$ is an infinite monomial, it is $\omega^{\mu+1}$-truncated, so $E_\beta(\mathfrak{a}) \in \mathfrak{M}_{\omega^{\mu+1}}$ so long as it is defined. Thus, $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])$ is either equal to $\mathrm{e}_\beta^{\mathfrak{a}}$ or $E_\beta(\mathfrak{a})$.

If $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = E_\beta(\mathfrak{a})$, then $\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^\eta}(E_\beta(\mathfrak{a}))$ for $\eta \in \mathbf{On}$ with $\mu + 1 \leqslant \eta \leqslant \boldsymbol{\nu}$. On the other hand, if $\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathrm{e}_\beta^{\mathfrak{a}}$, then

$$L_{\omega^{\mu+1}}(\mathfrak{d}_{\omega^{\mu+1}}(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = L_{\omega^{\mu+1}}(\mathrm{e}_\beta^{\mathfrak{a}}) = L_{\omega^{\mu+1}}(\mathfrak{a}) + 1 \asymp L_{\omega^{\mu+1}}(\mathfrak{a}).$$

Take $n \in \mathbb{N}$ with $(L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ n}(\mathfrak{a}) \asymp (L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ n}(\mathfrak{d}_{\omega^{\mu+2}}(\mathfrak{a}))$. Then

$$(L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ(n+1)}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) \asymp (L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ(n+1)}(\mathfrak{a}) \asymp (L_{\omega^{\mu+1}} \circ \mathfrak{d}_{\omega^{\mu+1}})^{\circ(n+1)}(\mathfrak{d}_{\omega^{\mu+2}}(\mathfrak{a})),$$

so $\mathfrak{d}_{\omega^{\mu+2}}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^{\mu+2}}(\mathfrak{a})$ and, more generally, $\mathfrak{d}_{\omega^\eta}(\ell_\gamma[\mathrm{e}_\beta^\varphi]) = \mathfrak{d}_{\omega^\eta}(\mathfrak{a})$ when $\eta \in \mathbf{On}$ and $\mu + 2 \leqslant \eta \leqslant \boldsymbol{\nu}$. □

Propositions 6.4.1 and 6.4.2 yield:

Corollary 6.4.3. *The structure $(\mathbb{T}_{(\mu)}, (L_{\omega^\eta})_{\eta < \boldsymbol{\nu}})$ is a confluent hyperserial skeleton of force $\boldsymbol{\nu}$.*

Remark 6.4.4. Let $0 < \eta \leqslant \mu_-$. Then

$$(\mathfrak{M}_{(\mu)})_{\omega^\eta} = \mathfrak{M}_{\omega^\eta} \cup \{\mathfrak{l}[\mathrm{e}_\beta^\varphi] : \mathfrak{l} \in (\mathfrak{L}_{<\theta})_{\omega^\eta} \text{ and } \varphi \in \mathbf{T}\}.$$

Given $\gamma < \omega^\eta$ and $\mathfrak{l}[\mathrm{e}_\beta^\varphi] \in (\mathfrak{M}_{(\mu)})_{\omega^\eta} \setminus \mathfrak{M}_{\omega^\eta}$, we have $L_\gamma(\mathfrak{l}[\mathrm{e}_\beta^\varphi]) = L_\gamma(\mathfrak{l})[\mathrm{e}_\beta^\varphi]$. Given $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$, we have $\mathfrak{d}_{\omega^\eta}(\mathfrak{t}) = \mathfrak{d}_{\omega^\eta}(\mathfrak{t}_{\varphi_{\mathfrak{t}}})[\mathrm{e}_\beta^{\varphi_{\mathfrak{t}}}]$.

Let us now show that $\mathbb{T}_{(\mu)}$ satisfies a universal property. We start with a lemma.

Lemma 6.4.5. *For any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_\beta$ with $\mathfrak{a} \prec \mathfrak{b}$ and any $\gamma, \sigma < \theta$, we have $L_\sigma(\mathfrak{a}) \prec L_\gamma(\mathfrak{b})$.*

Proof. Choose $\eta < \mu_-$ and $n \in \mathbb{N}$ such that $\gamma, \sigma < \omega^\eta n$. Then $L_\sigma(\mathfrak{a}) \prec \mathfrak{a}$ and $L_{\omega^\eta n}(\mathfrak{b}) \prec L_\gamma(\mathfrak{b})$ so it suffices to show that $\mathfrak{a} \prec L_{\omega^\eta n}(\mathfrak{b})$. Since $L_{\omega^{\eta+1}}(\mathfrak{a}), L_{\omega^{\eta+1}}(\mathfrak{b})$ are monomials and $L_{\omega^{\eta+1}}(\mathfrak{a}) < L_{\omega^{\eta+1}}(\mathfrak{b})$, we have

$$L_{\omega^{\eta+1}}(\mathfrak{a}) \prec L_{\omega^{\eta+1}}(\mathfrak{b}) \asymp L_{\omega^{\eta+1}}(\mathfrak{b}) - n = L_{\omega^{\eta+1}}(L_{\omega^\eta n}(\mathfrak{b})).$$

The monotonicity of $L_{\omega^{\eta+1}}$ gives $\mathfrak{a} < L_{\omega^\eta n}(\mathfrak{b})$. We conclude that $\mathfrak{a} \prec L_{\omega^\eta n}(\mathfrak{b})$, since $\mathfrak{a}$ and $L_{\omega^\eta n}(\mathfrak{b})$ are monomials. □

Proposition 6.4.6. *Let $\mathbb{U} = \mathbb{R}[[\mathfrak{N}]]$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ be an embedding of force $\boldsymbol{\nu}$. If $\Phi(\mathbf{T}) \subseteq L_\beta(\mathbb{U}^{>,\succ})$, then there is a unique embedding*

$$\Psi: \mathbb{T}_{(\mu)} \longrightarrow \mathbb{U}$$

of force $\boldsymbol{\nu}$ that extends Φ.

Proof. Since $\mathbb{U}$ is confluent, we have an external composition $\circ: \mathbb{L}_{<\boldsymbol{\alpha}} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$. Given $\varphi \in \mathbf{T}$, the series $\Phi(\varphi)$ is β-truncated, so $E_\beta(\Phi(\varphi))$ is β-atomic, by Remark 5.3.10. We set $\mathfrak{a}_\varphi := E_\beta(\Phi(\varphi)) \in \mathfrak{N}_\beta$. Note that for $\varphi \in \mathbf{T}$ and $\mathfrak{l} = \prod_{\gamma<\beta} \ell_\gamma^{\mathfrak{l}_\gamma} \in \mathfrak{L}_{<\beta}$, the series

$$\mathfrak{l} \circ \mathfrak{a}_\varphi = \exp\left(\sum_{\gamma<\beta} \mathfrak{l}_\gamma L_{\gamma+1}(\mathfrak{a}_\varphi)\right)$$

exists in $\mathfrak{N}$ by $\mathbf{P}_{\boldsymbol{\mu}}$. Let us define a map $\Psi: \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}] \longrightarrow \mathfrak{N}$. Let $\mathfrak{t} \in \mathfrak{L}_\theta[\mathrm{e}_\beta^{\mathbf{T}}]$. If μ is a limit, then $\operatorname{hsupp} \mathfrak{t}$ is finite and we define

$$\Psi(\mathfrak{t}) := \prod_{\varphi \in \mathbf{T}} \mathfrak{t}_\varphi \circ \mathfrak{a}_\varphi \in \mathfrak{N}.$$

If μ is a successor, let $\varphi_1 > \cdots > \varphi_n \in \mathbf{T}$ and $\mathfrak{t}_{\varphi_i}^*$ be as in Remark 6.2.2. We define

$$\Psi(\mathfrak{t}) := \prod_{i=1}^n \mathfrak{t}_{\varphi_i}^* \circ \mathfrak{a}_{\varphi_i}.$$

Note that in both the limit and successor case, we have

$$\log \Psi(\mathfrak{t}) = \sum_{\varphi \in \mathbf{T}} \log(\mathfrak{t}_\varphi \circ \mathfrak{a}_\varphi) = \sum_{\varphi \in \mathbf{T}} \sum_{\gamma<\theta} (\mathfrak{t}_\varphi)_\gamma L_{\gamma+1}(\mathfrak{a}_\varphi).$$

Given $\varphi < \psi \in \mathbf{T}$ and $\gamma, \sigma < \theta$, we have $L_\sigma(\mathfrak{a}_\varphi) \prec L_\gamma(\mathfrak{a}_\psi)$ by Lemma 6.4.5 and, if $\gamma < \sigma$, then $L_\sigma(\mathfrak{a}_\varphi) \prec L_\gamma(\mathfrak{a}_\varphi)$. Thus, $\log \Psi(\mathfrak{t}) \sim (\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}})$ for $\mathfrak{t} \neq 1$. In particular, Ψ is order preserving, since

$$\mathfrak{t} \succ 1 \Longleftrightarrow (\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} > 0 \Longleftrightarrow \log \Psi(\mathfrak{t}) > 0 \Longleftrightarrow \Psi(\mathfrak{t}) \succ 1.$$

Next, we extend Ψ to all of $\mathfrak{M}_{(\mu)}$ by setting $\Psi(\mathfrak{t}\mathfrak{m}) = \Psi(\mathfrak{t})\,\Phi(\mathfrak{m})$ for $\mathfrak{t}\mathfrak{m} \in \mathfrak{M}_{(\mu)}$. Note that Ψ extends Φ. It is straightforward to check that $\Psi: \mathfrak{M}_{(\mu)} \longrightarrow \mathfrak{N}$ is an embedding of monomial groups which respects real powers. We need to show that Ψ preserves the ordering. Let $\mathfrak{t}\mathfrak{m} \in \mathfrak{M}_{(\mu)}^\succ$. If both $\mathfrak{t}$, $\mathfrak{m} \succcurlyeq 1$, then $\Psi(\mathfrak{t}\mathfrak{m}) = \Psi(\mathfrak{t})\,\Phi(\mathfrak{m}) \succ 1$. This leaves us two cases to consider:

1. Suppose $\mathfrak{t} \succ 1$, $\mathfrak{m} \prec 1$, and $\varphi_\mathfrak{t} > L_\beta(\mathfrak{m}^{-1})$. Set $r := (\mathfrak{t}_{\varphi_\mathfrak{t}})_{\gamma_\mathfrak{t}} > 0$. We claim that $L_\beta(\mathfrak{m}^{-1}) =_\beta L_\beta^{\uparrow \gamma_\mathfrak{t}+1}(2\,r^{-1} L_1(\mathfrak{m}^{-1}))$. If $\mu = 1$, then $\gamma_\mathfrak{t} = 0$, so this follows from Lemma 5.3.14. If $\mu > 1$, then $1, \gamma_\mathfrak{t} + 1 < \theta$, so this follows from Lemmas 5.3.14 and 5.3.16. In either case, we have $\varphi_\mathfrak{t} > L_\beta^{\uparrow \gamma_\mathfrak{t}+1}(2\,r^{-1}\,L_1(\mathfrak{m}^{-1}))$, so $\Phi(\varphi_\mathfrak{t}) > L_\beta^{\uparrow \gamma_\mathfrak{t}+1}(2\,r^{-1}\,L_1(\Phi(\mathfrak{m}^{-1})))$. From this, we see that

$$L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}}) = L_{\gamma_\mathfrak{t}+1}(E_\beta(\Phi(\varphi_\mathfrak{t}))) > 2\,r^{-1}\,L_1(\Phi(\mathfrak{m}^{-1})),$$

so $\frac{1}{2}\,r\,L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}}) > L_1(\Phi(\mathfrak{m}^{-1}))$. Since $L_1(\Psi(\mathfrak{t})) \sim r\,L_{\gamma_\mathfrak{t}+1}(\mathfrak{a}_{\varphi_\mathfrak{t}})$, this gives $L_1(\Psi(\mathfrak{t})) > L_1(\Phi(\mathfrak{m}^{-1}))$. Thus

$$\log(\Psi(\mathfrak{t}\mathfrak{m})) = L_1(\Psi(\mathfrak{t})) - L_1(\Phi(\mathfrak{m}^{-1})) > 0,$$

so $\Psi(\mathfrak{t}\mathfrak{m}) \succ 1$.

2. Suppose $\mathfrak{t} \prec 1$, $\mathfrak{m} \succ 1$, and $\varphi_{\mathfrak{t}} < L_\beta(\mathfrak{m})$. Set $r := (\mathfrak{t}_{\varphi_{\mathfrak{t}}})_{\gamma_{\mathfrak{t}}} < 0$. As before, Lemmas 5.3.14 and 5.3.16 give $\Phi(\varphi_{\mathfrak{t}}) < L_\beta^{\uparrow \gamma_{\mathfrak{t}}+1}\left(-\frac{1}{2} r^{-1} L_1(\Phi(\mathfrak{m}))\right)$, so

$$-2\, r\, L_{\gamma_{\mathfrak{t}}+1}(\mathfrak{a}_{\varphi_{\mathfrak{t}}}) < L_1(\Phi(\mathfrak{m})).$$

Since $L_1(\Psi(\mathfrak{t})^{-1}) = -\log(\Psi(\mathfrak{t})) \sim -r\, L_{\gamma_{\mathfrak{t}}+1}(\mathfrak{a}_{\varphi_{\mathfrak{t}}})$, this gives $L_1(\Psi(\mathfrak{t})^{-1}) < L_1(\Phi(\mathfrak{m}))$, so

$$\log(\Psi(\mathfrak{t}\,\mathfrak{m})) = L_1(\Phi(\mathfrak{m})) - L_1(\Psi(\mathfrak{t})^{-1}) > 0$$

and $\Psi(\mathfrak{t}\,\mathfrak{m}) \succ 1$.

By Proposition 1.3.2, the function $\Psi: \mathfrak{M}_{(\mu)} \longrightarrow \mathfrak{N}$ extends uniquely into a strongly linear strictly increasing embedding $\mathbb{T}_{(\mu)} \longrightarrow \mathbb{U}$, which we still denote by Ψ.

We claim that Ψ is an embedding of force $\boldsymbol{\nu}$. By Lemma 6.1.3, we need only show that Ψ commutes with logarithms and hyperlogarithms. We begin with logarithms. Let $\mathfrak{l} \in \mathfrak{L}_{<\theta}$ and $\varphi \in \mathbf{T}$. If $\mu = 1$, then $\mathfrak{l} = \ell_0^r$ for some $r \in \mathbb{R}$ and

$$\log(\Psi(\ell_0^r[\mathrm{e}_\omega^\varphi])) = r\, L_1(\mathfrak{a}_\varphi) = r\, L_1(E_\omega(\Phi(\varphi))) = r\, E_\omega(\Phi(\varphi - 1)).$$

We have

$$r\, E_\omega(\Phi(\varphi - 1)) = r\, \mathfrak{a}_{\varphi-1} = \Psi(\log(\ell_0^r[\mathrm{e}_\omega^\varphi])).$$

If $\mu > 1$, then

$$\log(\Psi(\mathfrak{l}[\mathrm{e}_\beta^\varphi])) = \sum_{\gamma<\theta} \mathfrak{l}_\gamma\, L_{\gamma+1}(\mathfrak{a}_\varphi) = \Psi\left(\sum_{\gamma<\theta} \mathfrak{l}_\gamma\, \ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]\right) = \Psi(\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi])).$$

In all cases, we have, $\log(\Psi(\mathfrak{l}[\mathrm{e}_\beta^\varphi])) = \Psi(\log(\mathfrak{l}[\mathrm{e}_\beta^\varphi]))$. For $\mathfrak{t}\,\mathfrak{m} \in \mathfrak{M}_{(\mu)}$, we have

$$\begin{aligned}\log \Psi(\mathfrak{t}\,\mathfrak{m}) &= \log \Psi(\mathfrak{t}) + \log \Psi(\mathfrak{m}) = \sum_{\varphi \in \mathbf{T}} \log(\Psi(\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi])) + \log \Phi(\mathfrak{m}) \\ &= \sum_{\varphi \in \mathbf{T}} \Psi(\log(\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi])) + \Phi(\log \mathfrak{m}) = \Psi(\log \mathfrak{t}) + \Psi(\log \mathfrak{m}) = \Psi(\log(\mathfrak{t}\,\mathfrak{m})).\end{aligned}$$

Now, let $0 < \eta \leqslant \mu + 1$ and let $\mathfrak{t} = \ell_\gamma[\mathrm{e}_\beta^\varphi] \in \operatorname{dom} L_{\omega^\eta} \setminus \mathfrak{M}_{\omega^\eta}$. Note that $\Psi(\mathfrak{t}) = L_\gamma(\mathfrak{a}_\varphi)$, so we need to show that $\Psi(L_{\omega^\eta}(\mathfrak{t})) = L_{\omega^\eta}(L_\gamma(\mathfrak{a}_\varphi))$. Write $\gamma = \gamma_{\geqslant \omega^\eta} + \omega^{\eta_-} n$. If $\eta < \mu_-$, then

$$\Psi(L_{\omega^\eta}(\mathfrak{t})) = \Psi(\ell_{\gamma_{\geqslant \omega^\eta} + \omega^\eta}[\mathrm{e}_\beta^\varphi] - n) = L_{\gamma_{\geqslant \omega^\eta} + \omega^\eta}(\mathfrak{a}_\varphi) - n = L_{\omega^\eta}(L_\gamma(\mathfrak{a}_\varphi)).$$

If $\eta = \mu_- < \mu$, then $\gamma = \omega^{\mu_{--}} n$. We have

$$\Psi(L_\theta(\ell_\gamma[\mathrm{e}_\beta^\varphi])) = \Psi(\mathrm{e}_\beta^{\varphi-1}) - n = \mathfrak{a}_{\varphi-1} - n = L_\theta(\mathfrak{a}_\varphi) - n = L_\theta(L_\gamma(\mathfrak{a}_\varphi)).$$

If $\eta = \mu$, then $\gamma = 0$ and

$$\Psi(L_\beta(\mathfrak{t})) = \Psi(\varphi) = \Phi(\varphi) = L_\beta(\mathfrak{a}_\varphi).$$

If $\eta = \mu + 1$, then $\gamma = 0$ and

$$\Psi(L_{\omega^{\mu+1}}(\mathfrak{t})) = \Psi(L_{\omega^{\mu+1}}(\varphi) + 1) = \Phi(L_{\omega^{\mu+1}}(\varphi) + 1) = L_{\omega^{\mu+1}}(\Phi(\varphi)) + 1 = L_{\omega^{\mu+1}}(\mathfrak{a}_\varphi).$$

Since $\Psi(L_{\omega^\eta}(\mathfrak{m})) = \Phi(L_{\omega^\eta}(\mathfrak{m})) = L_{\omega^\eta}(\Phi(\mathfrak{m})) = L_{\omega^\eta}(\Psi(\mathfrak{m}))$ for $\mathfrak{m} \in \mathfrak{M}_{\omega^\eta}$ and since $\operatorname{dom} L_{\omega^\eta} = \mathfrak{M}_{\omega^\eta}$ for $\eta > \mu + 1$, this completes the proof of our claim that Ψ is an embedding of force $\boldsymbol{\nu}$.

We finish with the uniqueness of Ψ. Let $\Lambda: \mathbb{T}_{(\mu)} \longrightarrow \mathbb{U}$ be another embedding of force $\boldsymbol{\nu}$ that extends Φ. To see that $\Lambda = \Psi$, we only need to show that $\Lambda(\mathfrak{t}) = \Psi(\mathfrak{t})$ for all $\mathfrak{t} \in \mathfrak{L}_{<\theta}[\mathrm{e}_\beta^{\mathbf{T}}]$. For $\varphi \in \mathbf{T}$, we have

$$L_\beta(\Lambda(\mathrm{e}_\beta^\varphi)) = \Lambda(L_\beta(E_\beta(\varphi))) = \Lambda(\varphi) = \Phi(\varphi),$$

so $\Lambda(\mathrm{e}_\beta^\varphi) = \mathfrak{a}_\varphi$. For $\gamma < \theta$, we deduce that

$$\Lambda(\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]) = \Lambda(L_{\gamma+1}(\mathrm{e}_\beta^\varphi)) = L_{\gamma+1}(\Lambda(\mathrm{e}_\beta^\varphi)) = L_{\gamma+1}(\mathfrak{a}_\varphi) = \Psi(\ell_{\gamma+1}[\mathrm{e}_\beta^\varphi]).$$

Since Λ is strongly linear, this gives $\log \Lambda(\mathfrak{t}) = \Lambda(\log \mathfrak{t}) = \Psi(\log \mathfrak{t}) = \log \Psi(\mathfrak{t})$ for $\mathfrak{t} \in \mathfrak{L}_{<\beta}[\mathrm{e}_\beta^{\mathbf{T}}]$, so $\Lambda(\mathfrak{t}) = \Psi(\mathfrak{t})$ by the injectivity of log. □

6.4.2 Hyperexponential closure

In this section, we prove Theorem 5.1.5 for $\boldsymbol{\pi} = \mu + 1$. We fix a confluent hyperserial skeleton $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ of force $\boldsymbol{\nu}$.

Definition 6.4.7. *For $\gamma \in \mathbf{On}$, we define $\mathfrak{M}_{(\mu+1,\gamma)}$ as follows:*

- $\mathfrak{M}_{(\mu+1,0)} := \mathfrak{M}$.
- $\mathfrak{M}_{(\mu+1,\gamma+1)} := ((\mathfrak{M}_{(\mu+1,\gamma)})_{(<\mu)})_{(\mu)}$.
- $\mathfrak{M}_{(\mu+1,\gamma)} := \bigcup_{\rho<\gamma} \mathfrak{M}_{(\mu+1,\rho)}$ *if γ is a non-zero limit*

Where $\mathfrak{N} \mapsto \mathfrak{N}_{(\mu)}$ is given by Section 6.2.2 (for $\mathbb{R}[[\mathfrak{N}]]$ of force $(\boldsymbol{\nu}, \mu)$), and $\mathfrak{N} \mapsto \mathfrak{N}_{(<\mu)}$ is given by 5.1.5 (in general).

We set $\mathbb{T}_{(\mu+1,\gamma)} := \mathbb{R}[[\mathfrak{M}_{(\mu+1,\gamma)}]]$, so $\mathbb{T}_{(\mu+1,0)} = \mathbb{T}$ and we have the force $\boldsymbol{\nu}$ inclusion $\mathbb{T}_{(\mu+1,\gamma)} \subseteq \mathbb{T}_{(\mu+1,\rho)}$ whenever $\gamma < \rho$. We set

$$\mathfrak{M}_{(<\mu+1)} := \bigcup_{\gamma \in \mathbf{On}} \mathfrak{M}_{(\mu+1,\gamma)}, \qquad \mathbb{T}_{(<\mu+1)} := \bigcup_{\gamma \in \mathbf{On}} \mathbb{T}_{(\mu+1,\gamma)}$$

Note that $\mathbb{T}_{(<\mu+1)} = \mathbb{R}[[\mathfrak{M}_{(<\mu+1)}]]$ by Lemma 1.1.9.

Proposition 6.4.8. *$\mathbb{T}_{(<\mu+1)}$ is a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \mu+1)$.*

Proof. By Corollary 5.3.13, it suffices to show that

$$(\mathbb{T}_{(<\mu+1)})_{\succ,\omega^\eta} \subseteq L_{\omega^\eta}(\mathbb{T}^{>,\succ}_{(<\mu+1)})$$

for all $\eta < \mu + 1$. Fix $\eta < \mu + 1$ and fix $s \in (\mathbb{T}_{<\mu+1})_{\succ,\omega^\eta}$. Fix also an ordinal γ with $s \in \mathbb{T}^{>,\succ}_{(\mu+1,\gamma)}$. Then either $E_{\omega^\eta}(s)$ exists in $\mathbb{T}_{(\mu+1,\gamma)}$ or $E_{\omega^\eta}(s)$ exists in $\mathbb{T}_{(\mu+1,\gamma+1)}$. In either case, $E_{\omega^\eta}(s) \in \mathbb{T}_{(<\mu+1)}$. □

Proposition 6.4.9. *Let $\mathbb{U}$ be a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \mu+1)$ and let $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ be a force $\boldsymbol{\nu}$ embedding. Then there is a unique force $\boldsymbol{\nu}$ embedding $\Psi: \mathbb{T}_{(<\mu+1)} \longrightarrow \mathbb{U}$ extending Φ.*

Proof. The proof is the same as that of Proposition 5.1.6, using Proposition 6.1.4 if $\mu = 1$ or Proposition 6.4.6 if $\mu > 1$ in the case of successor γ. □

Theorem 5.1.5 at $\boldsymbol{\pi} = \mu + 1$ follows from Propositions 6.4.8 and 6.4.9. We conclude by induction that Theorem 5.1.5 holds.

Chapter 7
Hyperserial fields

In this chapter, we define hyperserial fields and establish the equivalence between confluent hyperserial fields and skeletons.

7.1 Hyperserial fields

We now define hyperserial fields. Throughout the section, we fix a $\boldsymbol{\nu} \leqslant \mathbf{On}$ and set $\boldsymbol{\lambda} := \omega^{\boldsymbol{\nu}}$.

7.1.1 Axioms for hyperserial fields

Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be an ordered field of well-based series and let $\circ : \mathbb{L}_{<\omega^{\nu}} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ be a function. For $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}$, we define $\mathfrak{m}^r$ as follows: set

$$\begin{aligned} 1^r &:= 1, \\ \mathfrak{m}^r &:= \ell_0^r \circ \mathfrak{m} \quad \text{if } \mathfrak{m} \succ 1, \text{ and} \\ \mathfrak{m}^r &:= \ell_0^{-r} \circ \mathfrak{m}^{-1} \quad \text{if } \mathfrak{m} \prec 1. \end{aligned}$$

For $\mu \leqslant \mathbf{On}$, as in the case of hyperserial skeletons, we define $\mathfrak{M}_{\omega^\mu}$ to be the class of series $s \in \mathbb{T}^{>,\succ}$ with $\ell_\gamma \circ s \in \mathfrak{M}^{\succ}$ for all $\gamma < \omega^\mu$. The elements of $\mathfrak{M}_{\omega^\mu}$ are said *$L_{<\omega^\mu}$-atomic*, and $L_{<\omega}$-atomic series are said *log-atomic*. Finally the elements of $\mathfrak{M}_{\omega^\nu}$ are said *atomic*.

We say that $(\mathbb{T}, \circ)$ is a *hyperserial field of force $\boldsymbol{\nu}$* if the following axioms are satisfied:

HF1. $\mathbb{L}_{<\omega^{\nu}} \longrightarrow \mathbb{T}; f \mapsto f \circ s$ is a strongly linear embedding of ordered fields for all $s \in \mathbb{T}^{>,\succ}$.

HF2. $f \circ (g \circ s) = (f \circ g) \circ s$ for all $f \in \mathbb{L}_{<\omega^{\nu}}$, $g \in \mathbb{L}_{<\omega^{\nu}}^{>,\succ}$, and $s \in \mathbb{T}^{>,\succ}$.

HF3. $f \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $f \in \mathbb{L}_{<\omega^{\nu}}$, $t \in \mathbb{T}^{>,\succ}$, and $\delta \in \mathbb{T}$ with $\delta \prec t$.

HF4. $\ell_{\omega^\mu}^{\uparrow\gamma} \circ s < \ell_{\omega^\mu}^{\uparrow\gamma} \circ t$ for all ordinals $\mu < \boldsymbol{\nu}$, all $\gamma < \omega^\mu$, and all $s, t \in \mathbb{T}^{>,\succ}$ with $s < t$.

HF5. The map $\mathbb{R} \times \mathfrak{M} \rightarrow \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$ is a law of ordered $\mathbb{R}$-vector field (i.e. real power operation) on $\mathfrak{M}$.

HF6. $\ell_1 \circ (s\, t) = \ell_1 \circ s + \ell_1 \circ t$ for all $s, t \in \mathbb{T}^{>,\succ}$.

HF7. $\operatorname{supp} \ell_1 \circ \mathfrak{m} \succ 1$ for all $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and $\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ (\ell_\gamma \circ \mathfrak{a})^{-1}$ for all $1 \leqslant \mu < \boldsymbol{\nu}$, all $\gamma < \omega^\mu$, and all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

7.1.2 Elementary consequences of the axioms

The axioms **HF6** and **HF7** are assumed to hold trivially when $\boldsymbol{\nu} = 0$. In most cases we will assume that $\boldsymbol{\nu} > 0$. A consequence of the axioms is that ℓ_0 acts as the identity function:

Lemma 7.1.1. *Let $\mathbb{T}$ be a hyperserial field of force $\boldsymbol{\nu}$. For all $s \in \mathbb{T}^{>,\succ}$, we have $\ell_0 \circ s = s$.*

Proof. Let $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and $r \in \mathbb{R}^{>}$. We have $\ell_0 \circ \mathfrak{m} = \mathfrak{m}^1$ and $(\mathfrak{m}^1)^1 = \mathfrak{m}^{1 \times 1} = \mathfrak{m}^1$ by **HF5**. The function $\mathfrak{M} \longrightarrow \mathfrak{M}; \mathfrak{n} \mapsto \mathfrak{n}^1$ is strictly increasing by **HF5**, hence injective. Thus $\mathfrak{m}^1 = \mathfrak{m}$. We obtain $(r\, \ell_0) \circ \mathfrak{m} = r\, \mathfrak{m}$ by **HF1**. In $\mathbb{L}$, we have $\ell_0 \circ (r\, \ell_0) = r\, \ell_0$, so **HF2** yields $\ell_0 \circ (r\, \mathfrak{m}) = r\, \mathfrak{m}$. Now let $s \in \mathbb{T}^{>,\succ}$ and write $s = r\, \mathfrak{d}_s + \delta$ where $r \in \mathbb{R}^{>}$ and $\delta \prec \mathfrak{d}_s$. By **HF3**, we have

$$\ell_0 \circ s = \sum_{k \in \mathbb{N}} \frac{\ell_0^{(k)} \circ (r\, \mathfrak{d}_s)}{k!} \delta^k = r\, \mathfrak{d}_s + \delta = s. \qquad \square$$

The hyperserial field $(\mathbb{T},\circ)$ is said *confluent* if $\mathfrak{M}\neq 1$ and if for all $\mu\in\mathbf{On}$ with $\mu\leqslant\boldsymbol{\nu}$ and all $s\in\mathbb{T}^{>,\succ}$, there are an $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$ and a $\gamma<\omega^\mu$ with

$$\ell_\gamma\circ s\asymp\ell_\gamma\circ\mathfrak{a}. \tag{7.1.1}$$

For $\gamma<\boldsymbol{\lambda}$, we write L_γ for the partial function $\mathbb{T}^{>,\succ}\longrightarrow\mathbb{T}; s\mapsto\ell_\gamma\circ s$, and we call $(\mathbb{T},(L_{\omega^\mu}\upharpoonright\mathfrak{M}_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ the hyperserial skeleton of $(\mathbb{T},\circ)$. We will see in the next section that if $(\mathbb{T},\circ)$ is confluent, then $(\mathbb{T},(L_{\omega^\mu}\upharpoonright\mathfrak{M}_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is indeed a confluent hyperserial skeleton of force $\boldsymbol{\nu}$.

7.2 Fields and skeletons

For the remainder of this section, we fix a hyperserial field $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ of force $\boldsymbol{\nu}$. For each $\mu<\boldsymbol{\nu}$, we define the function $L_{\omega^\mu}\colon\mathfrak{M}_{\omega^\mu}\longrightarrow\mathbb{T};\mathfrak{a}\longmapsto\ell_{\omega^\mu}\circ\mathfrak{a}$. The *skeleton* of $(\mathbb{T},\circ)$ is defined to be the structure $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ equipped with the real powering operation on $\mathfrak{M}$ given by **HF5**. The main purpose of this section is to prove the following theorem.

Theorem 7.2.1. *The skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ of $(\mathbb{T},\circ)$ is a hyperserial skeleton. Moreover, if $(\mathbb{T},\circ)$ is confluent, then so is its skeleton and $\circ$ coincides with the unique composition from Theorem 4.3.1.*

7.2.1 The skeleton of a hyperserial field

When $\boldsymbol{\nu}=0$, then the skeleton of $\mathbb{T}$ is just the field $\mathbb{T}$ itself with the real powering operation on $\mathfrak{M}$. Clearly, this is a hyperserial skeleton, as there are no axioms to verify. Moreover, it is 0-confluent so long as $(\mathbb{T},\circ)$ is, so Theorem 7.2.1 follows from Proposition 4.4.1, since $\circ$ clearly satisfies $\mathbf{C1}_0$, $\mathbf{C2}_0$, $\mathbf{C3}_0$, and $\mathbf{C4}_0$. Therefore, we may assume that $\boldsymbol{\nu}>0$. We will verify the various hyperserial skeleton axioms over the next few lemmas, beginning with the Domain of Definition axioms:

Lemma 7.2.2. *The skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ satisfies the axioms $\mathbf{DD}_\mu$ for all $\mu<\boldsymbol{\nu}$.*

Proof. By definition, $\mathfrak{D}_0$ is the class of $s\in\mathbb{T}^{>,\succ}$ with $\ell_0\circ s\in\mathfrak{M}^\succ$. Since $\ell_0\circ s=s$ by **HF5**, the axiom $\mathbf{DD_0}$ holds. Let us fix $0<\mu<\boldsymbol{\nu}$ and let us assume that $\mathbf{DD}_\eta$ holds for all $\eta<\mu$. If μ is a limit, then

$$\begin{aligned}\bigcap_{\eta<\mu}\operatorname{dom}L_{\omega^\eta} &= \bigcap_{\eta<\mu}\{s\in\mathbb{T}^{>,\succ}:\ell_\gamma\circ s\in\mathfrak{M}^\succ \text{ for all } \gamma<\omega^\eta\}\\ &= \{s\in\mathbb{T}^{>,\succ}:\ell_\gamma\circ s\in\mathfrak{M}^\succ \text{ for all } \gamma<\omega^\mu\}=\operatorname{dom}L_{\omega^\mu}.\end{aligned}$$

Suppose μ is a successor. The inclusion $\operatorname{dom}L_{\omega^\mu}\subseteq\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_{\omega^{\mu_-}}^{\circ n}$ holds by definition, so we show the other inclusion. Let $\gamma<\omega^\mu$ and let $s\in\bigcap_{n\in\mathbb{N}}\operatorname{dom}L_{\omega^{\mu_-}}^{\circ n}$. Take $n\in\mathbb{N}$ and $\sigma<_o\omega^{\mu_-}$ with $\gamma=\omega^{\mu_-}n+\sigma$. Then $L_{\omega^{\mu_-}}^{\circ n}(s)\in\operatorname{dom}L_{\omega^{\mu_-}}$, so $\ell_\sigma\circ L_{\omega^{\mu_-}}^{\circ n}(s)\in\mathfrak{M}^\succ$, by our inductive assumption. Repeated applications of **HF2** give $\ell_\sigma\circ L_{\omega^{\mu_-}}^{\circ n}(s)=\ell_\gamma\circ s$. Since $\gamma<\omega^\mu$ is arbitrary, this gives $s\in\operatorname{dom}L_{\omega^\mu}$. □

Now for the functional equations, asymptotics, regularity, and monotonicity axioms:

Lemma 7.2.3. *The skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ satisfies the axioms $\mathbf{FE}_\mu$, $\mathbf{A}_\mu$, and $\mathbf{R}_\mu$ for all $\mu<\boldsymbol{\nu}$.*

Proof. Given $r\in\mathbb{R}^>$ and $\mathfrak{m},\mathfrak{n}\in\mathfrak{M}$, we have

$$\begin{aligned}L_1(\mathfrak{m}^r) &= \ell_1\circ(\ell_0^r\circ\mathfrak{m})=(\ell_1\circ\ell_0^r)\circ\mathfrak{m}=(r\,\ell_1)\circ\mathfrak{m}=r\,(\ell_1\circ\mathfrak{m})=r\,L_1(\mathfrak{m}) && \text{(by HF2 and HF1)}\\ L_1(\mathfrak{m}\,\mathfrak{n}) &= \ell_1\circ(\mathfrak{m}\,\mathfrak{n})=\ell_1\circ\mathfrak{m}+\ell_1\circ\mathfrak{n}=L_1(\mathfrak{m})+L_1(\mathfrak{n}), && \text{(by HF6)}\end{aligned}$$

so $\mathbf{FE_0}$ holds. Let $0<\mu<\boldsymbol{\nu}$ be a successor ordinal and let $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$, so $L_{\omega^{\mu_-}}(\mathfrak{a})$ is defined and lies in $\mathfrak{M}_{\omega^\mu}$. The axiom **HF2** implies

$$L_{\omega^\mu}(L_{\omega^{\mu_-}}(\mathfrak{a}))=\ell_{\omega^\mu}\circ(\ell_{\omega^{\mu_-}}\circ\mathfrak{a})=(\ell_{\omega^\mu}\circ\ell_{\omega^{\mu_-}})\circ\mathfrak{a}=(\ell_{\omega^\mu}-1)\circ\mathfrak{a}=L_{\omega^\mu}(\mathfrak{a})-1,$$

so $\mathbf{FE}_\mu$ holds as well. The asymptotics axiom $\mathbf{A_0}$ follows from the relation $\ell_1 \prec \ell_0$ in $\mathbb{L}_{<\omega^\nu}$ and **HF1**. Likewise, $\mathbf{A}_\mu$ follows from the fact that $\ell_{\omega^\mu} \prec \ell_{\omega^\eta}$ for all $\eta < \mu$. By **HF1**, we note that the sets $(\ell_{<\omega^\mu} \circ s)^{-1}$ and $\{(\ell_{\omega^\eta n} \circ s)^{-1} : \eta < \mu \text{ and } n \in \mathbb{N}\}$ are mutually cofinal for each $s \in \mathbb{T}^{>,\succ}$. The regularity axioms $\mathbf{R}_\mu$ for $\mu < \boldsymbol{\nu}$ therefore follow from **HF7**. □

Lemma 7.2.4. *The skeleton $(\mathbb{T}, (L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ satisfies the axioms $\mathbf{M}_\mu$ for all $\mu < \boldsymbol{\nu}$.*

Proof. The axiom $\mathbf{M_0}$ follows from the fact that $\ell_1 > 0$. For $0 < \mu < \boldsymbol{\nu}$, let $\gamma < \omega^\mu$ and take $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_{\omega^\mu}$ with $\mathfrak{a} \prec \mathfrak{b}$. We need to show

$$\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > (\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1}.$$

We first consider the case that $\mathfrak{a} \prec \ell_{\omega^\eta} \circ \mathfrak{b}$ for some $\eta < \mu$ with $\gamma < \omega^{\eta+1}$. Then **HF4** gives us that $\ell_{\omega^\mu}^{\uparrow\omega^\eta} \circ \mathfrak{a} < \ell_{\omega^\mu}^{\uparrow\omega^\eta} \circ (\ell_{\omega^\eta} \circ \mathfrak{b}) = \ell_{\omega^\mu} \circ \mathfrak{b}$. By (4.1.4), we have $\ell_{\omega^\mu}^{\uparrow\omega^\eta} = \ell_{\omega^\mu} + \mathfrak{l} + \varepsilon$, where $\mathfrak{l} = \frac{1}{\ell'_{\omega^{\eta+1}}} \ell'_{\omega^\mu} = \prod_{\omega^{\eta+1} \leqslant \sigma < \omega^\mu} \ell_\sigma^{-1}$ and $\varepsilon \prec \mathfrak{l}$. Since $\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu}^{\uparrow\omega^\eta} \circ \mathfrak{a} > 0$, we have

$$\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > \mathfrak{l} \circ \mathfrak{a} + \varepsilon \circ \mathfrak{a}.$$

Since $\gamma < \omega^{\eta+1}$, we have $\ell_\gamma^{-1} \prec \mathfrak{l}$, so $(\ell_\gamma \circ \mathfrak{a})^{-1} = \ell_\gamma^{-1} \circ \mathfrak{a} \prec \mathfrak{l} \circ \mathfrak{a}$. The axiom **HF4** gives $\ell_\gamma \circ \mathfrak{a} < \ell_\gamma \circ \mathfrak{b}$, so $(\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1} < 2(\ell_\gamma \circ \mathfrak{a})^{-1} \prec \mathfrak{l} \circ \mathfrak{a}$. Thus,

$$\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > (\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1}.$$

Now we handle the case that $\mathfrak{a} \succcurlyeq \ell_{\omega^\eta} \circ \mathfrak{b}$ for all $\eta < \mu$ with $\gamma < \omega^{\eta+1}$. We claim that the sets

$$\{(\ell_\sigma \circ \mathfrak{a})^{-1} : \sigma < \omega^\mu\} \text{ and } \{(\ell_\sigma \circ \mathfrak{b})^{-1} : \sigma < \omega^\mu\}$$

are mutually cofinal. Let $\sigma < \omega^\mu$ be given and take $\eta < \mu$ with $\gamma < \omega^{\eta+1}$ and $\sigma \leqslant_o \omega^\eta$. Then $\mathfrak{a} \succcurlyeq \ell_{\omega^\eta} \circ \mathfrak{b}$ by assumption, so $\mathfrak{a} > \ell_{\omega^\eta 2} \circ \mathfrak{b}$ and **HF4** gives $\ell_\sigma \circ \mathfrak{b} > \ell_\sigma \circ \mathfrak{a} > \ell_\sigma \circ (\ell_{\omega^\eta 2} \circ \mathfrak{b}) = \ell_{\omega^\eta 2 + \sigma} \circ \mathfrak{b}$. This proves the cofinality claim. Now **HF7** gives $\operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{a}) \succ \{(\ell_\sigma \circ \mathfrak{a})^{-1} : \sigma < \omega^\mu\}$ and likewise, $\operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{b}) \succ \{(\ell_\sigma \circ \mathfrak{b})^{-1} : \sigma < \omega^\mu\}$. Thus

$$\operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a}) \subseteq \operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{a}) \cup \operatorname{supp}(\ell_{\omega^\mu} \circ \mathfrak{b}) \succ \{(\ell_\sigma \circ \mathfrak{a})^{-1}, (\ell_\sigma \circ \mathfrak{b})^{-1} : \sigma < \omega^\mu\}.$$

In particular, we have $\ell_{\omega^\mu} \circ \mathfrak{b} - \ell_{\omega^\mu} \circ \mathfrak{a} > (\ell_\gamma \circ \mathfrak{a})^{-1} + (\ell_\gamma \circ \mathfrak{b})^{-1}$, as desired. □

Before proving the infinite powers axioms, we need a lemma:

Lemma 7.2.5. *Let $s = c\,\mathfrak{m}\,(1+\varepsilon) \in \mathbb{T}^{>,\succ}$ with $c \in \mathbb{R}^>$, $\mathfrak{m} := \mathfrak{d}_s$, and $\varepsilon \prec 1$. Then*

$$\ell_1 \circ s = \ell_1 \circ \mathfrak{m} + \log c + \tilde{L}(\varepsilon),$$

where L is as defined in Section 3.1.

Proof. Set $\delta := c\,\mathfrak{m}\,\varepsilon$, so $\delta \prec c\,\mathfrak{m}$ and $s = c\,\mathfrak{m} + \delta$. The axiom **HF3** gives

$$\ell_1 \circ s = \ell_1 \circ (c\,\mathfrak{m}) + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta.$$

We have $\ell_1 \circ (c\,\mathfrak{m}) = \ell_1 \circ ((c\,\ell_0) \circ \mathfrak{m}) = (\ell_1 \circ (c\,\ell_0)) \circ \mathfrak{d}_{\mathfrak{l} \circ \mathfrak{a}}$ by **HF2**, and $\ell_1 \circ (c\,\ell_0) = \ell_1 + \log c$. Hence

$$\ell_1 \circ s = (\ell_1 + \log c) \circ \mathfrak{m} + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta = \ell_1 \circ \mathfrak{m} + \log c + \sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta.$$

Given $k > 0$, we have $\ell_1^{(k)} \circ t = (-1)^{k-1} (k-1)!\, t^{-k}$. So for $\delta \prec t$, we have

$$\frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta = \frac{(-1)^{k-1}}{k} \left(\frac{\delta}{c\,\mathfrak{m}}\right)^k = \frac{(-1)^{k-1}}{k} \varepsilon.$$

Thus, $\sum_{k \in \mathbb{N}^>} \frac{\ell_1^{(k)} \circ (c\,\mathfrak{m})}{k!} \delta^k = \tilde{L}(\varepsilon)$. □

By **HF4**, **HF6** and **HF7**, the function $\mathfrak{M} \longrightarrow \mathbb{T}; \mathfrak{m} \mapsto \ell_1 \circ \mathfrak{m}$ satisfies the hypotheses of Proposition 3.1.10. Furthermore Lemma 7.2.5 and the corresponding logarithm on $\mathbb{T}^>$ coincides with $s \mapsto \ell_1 \circ s$ on $\mathbb{T}^{>,\succ}$.

Lemma 7.2.6. *Let $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ be a hyperserial field of force ν, let $\mu<\nu$ and $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$. We have*

$$\forall\mathfrak{m}\in\operatorname{supp}L_{\omega^\mu}(\mathfrak{a}),\mathfrak{m}\lessdot\!\!\!\lessdot\mathfrak{a}.$$

Proof. If $\mu=0$, then this follows from the fact that $\operatorname{supp}L_1(\mathfrak{a})\succ 1$ by **HF7** and from Lemma 3.1.7. Assume that $\mu>0$ and let $\mathfrak{m}\in\operatorname{supp}L_{\omega^\mu}(\mathfrak{a})$. Then $\mathfrak{m}\preccurlyeq L_{\omega^\mu}(\mathfrak{a})\ll\mathfrak{a}$ so $\mathfrak{m}\ll\mathfrak{a}$. We have

$$\operatorname{supp}L_{\omega^\mu}(\mathfrak{a})\succ(L_{<\omega^\mu}\mathfrak{a})^{-1}$$

by **HF7**, whence in particular $\mathfrak{m}\succ(L_1\mathfrak{a})^{-1}$. We deduce that $\mathfrak{m}\gg\mathfrak{a}^{-1}$. Therefore $\mathfrak{m}\lessdot\!\!\!\lessdot\mathfrak{a}$. □

Lemma 7.2.7. *The skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ satisfies the axioms $\mathbf{P}_{\boldsymbol{\mu}}$ for all $\mu\in\mathbf{On}$ with $\mu\leqslant\boldsymbol{\nu}$.*

Proof. let $\mu\in\mathbf{On}$ with $\mu\leqslant\boldsymbol{\nu}$, let $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$ and let $(r_\gamma)_{\gamma<\omega^\mu}$ be a sequence of real numbers. We need to show that $\sum_{\gamma<\omega^\mu}r_\gamma L_{\gamma+1}(\mathfrak{a})\in\log\mathfrak{M}$, where $\log\mathfrak{m}:=-\ell_1\circ\mathfrak{m}^{-1}$ for $\mathfrak{m}\in\mathfrak{M}^{\prec}$ and where $\log 1:=0$. Set $\mathfrak{l}:=\prod_{\gamma<\omega^\mu}\ell_\gamma^{r_\gamma}$. We may assume that $\mathfrak{l}\neq 1$ and, by negating each r_γ if need be, we further assume that $\mathfrak{l}\succ 1$. Hence $\ell_1\circ\mathfrak{l}$ is defined. The axioms **HF1** and **HF2** give

$$\sum_{\gamma<\omega^\mu}r_\gamma L_{\gamma+1}(\mathfrak{a})=(\ell_1\circ\mathfrak{l})\circ\mathfrak{a}=\ell_1\circ(\mathfrak{l}\circ\mathfrak{a}),$$

so it remains to show that $\mathfrak{l}\circ\mathfrak{a}\in\mathfrak{M}^{\succ}$. For each $\gamma<\omega^\mu$, we have $L_{\gamma+1}(\mathfrak{a})\in\mathfrak{M}^{\succ}$. This gives $\operatorname{supp}\ell_1\circ(\mathfrak{l}\circ\mathfrak{a})\subseteq\mathfrak{M}^{\succ}$. Take $r\in\mathbb{R}^{>}$ and $\varepsilon\prec 1$ with $\mathfrak{l}\circ\mathfrak{a}=r\,\mathfrak{d}_{\mathfrak{l}\circ\mathfrak{a}}(1+\varepsilon)$. Lemma 7.2.5 yields

$$\ell_1\circ(\mathfrak{l}\circ\mathfrak{a})=\ell_1\circ\mathfrak{d}_{\mathfrak{l}\circ\mathfrak{a}}+\log r+\tilde{L}(\varepsilon).$$

We have $\operatorname{supp}(\ell_1\circ\mathfrak{d}_{\mathfrak{l}\circ\mathfrak{a}})\succ 1$ by **HF7**. If $\varepsilon\neq 0$, then $\tilde{L}(\varepsilon)\sim\varepsilon$, so $\mathfrak{d}_\varepsilon\in\operatorname{supp}\tilde{L}(\varepsilon)$. If $r\neq 1$, we have $\operatorname{supp}\log r=\{1\}$. As we have established that $\operatorname{supp}\ell_1\circ(\mathfrak{l}\circ\mathfrak{a})\subseteq\mathfrak{M}^{\succ}$, it follows that $r=1$ and $\varepsilon=0$. Thus $\mathfrak{l}\circ\mathfrak{a}=\mathfrak{d}_{\mathfrak{l}\circ\mathfrak{a}}\in\mathfrak{M}$, as desired. □

This shows that $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is a hyperserial skeleton of force $\boldsymbol{\nu}$.

7.2.2 Equivalence between confluent fields and skeletons

Now we turn to confluence. First, we need a lemma:

Lemma 7.2.8. *Let $s,t\in\mathbb{T}^{>,\succ}$ and let $\gamma<\omega^{\boldsymbol{\nu}}$. If $\ell_\gamma\circ s\asymp\ell_\gamma\circ t$, then $\ell_{\gamma+1}\circ s-\ell_{\gamma+1}\circ t\preccurlyeq 1$ and $\ell_\sigma\circ s-\ell_\sigma\circ t\prec 1$ for all σ with $\gamma+2\leqslant\sigma<\omega^{\boldsymbol{\nu}}$. In particular, $\ell_\sigma\circ s\asymp\ell_\sigma\circ t$ for all σ with $\gamma\leqslant\sigma<\omega^{\boldsymbol{\nu}}$.*

Proof. The proof is essentially the same as the proof of Lemma 4.3.4. Take $c\in\mathbb{R}^{>}$ and $\varepsilon\prec 1$ with $\ell_\gamma\circ s=c\,(\ell_\gamma\circ t)\,(1+\varepsilon)$. By Lemma 7.2.5, we have

$$\ell_{\gamma+1}\circ s=\ell_1\circ(\ell_\gamma\circ s)=\ell_1\circ(c\,(\ell_\gamma\circ t)\,(1+\varepsilon))=\ell_{\gamma+1}\circ t+\log c+\tilde{L}(\varepsilon),$$

so $\ell_{\gamma+1}\circ s\sim\ell_{\gamma+1}\circ t$. Set $\delta:=(\ell_{\gamma+1}\circ t)^{-1}(\log c+\tilde{L}(\varepsilon))\prec 1$, so $\ell_{\gamma+1}\circ s=(\ell_{\gamma+1}\circ t)\,(1+\delta)$. Again, Lemma 7.2.5 gives

$$\ell_{\gamma+2}\circ s=\ell_1\circ(\ell_{\gamma+1}\circ s)=\ell_1\circ((\ell_{\gamma+1}\circ t)\,(1+\delta))=\ell_{\gamma+2}\circ t+\tilde{L}(\delta),$$

so $\ell_{\gamma+2}\circ s-\ell_{\gamma+2}\circ t=L(1+\delta)\sim\delta\prec 1$. Now set $h:=(\ell_{\gamma+2}\circ s-\ell_{\gamma+2}\circ t)\prec 1$ and fix σ with $\gamma+2\leqslant\sigma<\omega^{\boldsymbol{\nu}}$. We have

$$\begin{aligned}\ell_\sigma\circ s-\ell_\sigma\circ t &= \ell_\sigma^{\uparrow\gamma+2}\circ(\ell_{\gamma+2}\circ s)-\ell_\sigma^{\uparrow\gamma+2}\circ(\ell_{\gamma+2}\circ t)\\ &= \ell_\sigma^{\uparrow\gamma+2}\circ((\ell_{\gamma+2}\circ t)+h)-\ell_\sigma^{\uparrow\gamma+2}\circ(\ell_{\gamma+2}\circ t)\\ &= \mathcal{T}_{\ell_\sigma^{\uparrow\gamma+2}}(\ell_{\gamma+2}\circ t,h)\sim(\ell_\sigma^{\uparrow\gamma+2})'\circ(\ell_{\gamma+2}\circ t)\,h.\end{aligned}$$

Since $(\ell_\sigma^{\uparrow\gamma+2})',h\prec 1$, we have $\ell_\sigma\circ s-\ell_\sigma\circ t\prec 1$. □

Lemma 7.2.9. *Suppose $(\mathbb{T},\circ)$ is confluent. Then $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is confluent as well.*

Proof. The skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is 0-confluent since $\mathfrak{M}$ is non-trivial. Let $\mu\in\mathbf{On}$ with $0<\mu\leqslant\boldsymbol{\nu}$ and assume that $(\mathbb{T},(L_{\omega^\eta})_{\eta<\boldsymbol{\nu}})$ is η-confluent for all $\eta<\mu$. We also make the inductive assumption that for $s\in\mathbb{T}^{>,\succ}$ and $\eta<\mu$, we have $\ell_\gamma\circ s\asymp\ell_\gamma\circ\mathfrak{d}_{\omega^\eta}(s)$ for some $\gamma<\omega^\eta$. Let $s\in\mathbb{T}^{>,\succ}$ and take $\gamma<\omega^\mu$ and $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$ with $\ell_\gamma\circ s\asymp\ell_\gamma\circ\mathfrak{a}$. We will show that $\mathfrak{d}_{\omega^\mu}(s)=\mathfrak{a}$. We first handle the case that μ is a successor. Take $n\in\mathbb{N}^>$ with $\gamma<\omega^{\mu_-}n$. Lemma 7.2.8 gives $\ell_{\omega^{\mu_-}n}\circ s\asymp\ell_{\omega^{\mu_-}n}\circ\mathfrak{a}$. By assumption, we have $\ell_\rho\circ\mathfrak{d}_{\omega^{\mu_-}}(s)\asymp\ell_\rho\circ s$ for some $\rho<\omega^{\mu_-}$, so $\ell_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}}(s)\asymp\ell_{\omega^{\mu_-}}\circ s$, again by Lemma 7.2.8. Induction on m gives $(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ m}(s)\asymp\ell_{\omega^{\mu_-}m}\circ s$ for all $m\in\mathbb{N}^>$, so

$$(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(s)\asymp\ell_{\omega^{\mu_-}n}\circ s\asymp\ell_{\omega^{\mu_-}n}\circ\mathfrak{a}=(L_{\omega^{\mu_-}}\circ\mathfrak{d}_{\omega^{\mu_-}})^{\circ n}(\mathfrak{a}),$$

and $\mathfrak{d}_{\omega^\mu}(s)=\mathfrak{a}$. The case that μ is a limit is similar, though this time we take $\eta<\mu$ with $\gamma<\omega^\eta$ and use that

$$L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(s))\asymp\ell_{\omega^\eta}\circ s\asymp\ell_{\omega^\eta}\circ\mathfrak{a}\asymp L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{a}))$$

to see that $\mathfrak{d}_{\omega^\mu}(s)=\mathfrak{a}$. Since s was arbitrary, this gives that $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is μ-confluent. □

Proof of Theorem 7.2.1. Lemmas 7.2.2, 7.2.3, 7.2.4, and 7.2.7 show that $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ of $(\mathbb{T},\circ)$ is a hyperserial skeleton. The composition $\circ$ clearly satisfies **C1**$_{\boldsymbol{\nu}}$, **C2**$_{\boldsymbol{\nu}}$, **C3**$_{\boldsymbol{\nu}}$, and **C4**$_{\boldsymbol{\nu}}$. If $(\mathbb{T},\circ)$ is confluent, then $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is confluent by Lemma 7.2.9 and Proposition 4.4.14 implies that $\circ$ coincides with the unique composition from Theorem 4.3.1. □

Given a confluent hyperserial skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ of force $\boldsymbol{\nu}$, it is clear that the unique composition $\circ:\mathbb{L}_{<\omega^{\boldsymbol{\nu}}}\times\mathbb{T}^{>,\succ}\longrightarrow\mathbb{T}$ in Theorem 4.3.1 satisfies all of the hyperserial field axioms, where **HF4** follows from Lemma 5.1.7. This gives us the following result:

Theorem 7.2.10. *If $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$ is a confluent hyperserial skeleton of force $\boldsymbol{\nu}$, then there is a unique function $\circ$ such that $(\mathbb{T},\circ)$ is a confluent hyperserial field of force $\boldsymbol{\nu}$ with skeleton $(\mathbb{T},(L_{\omega^\mu})_{\mu<\boldsymbol{\nu}})$.*

In the sequel of the thesis, we will always equip confluent hyperserial skeletons with the corresponding composition law, and equip confluent hyperserial fields with their skeleton. We will no longer use the notation $\mathfrak{D}_{\omega^\mu}$ for the class of $L_{<\omega^\mu}$-atomic series in $\mathbb{T}$, since it is redundant with the notation $\mathfrak{M}_{\omega^\mu}$ which we will use instead.

We also deduce that our results on hyperexponentiation in confluent hyperserial skeletons apply for confluent hyperserial fields. In particular, we have the following characterizations of hyperlogarithms and hyperexponentials of force μ:

Let $\mathbb{T}$ be a hyperserial field of force $\boldsymbol{\nu}$, let $\mu\in(0,\boldsymbol{\nu})$ and write $\beta:=\omega^\mu$. For all $s\in\mathbb{T}^{>,\succ}$, there is a $\gamma<\beta$ with $L_\gamma(s)\sim L_\gamma(\mathfrak{d}_\beta(s))$. For all such γ, writing $\delta:=L_\gamma(s)-L_\gamma(\mathfrak{d}_\beta(s))$, we have

$$L_\beta(s)=L_\beta(\mathfrak{d}_\beta(s))+\sum_{k>0}\frac{(\ell_\beta^{\uparrow\gamma})^{(k)}\circ L_\gamma(\mathfrak{d}_\beta(s))}{k!}\delta^k. \tag{7.2.1}$$

Assume that $s\in L_\beta(\mathbb{T}^{>,\succ})$ and let $\gamma<\beta$ and $\varepsilon\in\mathbb{T}^{\prec}$ with $\varepsilon\prec\frac{(L_\gamma\circ E_\beta)(\varphi)}{(L_\gamma\circ E_\beta)'(\varphi)}$. Then

$$E_\beta(s+\varepsilon)=E_\gamma\!\left(E_\beta(s)+\sum_{k>0}\frac{(L_\gamma\circ E_\beta)^{(k)}(s)}{k!}\varepsilon^k\right), \tag{7.2.2}$$

where each $(L_\gamma\circ E_\beta)^{(k)}(\varphi)=t_{\beta,\gamma,k}\circ E_\beta(s)$ for a certain series $t_{\beta,\gamma,k}\in\mathbb{L}_{<\beta}$. The series $t_{\beta,\gamma,k}$ are given by induction by

$$\begin{aligned} t_{\beta,\gamma,0} &= \ell_\gamma \quad\text{and}\\ \forall k\in\mathbb{N}, t_{\beta,\gamma,k+1} &= \frac{t'_{\beta,\gamma,k}}{\ell'_\beta}.\end{aligned}$$

7.2.3 Confluent hyperserial subfields

We next introduce a notion of hyperserial embedding which is more appropriate to the context of hyperserial fields. We fix a $\boldsymbol{\nu}\leqslant\mathbf{On}$ and set $\boldsymbol{\lambda}:=\omega^{\boldsymbol{\nu}}$.

Definition 7.2.11. *Let* $(\mathbb{T}, \circ_{\mathbb{T}})$ *and* $(\mathbb{U}, \circ_{\mathbb{U}})$ *be hyperserial fields of force* $\boldsymbol{\nu}$ *with monomial groups* $\mathfrak{M}$ *and* $\mathfrak{N}$ *respectively. We say that a strongly linear morphism of ordered rings* $\Phi: \mathbb{T} \longrightarrow \mathbb{U}$ *is an* **embedding** *of force* $\boldsymbol{\nu}$ *if we have*

$$\begin{aligned} \Phi(\mathfrak{M}) &\subseteq \mathfrak{N}, \quad \textit{and} \\ \forall f \in \mathbb{L}, \forall s \in \mathbb{T}^{>,\succ}, \Phi(f \circ_{\mathbb{T}} s) &= f \circ_{\mathbb{U}} \Phi(s). \end{aligned}$$

We then say that $(\mathbb{U}, \Psi)$ *is an* **extension** *of* $\mathbb{T}$ *of force* $\boldsymbol{\nu}$*.*

We say that $(\mathbb{T}, \circ_{\mathbb{T}})$ *is a* **subfield** *of* $(\mathbb{U}, \circ_{\mathbb{U}})$ *of force* $\boldsymbol{\nu}$ *and we write* $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{U}, \circ_{\mathbb{U}})$ *if* $\mathbb{T} \subseteq \mathbb{U}$ *and* $\mathrm{Id}_{\mathbb{T}}: \mathbb{T} \longrightarrow \mathbb{U}$ *is an embedding of force* $\boldsymbol{\nu}$*.*

We will add the adjective "confluent" when appropriate, e.g. confluent subfields are subfields that are confluent as hyperserial fields. We allow ourselves to use the same vocabulary as in the case of hyperserial skeletons (see Definition 4.2.4) because we will next show that those two notions coincide (through the correspondence given by Theorems 7.2.1 and 7.2.10) in the confluent case.

Proposition 7.2.12. *Let* $(\mathbb{U}, \circ_{\mathbb{U}})$ *be a hyperserial field of force* $\boldsymbol{\nu}$*. Let* $\triangle: \mathbb{T} \longrightarrow \mathbb{U}$ *be a strongly linear function with*

$$\begin{aligned} \triangle(\log \mathfrak{m}) &= \log \triangle(\mathfrak{m}) \quad \textit{for all } \mathfrak{m} \in \mathfrak{U}, \textit{ and} \\ \triangle(L_{\omega^\mu}(\mathfrak{a})) &= L_{\omega^\mu}(\triangle(\mathfrak{a})) \quad \textit{for all } 0 < \eta < \boldsymbol{\nu} \textit{ and } \mathfrak{a} \in \mathfrak{U}_{\omega^\eta}. \end{aligned}$$

Then $\triangle$ *is a strictly increasing ring morphism with*

$$\begin{aligned} \triangle(f \circ s) &= f \circ_{\mathbb{U}} \triangle(s) \quad \textit{for all } f \in \mathbb{L}_{<\boldsymbol{\lambda}} \textit{ and } s \in \mathbb{T}^{>,\succ}, \textit{ and} \\ \triangle(\log t) &= \log \triangle(t) \quad \textit{for all } t \in \mathbb{T}^{>}. \end{aligned}$$

Proof. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. We have

$$\begin{aligned} \log \triangle(\mathfrak{m}\,\mathfrak{n}) &= \triangle(\log \mathfrak{m}\,\mathfrak{n}) \\ &= \triangle(\log \mathfrak{m}) + \triangle(\log \mathfrak{n}) \\ &= \log \triangle(\mathfrak{m}) + \log \triangle(\mathfrak{n}) \\ &= \log(\triangle(\mathfrak{m})\,\triangle(\mathfrak{n})). \end{aligned}$$

We deduce since log is injective that $\triangle(\mathfrak{m}\,\mathfrak{n}) = \triangle(\mathfrak{m})\,\triangle(\mathfrak{n})$. By strong linearity, it follows that $\triangle(s\,t) = \triangle(s)\,\triangle(t)$ for all $s, t \in \mathbb{U}$, hence that $\triangle$ is a morphism of rings. We deduce with Proposition 2.5.3 that $\triangle$ is strictly increasing.

Let $\mathbf{C}$ denote the class of series $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ with $\triangle(f \circ s) = f \circ \triangle(s)$ for all $s \in \mathbb{U}^{>,\succ}$. We will prove that we have $\mathbb{L}_{<\omega^{\boldsymbol{\mu}}} \subseteq \mathbf{C}$ by induction on $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$, starting with $\boldsymbol{\mu} = 1$. We have $\triangle(\log s) = \log \triangle(s)$ for all $s > 0$ by Proposition 3.1.9. We deduce that $\mathbf{C}$ contains $\mathfrak{l} \in \mathfrak{L}_{<\boldsymbol{\lambda}}$ if and only if it contains $\log \mathfrak{l}$. Note that by strong linearity, the class $\mathbf{C}$ is closed under sums of well-based families. Moreover, for $f, h \in \mathbf{C}$ with $h > \mathbb{R}$, we have $f \circ h \in \mathbf{C}$. So we need only prove that we have $\ell_{\omega^\eta} \in \mathbf{C}$ for all $\eta \in (0, \boldsymbol{\nu})$. Let $\eta \in (0, \boldsymbol{\nu})$ such that this holds for all $\iota < \eta$. So $\mathbb{L}_{<\omega^\eta} \subseteq \mathbf{C}$ by the previous arguments. Let $s \in \mathbb{T}^{>,\succ}$ and write $\mathfrak{s} := \mathfrak{d}_{\omega^\eta}(s)$. Recall that there is a $\gamma < \omega^\eta$ such that the series $\varepsilon := \ell_\gamma \circ_{\mathbb{U}} s - \ell_\gamma \circ_{\mathbb{U}} \mathfrak{s}$ is infinitesimal, with

$$\ell_{\omega^\eta} \circ_{\mathbb{U}} s = \ell_{\omega^\eta} \circ_{\mathbb{U}} \mathfrak{s} + \sum_{k>0} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ_{\mathbb{U}} (\ell_\gamma \circ_{\mathbb{U}} \mathfrak{s})}{k!}\, \varepsilon^k.$$

Note that for $k \in \mathbb{N}^>$, we have $(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \in \mathbb{L}_{<\omega^\eta} \subseteq \mathbf{C}$. The induction hypothesis yields

$$\ell_\gamma \circ_{\mathbb{V}} \triangle(s) - \ell_\gamma \circ_{\mathbb{V}} \triangle(\mathfrak{s}) = \triangle(\ell_\gamma \circ_{\mathbb{U}} s - \ell_\gamma \circ_{\mathbb{U}} \mathfrak{s}) = \triangle(\varepsilon) \prec 1.$$

It follows that

$$\begin{aligned} \ell_{\omega^\eta} \circ_{\mathbb{V}} \triangle(s) &= \ell_{\omega^\eta} \circ_{\mathbb{V}} \triangle(\mathfrak{s}) + \sum_{k>0} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ_{\mathbb{V}} (\ell_\gamma \circ_{\mathbb{V}} \triangle(\mathfrak{s}))}{k!}\, \triangle(\varepsilon)^k. \\ &= \triangle(\ell_{\omega^\eta} \circ_{\mathbb{U}} \mathfrak{s}) + \triangle\left(\sum_{k>0} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ_{\mathbb{U}} (\ell_\gamma \circ_{\mathbb{U}} \mathfrak{s})}{k!}\, \varepsilon^k \right) \quad \text{(by the induction hypothesis)} \\ &= \triangle(\ell_{\omega^\eta} \circ_{\mathbb{U}} s). \end{aligned}$$

We conclude by induction that $\mathbf{C} = \mathbb{L}_{<\boldsymbol{\lambda}}$. □

Corollary 7.2.13. *Embeddings of force $\boldsymbol{\nu}$ between confluent hyperserial skeletons of force $\boldsymbol{\nu}$ are embeddings of force $\boldsymbol{\nu}$ for the corresponding confluent hyperserial fields, and* vice versa*.*

A subfield of $\mathbb{T}$ of force $\boldsymbol{\nu}$ is a hyperserial field $\mathbb{U} \subseteq \mathbb{T}$ such that $\mathrm{Id}_{\mathbb{S}}$ is an embedding of force $\boldsymbol{\nu}$. In particular, we have $f \circ s \in \mathbb{U}$ for all $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ and $s \in \mathbb{U}^{>,\succ}$. In view of the axioms for hyperserial fields, we have the following converse implication.

Lemma 7.2.14. *Let $\mathfrak{N} \subseteq \mathfrak{M}$ be a subgroup and assume that*

$$\mathbb{L}_{<\boldsymbol{\lambda}} \circ (\mathbb{R}[[\mathfrak{N}]])^{>,\succ} \subseteq \mathbb{R}[[\mathfrak{N}]].$$

Then $\mathbb{R}[[\mathfrak{N}]]$, equipped with the restriction of $\circ$ to $\mathbb{L}_{<\boldsymbol{\lambda}} \times \mathbb{R}[[\mathfrak{N}]]^{>,\succ}$ is a subfield of $\mathbb{T}$ of force $\boldsymbol{\nu}$.

Again, there is a skeletal version of this characterization in the case of confluent subfields.

Proposition 7.2.15. *Assume that $(\mathbb{T}, \circ)$ is confluent. Let $\mathfrak{N} \subseteq \mathfrak{M}$ be a subgroup and assume that*

$$\begin{aligned} \mathfrak{d}_{\omega^\mu}(\mathfrak{N}^\succ) &\subseteq \mathfrak{N} \quad \text{for all } \boldsymbol{\mu} \leqslant \boldsymbol{\nu}, \text{ and} \\ \mathfrak{L}_{<\omega^\mu} \circ (\mathfrak{d}_{\omega^\mu}(\mathfrak{N}^\succ)) &\subseteq \mathfrak{N} \quad \text{for all } \boldsymbol{\mu} \leqslant \boldsymbol{\nu}, \text{ and} \\ \operatorname{supp} L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{N}^\succ)) &\subseteq \mathfrak{N} \quad \text{for all } \eta < \boldsymbol{\nu}. \end{aligned}$$

Then $\circ$ restricts to a composition law $\circ : \mathbb{L}_{<\boldsymbol{\lambda}} \times \mathbb{R}[[\mathfrak{N}]]^{>,\succ} \longrightarrow \mathbb{R}[[\mathfrak{N}]]$ for which $\mathbb{R}[[\mathfrak{N}]]$ is a confluent subfield of $\mathbb{T}$ of force $\boldsymbol{\nu}$.

Proof. We first prove that $\mathbb{R}[[\mathfrak{N}]]$ is a subfield of $\mathbb{T}$ of force $\boldsymbol{\nu}$. By Lemma 7.2.14, it is enough to prove that for all $s \in \mathbb{T}^{>,\succ}$ and $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$, we have $\operatorname{supp} f \circ s \subseteq \mathfrak{N}$.

Fix $s \in \mathbb{T}^{>,\succ}$ and $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$. From the proof of Proposition 4.4.7 and by Theorem 7.2.1, we see that the support of $f \circ s$ is contained in the class of finite products of monomials in the class

$$(\operatorname{supp} s) \cup \bigcup_{\mu < \boldsymbol{\nu}} \bigcup_{\gamma \geqslant \omega^\mu, \gamma < \boldsymbol{\lambda}} \mathfrak{L}_{<\omega^\mu} \circ \mathfrak{d}_{\omega^\mu}(\mathfrak{d}_{L_\gamma(a)}).$$

For $\mu, \eta < \boldsymbol{\nu}$ with $\eta \geqslant \mu$, we have $\mathfrak{d}_{L_{\omega^\eta}(a)} = \mathfrak{d}_{L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(a))}$. So for $\gamma < \boldsymbol{\lambda}$ with $\gamma \geqslant_o \omega^\mu$, we have

$$\begin{aligned} \operatorname{supp} f \circ s &\subseteq (\operatorname{supp} s) \cup \bigcup_{\mu < \boldsymbol{\nu}} \bigcup_{\gamma \geqslant_o \omega^\mu, \gamma < \boldsymbol{\lambda}} \mathfrak{L}_{<\omega^\mu} \circ \mathfrak{d}_{\omega^\mu}(\mathfrak{d}_{L_\gamma(a)}) \\ &\subseteq (\operatorname{supp} s) \cup \bigcup_{\mu < \boldsymbol{\nu}} \bigcup_{\gamma \geqslant_o \omega^\mu, \gamma < \boldsymbol{\lambda}} \mathfrak{L}_{<\omega^\mu} \circ \mathfrak{d}_{\omega^\mu}\left(\bigcup_{\omega^\eta \not\preccurlyeq \gamma} \operatorname{supp} L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(\mathfrak{N}^\succ)) \right) \\ &\subseteq (\operatorname{supp} s) \cup \bigcup_{\mu < \boldsymbol{\nu}} \mathfrak{L}_{<\omega^\mu} \circ \mathfrak{d}_{\omega^\mu}(\mathfrak{N}^\succ) \\ &\subseteq \mathfrak{N}. \end{aligned}$$

This proves the first part of the proposition. To see that $\mathbb{R}[[\mathfrak{N}]]$ is confluent, consider a $s \in \mathbb{T}^{>,\succ}$ and a $\mu \leqslant \boldsymbol{\nu}$. The field $(\mathbb{T}, \circ)$ is confluent so there are an $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and a $\gamma < \omega^\mu$ with $\ell_\gamma \circ s \asymp \ell_\gamma \circ \mathfrak{a}$. We thus have $\mathfrak{a} = \mathfrak{d}_{\omega^\mu}(s) \in \mathfrak{N}$. So $\mathfrak{a} \in \mathfrak{N}_{\omega^\mu}$, which proves that $\mathbb{R}[[\mathfrak{N}]]$ is confluent. □

7.3 Examples of hyperserial fields

We conclude Part II by giving examples of important hyperserial fields.

7.3.1 Finitely nested hyperseries

Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ and write $\boldsymbol{\lambda} := \omega^{\boldsymbol{\nu}}$. As a consequence of Corollary 4.2.18 and Theorem 7.2.10, the field $\mathbb{L}_{<\boldsymbol{\lambda}}$ of logarithmic hyperseries of strength $<\boldsymbol{\lambda}$, together with its composition law

$$\circ : \mathbb{L}_{<\boldsymbol{\lambda}} \times \mathbb{L}_{<\boldsymbol{\lambda}}^{>,\succ} \longrightarrow \mathbb{L}_{<\boldsymbol{\lambda}},$$

is a confluent hyperserial field of force $\boldsymbol{\nu}$. Likewise, its hyperexponential closure $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ is a confluent hyperserial field of force $(\boldsymbol{\nu}, \boldsymbol{\nu})$. We call $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ the field of *finitely nested hyperseries of strength* $<\boldsymbol{\lambda}$. We simply call $\tilde{\mathbb{L}}$ the field of *finitely nested hyperseries*. This terminology will be justified in later work, indeed we intend to show in the future that under the natural embedding $f \mapsto f \circ \omega$ of $\tilde{\mathbb{L}}$ into $\mathbf{No}$, the numbers in $\tilde{\mathbb{L}} \circ \omega \subseteq \mathbf{No}$ are exactly those which contain no infinite path (see Section 13.1.2).

The reader might think that we have so far given a very conservative description of the structures of $\mathbb{L}$ and $\tilde{\mathbb{L}}$. Indeed, isn't the monotonicity entailed by the axiom **HF4** valid for more logarithmic hyperseries than just the series $\ell_{\omega^\mu}^{\uparrow\gamma}$? Doesn't the derivation $': \mathbb{L} \longrightarrow \mathbb{L}$ extend into a well behaved derivation $': \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$? Doesn't the composition law $\circ: \mathbb{L} \times \tilde{\mathbb{L}}^{>,\succ} \longrightarrow \tilde{\mathbb{L}}$ extend to $\tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>,\succ}$? Is $(\tilde{\mathbb{L}}^{>,\succ}, \ell_0, \circ, <)$ an ordered group? Do all conjugation equations

$$y \circ f = g \circ y$$

for fixed $f, g \in \tilde{\mathbb{L}}^{>\ell_0}$ have solutions y in $\tilde{\mathbb{L}}^{>,\succ}$? All the answers are positive (see [10, Introduction]), but require some efforts to be obtained and are not provided in this thesis.

7.3.2 Nested series

One could wonder whether $\tilde{\mathbb{L}}$, as the total hyperexponential closure of all logarithmic hyperseries, is the largest possible confluent hyperserial field. One way to try extending $\tilde{\mathbb{L}}$ is to fill initial cuts within it. As in [63, Chapter 9], we define an *initial cut* in $(\tilde{\mathbb{L}}, <)$ to be an initial subclass $\mathbf{L}$ of $(\tilde{\mathbb{L}}, <)$ without supremum in $(\tilde{\mathbb{L}}, <)$. Consider such an initial cut $\mathbf{L}$ and write $\mathbf{R} := \tilde{\mathbb{L}} \setminus \mathbf{L}$. Filling $\mathbf{L}$ means to construct an extension $\Phi: \tilde{\mathbb{L}} \longrightarrow \mathbb{U}\{\mathbf{L} \mid \mathbf{R}\}$ for which there is a $y \in \mathbb{U}\{\mathbf{L} \mid \mathbf{R}\}$ with

$$\Phi(\mathbf{L}) < y < \Phi(\mathbf{R}).$$

If one allows $\mathbf{L}$ to have no cofinal subset or $\mathbf{R}$ to have no coinitial subset, then one cannot expect that any such initial cut $(\mathbf{L}, \mathbf{R})$ be filled. Indeed taking

$$\mathbf{L}_0 = \{f \in \tilde{\mathbb{L}} : \exists n \in \mathbb{N}, f < n\} \quad \text{and} \tag{7.3.1}$$

$$\mathbf{L}_1 = \tilde{\mathbb{L}}, \tag{7.3.2}$$

we can fill $(\mathbf{L}_0, \tilde{\mathbb{L}}^{>,\succ})$ and $(\mathbf{L}_1, \varnothing)$ by adjoining suggestively denoted elements

$$\ell_{\mathbf{On}} \qquad \text{and} \qquad \mathrm{e}_{\mathbf{On}}^{\ell_0},$$

defining the appropriate structures on lexicographic products

$$\mathfrak{L} \times (\mathfrak{L} \circ \ell_{\mathbf{On}}) \qquad \text{and} \qquad (\mathfrak{L} \circ \mathrm{e}_{\mathbf{On}}^{\ell_0}) \times \mathfrak{L},$$

with prevalence of the first projection, then taking the hyperexponential closure. Other uninteresting initial cuts are what van der Hoeven calls *serial cuts*, which cannot be filled without breaking the condition that supports be set-sized.

A more interesting case is that when $\mathbf{L}$ has a cofinal subset L and $\mathbf{R}$ has a coinitial subset R. By taking convex hulls, we see that (L, R) determines $(\mathbf{L}, \mathbf{R})$. Then as in [63, Chapter 9], we expect that each such cut (L, R) can be expressed using cut operations and series in $\tilde{\mathbb{L}}$. In fact, we expect that each such initial cut is what we will informally call a *nested cut*. Let us give an example. For $n \in \mathbb{N}$, write

$$f_n := \sqrt{\ell_0} \mathbin{+\!\!\!+} \mathrm{e}^{\sqrt{\ell_1} \mathbin{+\!\!\!+} \mathrm{e}^{\cdot^{\cdot^{\cdot \sqrt{\ell_n}}}}}, \quad \text{and}$$

$$g_n := \sqrt{\ell_0} \mathbin{+\!\!\!+} \mathrm{e}^{\sqrt{\ell_1} \mathbin{+\!\!\!+} \mathrm{e}^{\cdot^{\cdot^{\cdot 2 \sqrt{\ell_n}}}}}.$$

It is easy to see that

$$f_0 < f_1 < \cdots < f_n < \cdots < \cdots < g_n < \cdots < g_1 < g_0.$$

We will show in later work that there is no series $f \in \tilde{\mathbb{L}}$ with

$$L := \{f_n : n \in \mathbb{N}\} < f < \{g_n : n \in \mathbb{N}\} =: R.$$

We thus have a corresponding initial cut in $\tilde{\mathbb{L}}$, which is in fact an initial cut in $(\mathbb{L}_{<\omega})_{(<1)}$.

Schmeling showed [92, Section 2.5] that there are transseries fields, and in fact also confluent hyperserial fields $\mathbb{T}(L \mid R)$ of force $(1,1)$, which contain a series f with $L < f < R$. One particular example of such series f_{nest} can be represented as a transfinite nested expansion

$$f_{\text{nest}} \coloneqq \sqrt{\ell_0} + \mathrm{e}^{\sqrt{\ell_1} + \mathrm{e}^{\cdot^{\cdot^{\cdot \sqrt{\ell_n} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}}, \tag{7.3.3}$$

in the sense that

$$\begin{aligned} f_{\text{nest}} &\sim \sqrt{\ell_0} \quad \text{and} \quad f_{\text{nest}} - \sqrt{\ell_0} \quad \text{is a monomial,} \\ \log(f_{\text{nest}} - \sqrt{\ell_0}) &\sim \sqrt{\ell_1} \quad \text{and} \quad \log(f_{\text{nest}} - \sqrt{\ell_0}) - \sqrt{\ell_1} \quad \text{is a monomial,} \\ \log(\log(f_{\text{nest}} - \sqrt{\ell_0}) - \sqrt{\ell_1}) &\sim \sqrt{\ell_2} \quad \text{and} \quad \text{so on}\ldots \end{aligned} \tag{7.3.4}$$

As we will see in Part III, it is known that the field of surreal numbers has a natural structure of confluent hyperserial field of force $(1,1)$. With van der Hoeven, we showed [11, Section 8] that such nested series satisfying (7.3.4) "already" exist in **No**. In fact, they form a proper subclass of **No**, which implies that the expansion (7.3.3) is ambiguous. In Chapter 14, we will generalize this to the hyperserial setting. It will turn out that $\tilde{\mathbb{L}}$ is naturally included in **No** (see Chapter 12), and we expect to show in future work that one obtains **No** by iteratively closing $\tilde{\mathbb{L}}$ under nested series such as f_{nest}, and hyperexponentials.

7.3.3 Non confluent hyperserial fields

We finish with remarks on the failure of confluence in hyperserial fields. One could wonder whether the confluence axioms are necessary to define composition laws on hyperserial fields, or if they simply impose convenient restrictions. The following lemma shows that in any case, a non-confluent hyperserial field which *can* be extended into a confluent can be thus extended *via* hyperexponential extensions.

Lemma 7.3.1. *Let $\mathbb{U}$ be a confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\nu})$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. Any subskeleton of $\mathbb{U}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ is $\boldsymbol{\mu}$-confluent.*

Proof. Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a subskeleton of $\mathbb{U}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$. We prove that $\mathbb{T}$ is $\boldsymbol{\eta}$-confluent by induction on $\boldsymbol{\eta} \leqslant \boldsymbol{\mu}$. Let $\boldsymbol{\eta} \in (0, \boldsymbol{\mu}]$ and assume that $\mathbb{T}$ is ι-confluent for all $\iota < \boldsymbol{\eta}$. If $\boldsymbol{\eta} = \mathbf{On}$, then it follows that $\mathbb{T}$ is $\boldsymbol{\eta}$-confluent; so we assume that $\boldsymbol{\eta} < \mathbf{On}$. Assume that $\boldsymbol{\eta} = \iota + 1$ is a successor. Consider an $s \in \mathbb{T}^{>,\succ}$. Since $\mathbb{U}$ is $\boldsymbol{\eta}$-confluent, there is a $k \in \mathbb{N}$ with $(L_{\omega^\iota} \circ \mathfrak{d}_{\omega^\iota})^{\circ k}(s) \asymp L_{\omega^\iota k}(\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s))$. So $L_{\omega^\iota k}(\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s))$ is the dominant monomial of $(L_{\omega^\iota} \circ \mathfrak{d}_{\omega^\iota})^{\circ k}(s) \in \mathbb{T}$, whence $L_{\omega^\iota k}(\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s)) \in \mathbb{T}$. But then $\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s) \in \mathbb{T}$ since $\mathbb{T}$ has force $(\boldsymbol{\nu}, \boldsymbol{\mu})$. Assume now that $\boldsymbol{\eta}$ is a limit, and let $s \in \mathbb{T}^{>,\succ}$. Since $\mathbb{U}$ is $\boldsymbol{\eta}$-confluent, there is a $\rho < \boldsymbol{\eta}$ with $L_{\omega^\rho}(\mathfrak{d}_{\omega^\rho}(s)) \asymp L_{\omega^\rho}(\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s))$. But likewise $\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s) = E_{\omega^\rho}(L_{\omega^\rho}(\mathfrak{d}_{\omega^{\boldsymbol{\eta}}}(s))) \in \mathbb{T}$. So in any case $\mathbb{T}$ is $\boldsymbol{\eta}$-confluent. This concludes the proof. □

Corollary 7.3.2. *Let $\mathbb{U}$ be a confluent hyperserial field of force $(\boldsymbol{\nu}, \boldsymbol{\nu})$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. Any subfield of $\mathbb{U}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ is $\boldsymbol{\mu}$-confluent.*

Proof. This follows from Lemma 7.3.1 and Theorems 7.2.1 and 7.2.10. □

We now give an example of a somewhat pathological non-confluent hyperserial field which cannot be extended into a confluent hyperserial field. We will also show that this particular field is ill-suited to the purpose of the thesis. Our example uses the axiom of choice by way of Zorn's lemma. For the remainder of this subsection, we fix a linear ordering $<$ of $\mathbb{N}^{\mathbb{N}}$ which extends the universal comparison

$$f < g \Longleftrightarrow (f \neq g \wedge (\forall n \in \mathbb{N}, (f(n) \leqslant g(n)))).$$

Write $+$ for the pointwise sum on $\mathbb{N}^{\mathbb{N}}$. Note that $(\mathbb{N}^{\mathbb{N}}, +, <)$ is an ordered monoid. Consider the (commutative) Hessenberg sum $\oplus$ on ω^ω (see Section 8.1.2). We have a strictly increasing morphism $(\omega^\omega, \oplus, \in) \longrightarrow (\mathbb{N}^{\mathbb{N}}, +, <)$, whereby each ordinal $\gamma < \omega^\omega$ with Cantor normal form $\gamma = \sum_{n \in \mathbb{N}} \omega^n \gamma_n$ is sent to the function $n \mapsto \gamma_n$ (which is zero outside of finite subset of $\mathbb{N}$). We identify each $\gamma < \omega^\omega$ with $(\gamma_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and see elements γ of $\mathbb{N}^{\mathbb{N}}$ as "non-standard ordinals"

$$\gamma \coloneqq \gamma_0 + \omega \gamma_1 + \cdots + \omega^n \gamma_n + \cdots.$$

Finally, for $n\in\mathbb{N}$ and $\gamma\in\mathbb{N}^{\mathbb{N}}$, we write $\gamma_{\geqslant n}$ for the function $\mathbb{N}\longrightarrow\mathbb{N}$ which coincides with γ on $[n,+\infty)$ and is identically zero on $[0;n)$, i.e. $\gamma\fallingdotseq\omega^n\gamma_n+\omega^{n+1}\gamma_{n+1}+\cdots$.

We write $\mathfrak{L}_*$ for the multiplicatively denoted Hahn product group $\prod_{(\mathbb{N}^{\mathbb{N}},>)}\mathbb{R}$, for the reverse ordering $>$ on $\mathbb{N}^{\mathbb{N}}$. Each $\mathfrak{f}\in\mathfrak{L}_*$ is a well-based product

$$\mathfrak{f}=\prod_{\gamma\in\mathbb{N}^{\mathbb{N}}}\ell_\gamma^{\mathfrak{f}_\gamma}$$

where $\mathfrak{f}_\gamma:=\mathfrak{f}(\gamma)$ and $\ell_\gamma^{\mathfrak{f}_\gamma}:=\mathfrak{f}_\gamma\,\chi_{\{\gamma\}}\in\mathfrak{L}_*$ (see Section 1.1.3).

Set $\mathbb{L}_*:=\mathbb{R}[[\mathfrak{L}_*]]$. Let us define a hyperserial skeleton of force ω on $\mathbb{L}_*$. We set

$$\begin{aligned}(\mathfrak{L}_*)_1 &:= \mathfrak{L}_*^{\succ} \quad\text{and}\\ (\mathfrak{L}_*)_{\omega^{n+1}} &:= \{\ell_\gamma:\gamma=\gamma_{\geqslant n}\}\quad\text{for all } n\in\mathbb{N}.\end{aligned}$$

Given $\mathfrak{f}\in(\mathfrak{L}_*)_1$, we define

$$L_1(\mathfrak{f}):=\sum_{\gamma\in\mathbb{N}^{\mathbb{N}}}\mathfrak{f}_\gamma\,\ell_{\gamma+1}.$$

Given $\ell_\gamma\in(\mathfrak{L}_*)_{\omega^{n+1}}$, we define

$$L_{\omega^{n+1}}(\ell_\gamma):=\ell_{\gamma_{\geqslant n+1}}-\gamma_n.$$

We leave it to the reader to check that $(\mathbb{L}_*,(L_{\omega^n})_{n<\omega})$ is a hyperserial skeleton of force ω and that it is n-confluent for all $n<\omega$. By Theorem 4.3.1, we have compositions laws $\circ^n:\mathbb{L}_{<\omega^n}\times\mathbb{L}_*^{>,\succ}\longrightarrow\mathbb{L}_*$ for each $n<\omega$. We let $\mathbb{L}_*^{\cup}$ denote the proper (regular) subfield $\mathbb{L}_{<\omega^\omega}^{\cup}:=\bigcup_{n<\omega}\mathbb{L}_{<\omega^n}$ of $\mathbb{L}_{<\omega^\omega}$ and we denote

$$\circ:\mathbb{L}_{<\omega^\omega}^{\cup}\times\mathbb{L}_*^{>,\succ}\longrightarrow\mathbb{L}_*$$

the function whose graph is the reunion of the graphs of all composition laws $\circ^n, n\in\mathbb{N}$. For the rest of this subsection, we fix $\iota:=\mathrm{Id}_{\mathbb{N}}\fallingdotseq 1+\omega+\omega^2+\cdots\in\mathbb{N}^{\mathbb{N}}$. We have

$$\ell_{\iota_{\geqslant n}}=\ell_{\omega^n}\circ\ell_{\iota_{\geqslant n+1}}\tag{7.3.5}$$

for each $n\in\mathbb{N}$, so ℓ_ι can be construed as a transfinite post-composition

$$\ell_\iota\fallingdotseq\ell_1\circ\ell_\omega\circ\cdots\circ\ell_{\omega^n}\circ\cdots.$$

The inclusion $(\omega^\omega,\ni)\longrightarrow(\mathbb{N}^{\mathbb{N}},>)$ induces an embedding of ordered groups $\mathfrak{L}_{<\omega^\omega}\longrightarrow\mathfrak{L}_*$ which in turn induces an embedding $\mathbb{L}_{<\omega^\omega}\longrightarrow\mathbb{L}_*$ of force ω. This last embedding is not a bijection because the field $\mathbb{L}_*$ contains non-standard expressions such as ℓ_ι. This prevents $\mathbb{L}_*$ from being ω-confluent. To justify this, we show that we cannot apply Theorem 4.3.1 for $\nu=\omega$ to obtain a composition $\mathbb{L}_{<\omega^\omega}\times\mathbb{L}_*^{>,\succ}\longrightarrow\mathbb{L}_*$.

Proposition 7.3.3. *There is* no *extension of* $\circ$ *into a function* $\circ:\mathbb{L}_{<\omega^\omega}\times\mathbb{L}_*^{>,\succ}\longrightarrow\mathbb{L}_*$ *satisfying the conditions* $\mathbf{C1}_\omega$*,* $\mathbf{C2}_\omega$*, and* $\mathbf{C3}_\omega$ *as per Theorem 4.3.1.*

Proof. Assume for contradiction that such a composition exists. By $\mathbf{C1}_\omega$, the family

$$(\ell_{\omega^n}\circ\ell_\iota)_{0<n<\omega}$$

is well-based. Let $n\in(0,\omega)$. For $k<n-1$, we have $\ell_{\omega^n}\circ\ell_{\omega^k}-\ell_{\omega^n}\prec 1$, whence

$$\varepsilon_n:=\ell_{\omega^n}\circ(\ell_1\circ\ell_\omega\circ\cdots\circ\ell_{\omega^{n-2}})-\ell_{\omega^n}\prec 1.$$

By (7.3.5), $\mathbf{C1}_\omega$ and $\mathbf{C3}_\omega$, we have

$$\ell_{\omega^n}\circ\ell_\iota=(\ell_{\omega^n}+\varepsilon_n)\circ\ell_{\iota_{\geqslant n-1}}=(\ell_{\omega^n}\circ(\ell_{\omega^{n-1}}\circ\ell_{\iota_{\geqslant n}}))+(\varepsilon_n\circ\ell_{\iota_{\geqslant n-1}})=(\ell_{\omega^n}\circ\ell_{\iota_{\geqslant n}})-1+(\varepsilon_n\circ\ell_{\iota_{\geqslant n-1}})$$

where $\varepsilon_n\circ\ell_{\iota_{\geqslant n-1}}$ is infinitesimal. By $\mathbf{C2}_\omega$, we have $\ell_{\omega^n}\circ\ell_{\iota_{\geqslant n}}=\ell_{\iota_{\geqslant n}+\omega^n}$, which is an infinite monomial. We deduce that 1 lies in $\operatorname{supp}\ell_{\omega^n}\circ\ell_\iota$ for all $n\in\mathbb{N}^{>}$, which contradicts the fact that $(\ell_{\omega^n}\circ\ell_\iota)_{0<n<\omega}$ is well-based. □

The existence of ℓ_ι also prevents $\mathbb{L}_*$ from enjoying a well-behaved derivation as the following shows.

Proposition 7.3.4. *There is no strongly linear derivation $\partial: \mathbb{L}_* \longrightarrow \mathbb{L}_*$ which extends the derivation on $\mathbb{L}_{<\omega^\omega}$ and which satisfies the chain rule*

$$\partial(f \circ g) = \partial(g)\,(f' \circ g) \quad \text{for} \quad f \in \mathbb{L}^{\cup}_{<\omega^\omega},\ g \in \mathbb{L}^{>,\succ}_*,$$

and the relation

$$1 \prec f \prec g \Longrightarrow \partial(f) \prec \partial(g) \quad \text{for} \quad f, g \in \mathbb{L}_*. \tag{7.3.6}$$

Proof. Assume for contradiction that such a derivation exists. By the chain rule, the derivative of ℓ_ι is given by

$$\begin{aligned}
\partial(\ell_\iota) &= \ell_\iota^{-1}\,\partial(\ell_{\tilde{\gamma}_{\geqslant 1}}) \\
&= \ell_\iota^{-1}\left(\prod_{n<\omega} \ell_{n+\iota_{\geqslant 1}}^{-1}\right)\partial(\ell_{\iota_{\geqslant 2}}) \\
&= \ldots \\
&= \left(\prod_{k<n}\prod_{\theta<\omega^k} \ell_{\theta+\iota_{\geqslant k+1}}^{-1}\right)\partial(\ell_{\iota_{\geqslant n+1}})
\end{aligned}$$

For $n<\omega$, we have $\ell_{\iota_{\geqslant n+1}} \prec \ell_{\omega^{n+1}}$ whence

$$\partial(\ell_{\iota_{\geqslant n+1}}) \prec \partial(\ell_{\omega^{n+1}}) = \prod_{k<n}\prod_{\theta<\omega^k} \ell_{\theta+\omega^k}^{-1}$$

by asymptoticity. For $k<n$ and $\theta<\omega^k$, we have $\theta+\omega^k < \theta+\iota_{\geqslant k+1}$ so

$$\partial(\ell_{\iota_{\geqslant n+1}}) \prec \prod_{k<n}\prod_{\theta<\omega^k} \ell_{\theta+\iota_{\geqslant k+1}}^{-1}.$$

We deduce that the support of $\mathfrak{d}_{\partial(\ell_\iota)} \in \mathfrak{L}_*$ in $\mathbb{N}^{\mathbb{N}}$ contains the strictly decreasing sequence

$$\iota > \iota_{\geqslant 1} > \iota_{\geqslant 2} > \cdots,$$

a contradiction. □

The relation (7.3.6) is a weaker version of properties of certain ordered, valued, differential fields called H-fields (see [2, 3, 4]). It is known [33, Lemma 3.2] that $\mathbb{L}$ itself with its standard derivation satisfies (7.3.6).

What Proposition 7.3.4 above suggests to us is that hyperserial fields ought not to contain those infinite post-compositions of hyperlogarithms, if they are to be equipped with well-behaved derivations.

Part III

Numbers

Seeing the number

The class **No** of *surreal numbers* was discovered by Conway and studied in his well-known monograph *On Numbers and Games* [28]. Conway's original definition of (surreal) number is somewhat informal and goes as follows:

> "If L and R are any two sets of numbers, and no member of L is $\geqslant$ any member of R, then there is a number $\{L \mid R\}$. All numbers are constructed in this way."

In order to gain some insight on this mysterious definition and on the corresponding notion of magnitude that it proposes, it is useful to find more explicit representations of such numbers. In Part III, we will (rather briefly) consider three such representations: as cuts, as sign sequences, and as transseries. Our motivation however is not mainly to understand surreal numbers, but rather to find a representation that is most convenient in order to operate on them as if they were germs of functions in Hardy fields. This is the task of representing numbers as hyperseries that we will only tackle in Part IV.

Conway's paradise

The magic of surreal numbers lies in the fact that many traditional operations on integers and real numbers can be defined in a very simple way on surreal numbers. Yet, the class **No** turns out to admit a surprisingly rich algebraic structure under these operations. For instance, the sum of two surreal numbers $x=\{x_L \mid x_R\}$ and $y=\{y_L \mid y_R\}$ is defined recursively by

$$x+y \;=\; \{x_L+y, x+y_L \mid x_R+y, x+y_R\}. \tag{1}$$

Similar definitions exist for subtraction and multiplication, which we will recall in Section 8.2. Despite the fact that the basic arithmetic operations can be defined in such an "effortless" way, Conway showed that **No** actually forms a real-closed field that contains $\mathbb{R}$.

Since Conway's seminal work, further algebraic structure has been defined, often in natural but non-trivial ways, on **No**. This includes an exponential function and a non-trivial derivation. There is a "simplicity heuristic", according to which it is sometimes possible to define an operation on **No** by picking the simplest (i.e. $\sqsubset$-minimal) solution to a given problem whenever several solutions exist. Not only has this approach been successful in defining the algebraic structure on **No**, but in a number of examples, it also turns **No** into a model of tame first-order theories, such as the theories of divisible linearly ordered Abelian groups, of real-closed fields, of the real exponential field, and of H-closed fields (see [5]).

We do not yet have a good model theoretic framework that could guide us in completing the task at-hand of defining a structure of hyperserial field on **No**. Some partial results exist in the form of Padgett's PhD thesis [83] which proposes in particular a first-order theory of the real ordered field with Kneser's hyperexponential function [66]. Nevertheless, we will see in Part IV that the simplicity heuristic, combined with the method of hyperserial skeletons of Chapter 4, gives a solution to our problem. Furthermore Part IV will illustrate that surreal numbers are a prime example of proper extension of $\tilde{\mathbb{L}}$ as a hyperserial field. For the sequel of the thesis, no one shall expel us from this paradise Conway has created.

Sign sequences

One convenient way to rigorously introduce surreal numbers a is to regard them as "sign sequences" $a=(a[\beta])_{\beta<\alpha}\in\{-1,+1\}^{\alpha}$ indexed by the elements $\beta<\alpha$ of an ordinal number $\alpha=\operatorname{bd}(a)$, called the *birth day* of a: see Section 8.1 below for details. Every ordinal α itself is represented as $\alpha=(\alpha[\beta])_{\beta<\alpha}$ with $\alpha[\beta]=1$ for all $\beta<\alpha$. The number $1/2$ is represented by the sign sequence $+1,-1$ of length 2. The ordering $<$ on **No** corresponds to the lexicographical ordering on sign sequences, modulo zero padding when comparing two surreal numbers of different lengths. The sign sequence representation also induces the important notion of *simplicity*: given $a,b\in\mathbf{No}$, we say that a is *simpler* as b, and write $a\sqsubset b$, if the sign sequence of a is a proper truncation of the sign sequence of b. The simplicity relation is denoted by $<_s$ in some previous works [18, 71, 6].

The sign sequence representation was introduced and studied systematically in Gonshor's book [55]. We will rely on it in order to give a simple definition of surreal numbers, as well as to give examples of important numbers and classes of numbers.

Surreal substructures

In the course of the above project to construct an isomorphism between **No** and a suitable class of hyperseries, one frequently encounters subclasses **S** of **No** that are naturally parametrized by **No** itself. For instance, Conway's generalized ordinal exponentiation $\mathbf{No} \longrightarrow \mathbf{Mo}; z \mapsto \dot{\omega}^z$ is bijective, hence we have a natural parameterization of the class **Mo** of monomials by **No**. Similarly, nested expressions such as (2.2):

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$$

do not give rise to a single surreal number, but rather to a class **Ne** of surreal numbers that is naturally parametrized by **No** (see Theorem 14.2.4). Yet another example is the class $\mathbf{Mo}_\omega$ of *log-atomic* surreal numbers [18, Section 5.2]. More general subclasses than surreal substructures such as initial subtrees as studied by Ehrlich [42] and subtrees of **No** as studied Lurie [75] play a role in investigating initial algebraic substructures of **No** and studying fixed points of certain surreal-valued functions.

In these examples, the parametrizations turn out to be more than mere bijective maps: they actually preserve both the ordering $<$ and the simplicity relation $\sqsubset$. This leads to the definition of a *surreal substructure* of **No** as being an isomorphic copy of $(\mathbf{No}, <, \sqsubset)$ inside itself. Surreal substructures such as **Mo**, **Ne**, and $\mathbf{Mo}_\omega$ behave similarly as the surreal numbers themselves in many regards. In particular, it is possible to define monotone functions between surreal substructures by well-based induction with respect to the simplicity relation.

Exponentiation, derivation, and hyperseries

In his book [55], Gonshor shows how to extend the real exponential function to **No**. This function actually has the same first-order properties as the usual exponential function: the class **No** is an elementary extension of $\mathbb{R}$ as an ordered exponential field. Berarducci and Mantova later showed that the field **No**, equipped with the functional inverse log of exp, is a *transseries field*, i.e. a transserial field (see Definition 3.1.4) satisfying Schmeling's axiom **T4** (see [92, Definition 2.2.1]). As a transseries field, the class **No** vastly extends the logarithmic-exponential transseries, and is in fact the largest transseries field [46, Proposition 8.4]. See also [46, Proposition 7.2] for a different proof of the validity of **T4** in **No**.

An important question concerns the possibility to define a natural transserial derivation ∂ on **No** with $\partial(\omega) = 1$. Such a derivation was first constructed by Berarducci and Mantova [18], using methods from [92]. It was shown in [6] that Berarducci and Mantova's derivation ∂ also turns surreal numbers into an elementary extension of the ordered valued differential field $\mathbb{T}_{\mathrm{LE}}$. We recall that the derivation ∂ cannot be compatible with a composition law on **No** [19, Theorem 8.4]. A tentative explanation of this fact is that ∂ is constructed by relying on a representation of surreal numbers as transseries, and that this representation fails to accurately describe those surreal numbers that will be best represented using the hyperexponential and hyperlogarithmic functions we will define in Part IV.

Chapter 8
Surreal numbers

In this chapter, we introduce surreal numbers as presented by Harry Gonshor in his book [55], and give their elementary properties as an ordered valued field with a simplicity relation.

8.1 Numbers as sign sequences

The sign sequence representation is most convenient for the rigorous development of the basic theory of surreal numbers. It was introduced by Gonshor [55, page 3] and we will actually use it to formally define surreal numbers as follows:

Definition 8.1.1. *A surreal number is a map* $a : \mathrm{bd}(a) \longrightarrow \{-1, 1\}; \alpha \mapsto a[\alpha]$*, where* $\mathrm{bd}(a) \in \mathbf{On}$ *is an ordinal number. We call* $\mathrm{bd}(a)$ *the* **birth day** *of* a *and the map* $a : \mathrm{bd}(a) \longrightarrow \{-1, 1\}$ *the* **sign sequence** *of* a*. We write* $\mathbf{No}$ *for the class of surreal numbers.*

It follows from this definition that $\mathbf{No}$ is a proper class. Given a surreal number $a \in \mathbf{No}$, it is convenient to extend its sign sequence with zeros to a map $\mathbf{On} \longrightarrow \{-1, 0, 1\}$ and still denote this extension by a. In other words, we take $a[\alpha] = 0$ for all $\alpha \geqslant \mathrm{bd}(a)$. Given $a \in \mathbf{No}$ and $\alpha \in \mathbf{On}$, we also introduce its *restriction* $b = a \upharpoonright \alpha \in \mathbf{No}$ to α as being the zero padded restriction of the map a to α: we set $b[\beta] = a[\beta]$ for $\beta < \alpha$ and $b[\beta] = 0$ for $\beta \geqslant \alpha$.

The first main relation on $\mathbf{No}$ is its *ordering* $\leqslant$. We define it to be the restriction of the lexicographical ordering on the set of all maps from $\mathbf{On}$ to $\{-1, 0, 1\}$. More precisely, given distinct elements $a, b \in \mathbf{No}$, there exists a smallest ordinal α with $a[\alpha] \neq b[\alpha]$. Then we define $a < b$ if and only if $a[\alpha] < b[\alpha]$.

The second main relation on $\mathbf{No}$ is the simplicity relation $\sqsubset$: given numbers $x, y \in \mathbf{No}$, we say that a is *strictly simpler* than b, and write $a \sqsubset b$ if $\mathrm{bd}(a) < \mathrm{bd}(b)$ and $a = b \upharpoonright \mathrm{bd}(a)$. We write $\sqsubseteq$ for the non-strict ordering corresponding to $\sqsubset$, and we say that a is *simpler* than b if $a \sqsubseteq b$. We write $a_{\sqsubset} = \{b \in \mathbf{No} : b \sqsubset a\}$ for the set of surreal numbers that are strictly simpler than a. The partially ordered class $(\mathbf{No}, \sqsubset)$ is well-founded, and $(a_{\sqsubset}, \sqsubset)$ is well-ordered with order type $\mathrm{ord}(a_{\sqsubset}, \sqsubset) = \mathrm{bd}(a)$.

Every linearly ordered—and thus well-ordered—subset X of $(\mathbf{No}, \sqsubset)$ has a *supremum* $s = \sup_{\prec} X$ in $(\mathbf{No}, \sqsubset)$: for any $a \in X$ and $\alpha < \mathrm{bd}(a)$, one has $s[\alpha] = a[\alpha]$; for any $\alpha \in \mathbf{On}$ with $\alpha \geqslant \mathrm{bd}(a)$ all $a \in X$, one has $s[\alpha] = 0$. We will only consider suprema in $(\mathbf{No}, \sqsubset)$ and never in $(\mathbf{No}, <)$. Numbers a that are equal to $\sup_{\sqsubseteq} a_{\sqsubset}$ are called *limit numbers*; other numbers are called *successor numbers*. Limit numbers are exactly the numbers whose birth day is a limit ordinal.

8.1.1 The fundamental property

We now introduce the fundamental property of the structure $(\mathbf{No}, <, \sqsubset)$:

Fundamental property. [55, Theorem 2.1] *Let* L, R *be* *<u>sets</u>* *of surreal numbers with* $L < R$*. Then there is a unique* $\sqsubset$*-minimal number* $\{L \mid R\} \in \mathbf{No}$ *with*

$$L < \{L \mid R\} < R.$$

We call $\{\}$ the *Conway bracket*. Note that $\{L \mid R\}$ is *the* simplest such number in the strong sense that for all $a \in \mathbf{No}$ with $L < a < R$, we have $\{L \mid R\} \sqsubseteq a$. The converse implication $\forall a \in \mathbf{No}$, $\{L \mid R\} \sqsubseteq a \Longrightarrow L < a < R$ may fail: see Remark 9.2.9 below.

Now consider two more sets L', R' of surreal numbers with $L' < R'$. If L has no strict upper bound in L' and R has no strict lower bound in R', then we say that (L, R) is *cofinal* with respect to (L', R'). We say that (L, R) and (L', R') are *mutually cofinal* if they are cofinal with respect to one another, in which case it follows that $\{L \mid R\} = \{L' \mid R'\}$. These definitions naturally extend to pairs $(\mathbf{L}, \mathbf{R})$ of classes with $\mathbf{L} < \mathbf{R}$. Note however that $\{\mathbf{L} \mid \mathbf{R}\}$ is not necessarily defined for such classes. Indeed, there may be no number a with $\mathbf{L} < a < \mathbf{R}$ (e.g. for $\mathbf{L} = \mathbf{No}$ and $\mathbf{R} = \varnothing$).

We call a pair (L, R) of sets with $L < R$ a *cut representation* of $\{L \mid R\}$. Such representations are not unique; in particular, we may replace (L, R) by any mutually cofinal pair (L', R'). For every surreal number a, we denote

$$\begin{aligned} a_L &= \{b \in \mathbf{No} : b < a, b \sqsubset a\} \\ a_R &= \{b \in \mathbf{No} : b > a, b \sqsubset a\}, \end{aligned}$$

which are *sets* of surreal numbers. We call a_L and a_R the sets of *left* and *right options* for a. By [55, Theorem 2.8], one has $a = \{a_L \mid a_R\}$ and the pair (a_L, a_R) is called the *canonical cut representation* of x.

This identity $a = \{a_L \mid a_R\}$ is the fundamental intuition behind Conway's definition of surreal numbers precisely as the simplest numbers lying in the "cut" defined by sets $L < R$ of simpler and previously defined surreal numbers. Of course, this is a highly recursive representation that implicitly relies on transfinite induction.

Conway's cut representation is attractive because it allows for the recursive definition of functions using by well-founded induction on $(\mathbf{No}, \sqsubset)$ or its powers. For instance, there is a unique bivariate function f such that for all a, $b \in \mathbf{No}$, we have

$$f(a, b) = \{f(a_L, b), f(a, b_L) \mid f(a_R, b), f(a, b_R)\}. \tag{8.1.1}$$

Here we understand that $f(a_L, b), f(a, b_L)$ denotes the set $\{f(a', b) : a' \in a_L\} \cup \{f(a, b') : b' \in b_L\}$ and similarly for $f(a_R, b), f(a, b_R)$. This recursive definition is justified by the fact that the elements of the sets $a_L \times \{b\}, \{a\} \times b_L$, $a_R \times \{b\}$, and $\{a\} \times b_R$ are all strictly smaller than (a, b) for the product order on $(\mathbf{No}, \sqsubset) \times (\mathbf{No}, \sqsubset)$. This precise equation is actually the one that Conway used to define the addition $+ = f$ on $\mathbf{No}$. We will recall similar definitions of a few other arithmetic operations in Section 8.2 below.

8.1.2 Ordinal numbers as surreal numbers

For ordinals α, β, we will denote their ordinal sum, product, and exponentiation by $\alpha \dotplus \beta$, $\alpha \dot{\times} \beta$ and $\dot{\alpha}^{\beta}$. Their Hessenberg sum and product coincide with their sum and product when seen as surreal numbers [55, Theorems 4.5 and 4.6]; accordingly, we denote them by $\alpha + \beta$ and $\alpha\,\beta$. Note that $\alpha + n = \alpha \dotplus n$ for all ordinals α and $n < \omega$. We assume that the reader is familiar with elementary computations in ordinal arithmetic. In this section, we define operations on surreal numbers which extend ordinal arithmetic.

For numbers a, b, we let $a \dotplus b$ denote the number, called the *concatenation sum* of a and b, whose sign sequence is the concatenation of that of b at the end of that of a. So $a \dotplus b$ is the number of birth day $\operatorname{bd}(a \dotplus b) = \operatorname{bd}(a) \dotplus \operatorname{bd}(b)$, which satisfies

$$\begin{aligned} (a \dotplus b)[\alpha] &= a[\alpha] && (\alpha < \operatorname{bd}(a)) \\ (a \dotplus b)[\operatorname{bd}(a) \dotplus \beta] &= b[\beta] && (\beta < \operatorname{bd}(b)) \end{aligned}$$

It is easy to check that this extends the definition of ordinal sums. Moreover, the concatenation sum is associative and satisfies $\sup_{\sqsubseteq} (a \dotplus b_{\sqsubset}) = a \dotplus b$ whenever $x \in \mathbf{No}$ and $y \in \mathbf{No}$ is a limit number.

We let $a \dot{\times} b$ denote the number of length $\operatorname{bd}(b) \dot{\times} \operatorname{bd}(b)$, called the *concatenation product* of a and b, whose sign sequence is defined by

$$(a \dot{\times} b)[\operatorname{bd}(a) \dot{\times} \alpha \dotplus \beta] = b[\alpha]\, a[\beta] \qquad (\alpha < \operatorname{bd}(y), \beta < \operatorname{bd}(x)).$$

Here we consider $b[\alpha]\, a[\beta]$ as a product in $\{-1, +1\}$. Informally speaking, given $a \in \mathbf{No}$ and $\alpha \in \mathbf{On}$, the number $a \dot{\times} \alpha$ is the α-fold right-concatenation of a with itself, whereas $\alpha \dot{\times} a$ is the number obtained from a by replacing each sign α times by itself. We note that $\dot{\times}$ extends Cantor's ordinal product.

The operations $\dotplus$ and $\dot{\times}$ will be useful in what follows for the construction of simple yet interesting examples of surreal substructures. The remainder of this section is devoted to the collection of basic properties of these operations. The proofs can be found in [11, Section 3.2]. We refer to [28, First Part] for a different extension of the ordinal product to the class of games (which properly contains **No**).

Lemma 8.1.2. [11, Lemma 3.1] *For $a,b,c\in\mathbf{No}$, we have*

a) $a\dot{\times}(b\dot{\times}c)=(a\dot{\times}b)\dot{\times}c$.

b) $a\dot{\times}1=a$ *and* $a\dot{\times}(-1)=-a$.

c) $a\dot{\times}(b\dotplus c)=(a\dot{\times}b)\dotplus(a\dot{\times}c)$.

d) $a\dot{\times}b=\sup_{\sqsubseteq}(a\dot{\times}b_{\sqsubset})$ *if b is limit.*

Remark 8.1.3. The previous lemma can be regarded as an alternative way to define the concatenation product. Yet another way is through the equation

$$\forall a>0,\forall b,\ a\dot{\times}b \ = \ \{a\dot{\times}b_L\dotplus a_L, a\dot{\times}b_R\dotplus(-a_R)\mid a\dot{\times}b_L\dotplus a_R, a\dot{\times}b_R\dotplus(-a_L)\}. \tag{8.1.2}$$

Likewise, the concatenation sum has the following equation [44, Proposition 2]:

$$\forall a>0,\forall b,\ a\dotplus b \ = \ \{a_L, a\dotplus b_L\mid a\dotplus b_R, a_R\}. \tag{8.1.3}$$

Note that these two equations are *not* uniform in the sense of Definition 9.2.16 below.

Proposition 8.1.4. [11, Proposition 3.3] *Let $a,b,c\in\mathbf{No}$.*

a) *If $a\neq 0$, then $b\sqsubseteq c$ if and only if $a\dot{\times}b\sqsubseteq a\dot{\times}c$.*

b) *If $0<a$, then $b<c$ if and only if $a\dot{\times}b<a\dot{\times}c$.*

8.2 Surreal arithmetic

We describe the arithmetic operations on surreal numbers as defined by Conway.

8.2.1 Surreal addition

As we mentioned above, the definition of the sum of two numbers $a,b\in\mathbf{No}$ is by induction on **No** by

$$a+b=\{a_L+b, a+b_L\mid a+b_R, a_R+b\}. \tag{8.2.1}$$

Recall that the ordinal 0 is identified in **No** with empty sign sequence, so $0_L=0_R=\varnothing$. Assume for a moment that the equation (8.2.1) is justified. Let us show as an exercise that $a+0=a$ for all numbers $a\in\mathbf{No}$, by induction on $(\mathbf{No},\sqsubset)$. So let $a\in\mathbf{No}$ such that $b+0=b$ for all $b\sqsubset a$. Since 0_L and 0_R are empty, the equation (8.2.1) for $b=0$ simplifies as

$$a+0=\{a_L+0,\ \varnothing\mid\varnothing, a_R+0\}=\{a_L+0\mid a_R+0\}.$$

We have $a_L+0=a_L$ and $a_R+0=a_R$ by the induction hypothesis, so

$$a+0=\{a_L\mid a_R\}=a$$

as claimed. We deduce the result in general by induction. Symmetric arguments yield $0+a=a$ for all $a\in\mathbf{No}$.

Let us now show as a second exercise that $1+1=2$. Our proof will be quicker than Russell's. Note that $1_L=\{0\}$, that $2_R=\{0,1\}$ and that $1_R=2_R=\varnothing$. We thus have

$$\begin{aligned} 1+1 &= \{1+1_L, 1_L+1\mid\varnothing\} \\ &= \{1+0, 0+1\mid\varnothing\} \\ &= \{1\mid\varnothing\} \\ &= \{0,1\mid\varnothing\} \\ &= 2. \end{aligned}$$

The reader can see that easy computations in $(\mathbf{No},+)$ or, in general, involving Conway brackets, can be more involved than they seem. This is why the theory of surreal numbers requires certain results in order to simplifying those computations. One important related tool is the notion of uniform equation, that we briefly mention here before giving more details in Section 9.2.4 below. By [55, Theorem 3.2], the equation (8.2.1) for the sum of two numbers is uniform in the sense that, given cut representations (L_a, R_a) and (L_b, R_b) of a, b in $\mathbf{No}$, we have

$$a+b=\{L_a+b, a+L_b \mid a+R_b, R_a+b\}. \tag{8.2.2}$$

The first-order properties of $(\mathbf{No},+)$ are summed up by the following result.

Proposition 8.2.1. [28, Theorems 5, 6 and 12] *The structure* $(\mathbf{No},+,0,<)$ *is a divisible, Abelian, linearly ordered group.*

Additive inverses $-a$ of numbers $a\in\mathbf{No}$ can be inductively computed using the identity

$$-a=\{-a_R \mid -a_L\}. \tag{8.2.3}$$

8.2.2 Surreal multiplication

The definition of surreal multiplication is a bit more involved than that of addition. It is based on the following intuition. Consider numbers $a=\{a_L \mid a_R\}$ and $b=\{b_L \mid b_R\}$. Then given $a'\in a_L$ and $b'\in b_L$, we have $a-a', b-b'>0$, so a sensible definition of the product should give that $(a-a')\,(b-b')>0$, which can be rewritten as

$$a\,b>a\,b'+a'\,b-a'\,b'.$$

Given $a''\in a_R$, we should have $(a''-a)\,(b-b')>0$, whence

$$a''\,b+a\,b'-a''\,b'>a\,b.$$

Considering the two other cases lead Conway to the following inductive definition

$$a\,b=\{a\,b'+a'\,b-a'\,b', a\,b''+a''\,b-a''\,b'' \mid a\,b''+a'\,b-a'\,b'', a''\,b+a\,b'-a''\,b'\},$$

where a', b', a'' and b'' respectively range in a_L, a_R, b_L and b_R.

Again this definition is justified, and the cut representations (a_L, a_R) and (b_L, b_R) can be replaced by arbitrary representations of a and b without changing the result. It can be shown that 1 is indeed a neutral element for $\times$. In fact, we have:

Proposition 8.2.2. [28, Theorem 26] *The structure* $(\mathbf{No},+,\times,0,1,<)$ *is a real-closed ordered field.*

Since the theory of real-closed fields is o-minimal, it follows from general model-theoretic arguments and from the fundamental property that $(\mathbf{No},+,\times,0,1)$ is κ-saturated for all infinite cardinals κ. Hence every real-closed field embeds into $(\mathbf{No},+,\times,0,1,<)$ as an ordered field, over any common set-sized subfield. See [41] for more details.

Conway showed [28, Theorem 12] that the set $\mathbb{D}=\left\{\frac{k}{2^n}: k,n\in\mathbb{N}\right\}$ of dyadic numbers in $\mathbf{No}$ is that of surreal numbers with finite birth day. In particular, for each $n\in\mathbb{N}$, the number

$$2^{-(n+1)}=\{0 \mid 2^{-n}\}$$

has birth day $n+2$, whereas the number $3^{-1}=\left\{0,\frac{1}{4},\frac{5}{16},\ldots \mid \ldots,\frac{11}{32},\frac{3}{8},\frac{1}{2}\right\}$ has birth day ω.

There is a specific embedding of $(\mathbb{R},+,\times,<)$ into $\mathbf{No}$ obtained by sending $r\in\mathbb{R}$ to the number $\{d\in\mathbb{D}: d<r \mid d\in\mathbb{D}: d>r\}$. The image of this embedding is the only subfield of $\mathbf{No}$ which is isomorphic to $\mathbb{R}$ and which is downward closed for $\sqsubset$ in $\mathbf{No}$ [42, Theorem 8]. In the sequel, we identify $\mathbb{R}$ with this subfield.

8.3 Valuation theory, and numbers as series

In this section, we sill see how surreal numbers can be represented as well-based series in a canonical way.

8.3.1 The natural valuation on an ordered field

Recall that any ordered field $(\mathbf{F},+,\times,0,1,<)$ can be equipped with a (possibly trivial) valuation, called the natural valuation, whose valuation ring is the subring

$$\mathcal{O}:=\mathbf{F}^{\preccurlyeq}=\{a\in\mathbf{F}:\exists n\in\mathbb{N},-n<a<n\}$$

of finite elements in $\mathbf{F}$, whose unique maximal ideal $\circ=\mathcal{O}\setminus\mathcal{O}^{\times}$ coincides with the class

$$\mathbf{F}^{\prec}=\left\{a\in\mathbf{F}:\forall n\in\mathbb{N}^{>},|a|<\frac{1}{n}\right\}$$

of infinitesimal elements in $\mathbf{F}$. The corresponding valuation $v a$ of $a\in\mathbf{F}^{\times}$ is its *Archimedean class*

$$v a:=a\,\mathcal{O}^{\times}=\left\{b\in\mathbf{F}:\exists n\in\mathbb{N}^{>},\left(\frac{1}{n}|a|<|b|<n\,|a|\right)\right\}.$$

If $\mathbf{F}$ is a set, then this is nothing but the quotient map $v:\mathbf{F}^{\times}\longrightarrow\mathbf{F}^{\times}/\mathcal{O}^{\times}$ corresponding to the quotient of $(\mathbf{F}^{\times},\times)$ by its subgroup $(\mathcal{O}^{\times},\times)$. The quotient group $v\,\mathbf{F}^{\times}=\mathbf{F}^{\times}/\mathcal{O}^{\times}$ is ordered by setting $a\,\mathcal{O}^{\times}<b\,\mathcal{O}^{\times}$ if and only if $a\,b\in\circ$, yielding the ordered value group of the valued field $(\mathbf{F},\mathcal{O})$.

The quotient field $\mathcal{O}/\circ$, called the residue field of $(\mathbf{F},\mathcal{O})$, is always Archimedean. Furthermore, if $(\mathbb{R},+,\times)$ embeds into $\mathcal{O}$, then $\mathcal{O}/\circ$ is necessarily isomorphic to $\mathbb{R}$. See [4, Section 3.5] for more details.

8.3.2 Valuation theory of surreal numbers

Since $(\mathbf{No},+,\times,0,1)$ is real-closed, Kaplansky's general theory of maximal valued fields, in this particular case already implies that $\mathbf{No}$ embeds, as a valued ordered field, into a field $\mathbb{R}[[\mathfrak{M}]]$ of well-based series. The value group $\mathfrak{M}=v\,\mathbf{No}^{\neq}$ can be identified with a subgroup of $(\mathbf{No}^{>},\times,1)$. In general, such so-called Kaplansky embeddings are defined using choice principles, both in order to find a copy of the value group $v\,\mathbf{No}^{\neq}$ inside $(\mathbf{No}^{>},\times,1)$ and to ascribe surreal numbers $\hat{f}\in\mathbf{No}$ to certain well-based series $f=\sum_{\mathfrak{m}\in\mathfrak{M}}f_{\mathfrak{m}}\,\mathfrak{m}$ in $\mathbb{R}[[\mathfrak{M}]]$. In the case of surreal number, the fundamental property allows one to circumvent choice and to define a canonical isomorphism $\mathbf{No}\simeq\mathbb{R}[[v\,\mathbf{No}^{\neq}]]$, as we next explain.

8.3.3 Monomials and the ω-map

It was noticed by Conway that for each $a\in\mathbf{No}^{\neq}$, the positive part $v a\cap\mathbf{No}^{>}$ of the Archimedean class of a, which can be seen to be the class

$$\{b\in\mathbf{No}^{>}:\exists r\in\mathbb{R}^{>},r^{-1}a<b<r\,a\},$$

has a simplest element denoted $\mathfrak{d}_a$, and that the class $\mathbf{Mo}:=\{\mathfrak{d}_a:a\in\mathbf{No}^{>}\}$ is a subgroup of $(\mathbf{No}^{>},\times,1)$. Furthermore, Gonshor showed [55, Theorems 5.2 and 5.3] that there is a unique isomorphism $(\mathbf{No},<,\sqsubset)\longrightarrow(\mathbf{Mo},<,\sqsubset)$ for the induced orderings on $\mathbf{Mo}$. In Chapter 9, we will systematically study subclasses of $\mathbf{No}$ that satisfy the property above, and refer to them as *surreal substructures*. We will see in Chapter 10 that this property is an instance of a more general phenomenon pertaining to convex partitions on surreal substructures.

The unique isomorphism $(\mathbf{No},<,\sqsubset)\longrightarrow(\mathbf{Mo},<,\sqsubset)$ is denoted $z\mapsto\dot{\omega}^z$ and called the ω-map. The exponent notation is used because of the following additional property of the ω-map:

Proposition 8.3.1. [28, Theorem 20] *The ω-map is an isomorphism of ordered groups. In other words, we have*

$$\dot{\omega}^{y+z}=\dot{\omega}^y\,\dot{\omega}^z\qquad\textit{and}\qquad y<z\Longleftrightarrow\dot{\omega}^y<\dot{\omega}^z$$

for all $y,z\in\mathbf{No}$.

Thus $\mathbf{Mo}$ can be understood as a multiplicative copy $\dot{\omega}^{\mathbf{No}}$ of the ordered group $(\mathbf{No},+,<)$ within $(\mathbf{No}^{>},\times,1)$, that contains exactly one element by Archimedean class. This means that $\mathbf{Mo}$ is a candidate for the monomial group $\mathfrak{M}$ in the identification $\mathbf{No}\simeq\mathbb{R}[[\mathfrak{M}]]$.

8.3.4 Numbers as well-based series

Let us next show how to define an isomorphism $\mathbb{R}[[\mathbf{Mo}]] \longrightarrow \mathbf{No}; f \mapsto \hat{f}$. We have the following inductive definition of $\hat{f}$ for each $f = \sum_{\mathfrak{m} \in \mathbf{Mo}} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathbf{Mo}]]$, by induction on the length of the series. Given $\mathfrak{n} \in \operatorname{supp} f$, the length of $f_{\succ \mathfrak{n}}$ is smaller than that of f, so we can assume that each such $\widehat{f_{\succ \mathfrak{n}}}$ is defined, and set

$$\hat{f} = \{ \widehat{f_{\succ \mathfrak{n}}} + r\,\mathfrak{n} : \mathfrak{n} \in \operatorname{supp} f \wedge r \in \mathbb{R}^{<f_{\mathfrak{n}}} \mid \widehat{f_{\succ \mathfrak{n}}} + r\,\mathfrak{n} : \mathfrak{n} \in \operatorname{supp} f \wedge r \in \mathbb{R}^{>f_{\mathfrak{n}}} \}.$$

Indeed the number $\hat{f}$ should lie between the left and right hand set. It turns out that this method is sound, as the following illustrates.

Theorem 8.3.2. [28, Theorem 21] *The function* $\mathbb{R}[[\mathbf{Mo}]] \longrightarrow \mathbf{No}; f \mapsto \hat{f}$ *is an isomorphism of ordered valued fields which sends each* $r \in \mathbb{R}$ *to the corresponding surreal real number* $r \in \mathbf{No}$.

See [55, Section 5.C] for a more detailed proof. In the sequel of the thesis, we will no longer distinguish between surreal numbers and well-based series in $\mathbb{R}[[\mathbf{Mo}]]$. In particular, each surreal number $a = \hat{f} \in \mathbf{No}$ has a support $\operatorname{supp} a := \operatorname{supp} f \subseteq \mathbf{Mo}$ and can be seen as a function

$$\operatorname{supp} a \longrightarrow \mathbb{R}; \mathfrak{m} \mapsto a_{\mathfrak{m}} := f_{\mathfrak{m}}$$

with well-based support. Note that under this identification, for all $a \in \mathbf{No}$, the family $(a_{\mathfrak{m}}\,\mathfrak{m})_{\mathfrak{m} \in \mathbf{Mo}}$ is well-based, with sum $\sum_{\mathfrak{m} \in \mathbf{Mo}} a_{\mathfrak{m}}\,\mathfrak{m} = a$. In this vein, we will also consider well-based families of surreal numbers and sums thereof, as well as strongly linear functions on surreal numbers.

8.3.5 Iterated expansions and fixed points

The representation of numbers as well-based series is the first step in representing surreal numbers as quantities amenable to various operations. Indeed, each number $a \neq 0$ is a well-based sum

$$a = \sum_{\mathfrak{m} \in \mathbf{Mo}} a_{\mathfrak{m}}\,\mathfrak{m} = \sum_{z \in \mathbf{No}} a_{\dot{\omega}^z}\,\dot{\omega}^z.$$

Expanding each exponent z with $\dot{\omega}^z \in \operatorname{supp} a$ itself as a sum, we obtain a representation

$$a = \sum_{z \in \mathbf{No}} a_{\dot{\omega}^z}\,\dot{\omega}^{\sum_{u \in \mathbf{No}} z_{\dot{\omega}^u} \dot{\omega}^u},$$

and this process can be repeated *ad infinitum*, as long as the exponents that appear are non-zero. One might hope that this should determine a as a nested expansion in ω-based exponentiations, with real coefficients as sole parameters.

This is far from the truth. Indeed it is known [55, Chapter 9] that there are numbers $\varepsilon \in \mathbf{No}$ for which $\dot{\omega}^{\varepsilon} = \varepsilon$. Moreover, such numbers form a proper class, containing the subclass of **On** of ε-numbers, that is isomorphic to $(\mathbf{No}, <, \sqsubset)$ for the induced orderings. In fact, as discussed in [28, Chapter 9], even higher order fixed points exist. What's more, as Lemire showed [73, 74], the problem is much more general, as other type of transfinitely nested expansions such as

$$a = 1 + \dot{\omega}^{-1/2 + \dot{\omega}^{-1/2 + \dot{\omega}^{\cdot^{\cdot^{\cdot}}}}}$$

are plenty in **No**. See [11, Section 5] for a more detailed discussion and expansions of those results about fixed points.

Chapter 9
Surreal substructures

We next introduce surreal substructures as tools to study surreal numbers in relation to transseries and hyperseries in Part IV. We hope to convey the sense that surreal substructures are at the same time very general and very rigid subclasses of **No** and that several problems regarding the enriched structure of **No** (highlighted in particular in the work of Gonshor [55], Kuhlmann–Matusinski [71], Berarducci–Mantova [18], and Aschenbrenner–van den Dries–van der Hoeven [6]) crucially involve surreal substructures.

9.1 Surreal substructures

9.1.1 Surreal substructures and their parametrizations

Let $\mathbf{X}$ be a subclass of **No** and let $\mathcal{R} = (<_i)_{i \in I}$ be a family of orderings on **No**. Then we say that a function $f: \mathbf{X} \longrightarrow \mathbf{No}$ is $\mathcal{R}$-*increasing* if f is increasing for each $<_i$ with $i \in I$. We say that it is *strictly* $\mathcal{R}$-*increasing* if it is strictly increasing for each $<_i$. If we have $x <_i y \Longleftrightarrow f(x) <_i f(y)$ for all $x, y \in \mathbf{X}$ and $i \in I$, then we call f an $\mathcal{R}$-*embedding* of $(\mathbf{X}, (<_i)_{i \in I})$ into $(\mathbf{No}, (<_i)_{i \in I})$. We simply say that f is an *embedding* if f is a $(<, \sqsubset)$-embedding.

Definition 9.1.1. *A* **surreal substructure** *is the image of an embedding of* **No** *into itself.*

Example 9.1.2. Given $a \in \mathbf{No}$, the function $x \mapsto a \dotplus x$ is an embedding of $(\mathbf{No}, <, \sqsubset)$ into itself. If $a > 0$, then so is the function $x \mapsto a \dot{\times} x$, by Proposition 8.1.4. Consequently:

- For $a \in \mathbf{No}$, the function $x \mapsto a \dotplus x$ gives rise to the surreal substructure $a \dotplus \mathbf{No}$ of numbers whose sign sequences begin with the sign sequence of a.
- For $0 < a \in \mathbf{No}$, the function $x \mapsto a \dot{\times} x$ induces the surreal substructure $a \dot{\times} \mathbf{No}$ of numbers whose sign sequences are (possibly empty or transfinite) concatenations of the sign sequences of a and $-a$.

Example 9.1.3. Let φ be an embedding of **No** into itself with image $\mathbf{S}$. Then the map $\psi: x \longmapsto -\varphi(-x)$ defines another embedding of **No** into itself with image $-\mathbf{S} = \{-x : x \in \mathbf{S}\}$. In other words, if $\mathbf{S}$ is a surreal substructure, then so is $-\mathbf{S}$.

We claim that any strictly $(<, \sqsubset)$-increasing map $f: \mathbf{No} \longrightarrow \mathbf{No}$ is automatically an embedding. We first need a lemma.

Lemma 9.1.4. *If x, y, z are numbers such that $x \sqsubseteq y$ and $x \not\sqsubseteq z$, then we have $x < z$ if and only if $y < z$, and $z < x$ if and only if $z < y$.*

Proof. Since $x \not\sqsubseteq z$, we have $x < z$ if and only if there is $\eta_x < \operatorname{bd}(x)$ with $x \upharpoonright \eta_x = z \upharpoonright \eta_x$ and $x[\eta_x] < z[\eta_x]$. Now $x \sqsubseteq y$ so $y \not\sqsubseteq z$ and likewise $y < z$ holds if and only if there is $\eta_y < \operatorname{bd}(y)$ with $y \upharpoonright \eta_y = z \upharpoonright \eta_y$ and $y[\eta_y] < z[\eta_y]$. Notice that $y \upharpoonright \eta_y = z \upharpoonright \eta_y$ and $y \sqsupseteq x \not\sqsubseteq z$ imply that $\eta_y < \operatorname{bd}(x)$. In both cases, since $x \sqsubseteq y$, we have $x[\eta_x] = y[\eta_x]$ and $x[\eta_y] = y[\eta_y]$. Therefore the existence of η_x yields that of $\eta_y = \eta_x$ and *vice versa*. The other equivalence follows by symmetry. □

Lemma 9.1.5. *Assume that $\mathbf{X}$ is a convex subclass of $(\mathbf{No}, <)$. Then every strictly $(<, \sqsubset)$-increasing function $\varphi: \mathbf{X} \longrightarrow \mathbf{No}$ is an embedding $(\mathbf{X}, <, \sqsubset) \longrightarrow (\mathbf{No}, <, \sqsubset)$.*

Proof. Since $(\mathbf{No}, <)$ is linearly ordered, the function φ is automatically an embedding for $<$, so we need only prove that it is an embedding for $\sqsubset$. Assume for contradiction that there are elements $x < y$ of $\mathbf{X}$ such that $x \not\sqsubseteq y$ and $\varphi(x) \sqsubseteq \varphi(y)$ (the case when $y < x$ is symmetric). Let z be the $\sqsubset$-maximal common initial segment of x and y. We have $x < z \leqslant y$, so $z \in \mathbf{X}$. Since φ is strictly $(<, \sqsubset)$-increasing, we have $\varphi(x) < \varphi(z) \leqslant \varphi(y)$ and $\varphi(x) \not\sqsubseteq \varphi(z)$, which given our assumption $\varphi(x) \sqsubseteq \varphi(y)$ contradicts the previous lemma. Hence $\varphi(x) \not\sqsubseteq \varphi(y)$, which concludes the proof. □

Since a surreal substructure $\mathbf{S}$ is an isomorphic copy of $\mathbf{No}$ into itself, it induces a natural Conway bracket $\{\}_{\mathbf{S}}$ on $\mathbf{S}$. This actually leads to an equivalent definition of surreal substructures. Let us investigate this in more detail.

Let $\mathbf{S}$ be an arbitrary subclass of $\mathbf{No}$. We say that $\mathbf{S}$ is *rooted* if it admits a simplest element, called its *root*, and which we denote by $\mathbf{S}^{\bullet}$. In other words, it is rooted if, as a subgraph of $(\mathbf{No}, \sqsubset)$, it has a root. Given subclasses $\mathbf{L} < \mathbf{R}$ of $\mathbf{S}$, we let $(\mathbf{L} \mid \mathbf{R})_{\mathbf{S}}$ denote the class of elements $x \in \mathbf{S}$ such that $\mathbf{L} < x < \mathbf{R}$. If $(\mathbf{L} \mid \mathbf{R})_{\mathbf{S}}$ is rooted, then we let $\{\mathbf{L} \mid \mathbf{R}\}_{\mathbf{S}}$ denote its root. If $L = \mathbf{L}$ and $R = \mathbf{R}$ are sets, then we call $(L \mid R)_{\mathbf{S}}$ the *cut* in $\mathbf{S}$ defined by L and R. If for any subsets $L < R$ of $\mathbf{S}$ the class $(L \mid R)_{\mathbf{S}}$ is rooted, then we say that $\mathbf{S}$ *admits an induced Conway bracket*.

Proposition 9.1.6. *Let* $\mathbf{S}$ *admit an induced Conway bracket. Then the map* $\Xi_{\mathbf{S}} : \mathbf{No} \longrightarrow \mathbf{S}$ *defined by*

$$\forall x \in \mathbf{No}, \Xi_{\mathbf{S}}\, x = \{\Xi_{\mathbf{S}}\, x_L \mid \Xi_{\mathbf{S}}\, x_R\}_{\mathbf{S}}$$

is an isomorphism $(\mathbf{No}, <, \sqsubset) \longrightarrow (\mathbf{S}, <, \sqsubset)$.

Proof. We first justify that $\Xi_{\mathbf{S}}$ is well defined. Let $x \in \mathbf{No}$ be such that $\Xi_{\mathbf{S}}$ is well-defined and strictly $<$-increasing on $x_{\sqsubset}$, with values in $\mathbf{S}$. We have $\Xi_{\mathbf{S}}\, x_L < \Xi_{\mathbf{S}}\, x_R$ where those sets are in $\mathbf{S}$ so $\Xi_{\mathbf{S}}\, x$ is a well-defined element of $(\Xi_{\mathbf{S}}\, x_L \mid \Xi_{\mathbf{S}}\, x_R)_{\mathbf{S}}$, and $\Xi_{\mathbf{S}}$ is strictly $<$-increasing on $\{x\} \cup x_L \cup x_R$. By induction, $\Xi_{\mathbf{S}}$ is a strictly increasing map $\mathbf{No} \longrightarrow \mathbf{S}$. Let $y \in \mathbf{No}$ with $x \sqsubseteq y$, so that $x_L < y < x_R$. By definition, the number $\Xi_{\mathbf{S}}\, x$ is the simplest element $u \in \mathbf{S}$ with $\Xi_{\mathbf{S}}\, x_L < u < \Xi_{\mathbf{S}}\, x_R$. Since $\Xi_{\mathbf{S}}\, y \in \mathbf{S}$ and $\Xi_{\mathbf{S}}\, x_L < \Xi_{\mathbf{S}}\, y < \Xi_{\mathbf{S}}\, y_L$, it follows that $\Xi_{\mathbf{S}}\, x \sqsubseteq \Xi_{\mathbf{S}}\, y$. We deduce from Lemma 9.1.5 that $\Xi_{\mathbf{S}}$ is an embedding of $(\mathbf{No}, <, \sqsubset)$ into itself.

We now prove that $\mathbf{S} \subseteq \Xi_{\mathbf{S}}\, \mathbf{No}$ by induction on $(\mathbf{S}, \sqsubset)$. Let $y \in \mathbf{S}$ be such that $y_{\sqsubset} \cap \mathbf{S}$ is a subset of $\Xi_{\mathbf{S}}\, \mathbf{No}$. Let $\Xi_{\mathbf{S}}\, L' = L = y_L \cap \mathbf{S}$ and $R = y_R \cap \mathbf{S} = \Xi_{\mathbf{S}}\, R'$ where since $\Xi_{\mathbf{S}}$ is strictly $\leqslant$-increasing and thus injective, the sets L', R' are uniquely determined and satisfy $L' < R'$. Since $\mathbf{S}$ admits an induced Conway bracket, the cut $(L \mid R)_{\mathbf{S}}$ is rooted and contains y, so $\{L \mid R\}_{\mathbf{S}} \sqsubseteq y$. Since $\{L \mid R\}_{\mathbf{S}} \notin L \cup R$, we necessarily have $y = \{L \mid R\}_{\mathbf{S}} = \Xi_{\mathbf{S}}\, \{L' \mid R'\}$. By induction, we conclude that $\mathbf{S} = \Xi_{\mathbf{S}}\, \mathbf{No}$. □

Proposition 9.1.7. *Let* $\mathbf{S}$ *be a subclass of* $\mathbf{No}$. *Then* $\mathbf{S}$ *is a surreal substructure if and only if it admits an induced Conway bracket.*

Proof. Assume that $\mathbf{S}$ admits an induced Conway bracket. By the previous proposition, $\mathbf{S}$ is the range of the strictly $(<, \sqsubset)$-increasing function $\Xi_{\mathbf{S}} : \mathbf{No} \longrightarrow \mathbf{No}$, whence $\mathbf{S}$ is a surreal substructure. Conversely, consider an embedding φ of $\mathbf{No}$ into itself with image $\mathbf{S}$. Let $L < R$ be subsets of $\mathbf{S}$ and define $(L', R') = (\varphi^{-1}(L), \varphi^{-1}(R))$. The function φ is strictly $<$-increasing so $L' < R'$, and we may consider the number $x = \{L' \mid R'\}$. Now let $y \in (L \mid R)_{\mathbf{S}}$. We have $\varphi^{-1}(y) \in (L' \mid R')$, so $x \sqsubseteq \varphi^{-1}(y)$. Since φ is $\sqsubset$-increasing, this implies $\varphi(x) \sqsubseteq y$, which proves that $\varphi(x) = \{L \mid R\}_{\mathbf{S}}$, so $\mathbf{S}$ admits an induced Conway bracket. □

Remark 9.1.8. More generally, one may discard the existence condition for the Conway bracket and consider subclasses $\mathbf{X}$ of $\mathbf{No}$ that satisfy the following condition:

IN. For all subsets L, R of $\mathbf{X}$ with $L < R$, the class $(L \mid R)_{\mathbf{X}}$ is either empty or rooted.

A subclass $\mathbf{X} \subseteq \mathbf{No}$ satisfies **IN** if and only if there is a (unique) $\sqsubset$-initial subclass $\mathbf{I}_{\mathbf{S}}$ of $\mathbf{No}$ and a (unique) isomorphism $(\mathbf{I}_{\mathbf{S}}, <, \sqsubset) \longrightarrow (\mathbf{S}, <, \sqsubset)$. This is in particular the case for the classes $\mathbf{Smp}_{\mathbf{\Pi}}$ described in Chapter 10 below. For more details on this more general kind of subclasses, we refer to [42].

In the thesis, we focus on surreal substructures. The characterizations given in Proposition 9.1.7 and Proposition 9.2.2 are known results. The second one was first proved (for more general types of ordinal sequences) by Lurie [75, Theorem 8.3], and both of them were proved by Ehrlich [42, Theorems 1 and 4].

Proposition 9.1.9. *Let* $\mathbf{S}$ *be a surreal substructure. The function* $\Xi_{\mathbf{S}}$ *is the unique surjective strictly* $(<,\sqsubset)$*-increasing function* $\mathbf{No}\longrightarrow\mathbf{S}$*.*

Proof. Let φ be a strictly $(<,\sqsubset)$-increasing function $\mathbf{No}\longrightarrow\mathbf{S}$ with image $\mathbf{S}$. By Lemma 9.1.5, it is an embedding. Given $x\in\mathbf{No}$ such that φ and $\Xi_{\mathbf{S}}$ coincide on $x_{\sqsubset}$, the numbers $\varphi(x)$ and $\Xi_{\mathbf{S}}\,x$ of $\mathbf{S}$ are both the simplest element of $(\Xi_{\mathbf{S}}\,x_L\mid\Xi_{\mathbf{S}}\,x_R)_{\mathbf{S}}$ and are thus equal. It follows by induction that $\varphi=\Xi_{\mathbf{S}}$. □

Given a surreal substructure $\mathbf{S}$, we call $\Xi_{\mathbf{S}}$ the *defining surreal isomorphism* or *parametrization* of $\mathbf{S}$. The above uniqueness property is fundamental; it allows us in particular to perform constructions on surreal substructures *via* their parametrizations.

9.1.2 Cut representations

Let $\mathbf{S}$ be a surreal substructure. Given an element $x\in\mathbf{S}$ and subsets L,R of $\mathbf{S}$ with $L<R$, we say that (L,R) is a *cut representation* of x in $\mathbf{S}$ if $x=\{L\mid R\}_{\mathbf{S}}$, i.e. if x is the simplest element of $(L\mid R)_{\mathbf{S}}$. We refer to elements in L and R as *left* and *right options* of the representation. For $x\in\mathbf{S}$, we write

$$(x_L^{\mathbf{S}},x_R^{\mathbf{S}}):=(x_L\cap\mathbf{S},x_R\cap\mathbf{S})$$

and call this pair the *canonical representation* of x in $\mathbf{S}$. We also write $x_{\sqsubset}^{\mathbf{S}}$ for the set $x_{\sqsubset}\cap\mathbf{S}$.

A $\sqsubset$*-final substructure* of $\mathbf{S}$ is a rooted final segment $\mathbf{T}$ of $\mathbf{S}$ for $\sqsubset$ (and thereby necessarily a substructure). It is easy to see that this is the case if and only if $\mathbf{T}$ is rooted and $\mathbf{T}$ is the class $\mathbf{S}^{\sqsupseteq\mathbf{T}^\bullet}$ of elements $x\in\mathbf{S}$ such that $\mathbf{T}^\bullet\sqsubseteq x$.

Proposition 9.1.10. *Let* $\mathbf{S}$ *be a surreal substructure and let* (L,R) *and* (L',R') *be cut representations in* $\mathbf{S}$*. For* $x\in\mathbf{S}$*, we have*

a) $\{L\mid R\}_{\mathbf{S}}\leqslant\{L'\mid R'\}_{\mathbf{S}}$ *if and only if* $\{L\mid R\}_{\mathbf{S}}<R'$ *and* $L<\{L'\mid R'\}_{\mathbf{S}}$*.*

b) $(x_L^{\mathbf{S}},x_R^{\mathbf{S}})$ *is a cut representation of* x *in* $\mathbf{S}$ *with respect to which any other cut representation of* x *in* $\mathbf{S}$ *is cofinal.*

c) $\mathbf{S}^{\sqsupseteq x}=(x_L^{\mathbf{S}}\mid x_R^{\mathbf{S}})_{\mathbf{S}}$*.*

Proof. The assertions *a*) and *b*) are true when $\mathbf{S}=\mathbf{No}$ by [55, Theorems 2.5 and 2.9]. By Proposition 9.1.6, the function $\Xi_{\mathbf{S}}$ is an isomorphism $(\mathbf{No},<,\sqsubset)\longrightarrow(\mathbf{S},<,\sqsubset)$, satisfying the relation $\forall a\in\mathbf{No},(\Xi_{\mathbf{S}}\,a_L,\Xi_{\mathbf{S}}\,a_R)=((\Xi_{\mathbf{S}}\,a)_L^{\mathbf{S}},(\Xi_{\mathbf{S}}\,a)_R^{\mathbf{S}})$, so *a*) and *b*) hold in general. We have $\mathbf{S}^{\sqsupseteq x}\supseteq(x_L^{\mathbf{S}}\mid x_R^{\mathbf{S}})_{\mathbf{S}}$, since $x=(x_L^{\mathbf{S}}\mid x_R^{\mathbf{S}})_{\mathbf{S}}^\bullet$. Conversely, for $y\in\mathbf{S}^{\sqsupseteq x}$ and $x'\in x_{\sqsubset}^{\mathbf{S}}$, we have $x'\sqsubset y$ and $y[\mathrm{bd}(x')]=x[\mathrm{bd}(x')]\in\{-1,1\}$, so $y-x'$ and $x-x'$ have the same sign. We conclude that $x_L^{\mathbf{S}}<y<x_R^{\mathbf{S}}$, which completes the proof of *c*). □

9.2 Operations on surreal substructures

9.2.1 Imbrications

Let $\mathbf{S},\mathbf{T}$ be two surreal substructures. Write $\Xi_{\mathbf{S}}^{\mathrm{inv}}$ for the functional inverse of $\Xi_{\mathbf{S}}\colon\mathbf{No}\longrightarrow\mathbf{S}$. Then there is a unique $(<,\sqsubset)$-isomorphism $\Xi_{\mathbf{T}}^{\mathbf{S}}:=\Xi_{\mathbf{T}}\circ\Xi_{\mathbf{S}}^{\mathrm{inv}}\colon\mathbf{S}\longrightarrow\mathbf{T}$ that we call the *surreal isomorphism* between $\mathbf{S}$ and $\mathbf{T}$. The composition $\Xi_{\mathbf{S}}\,\Xi_{\mathbf{T}}:=\Xi_{\mathbf{S}}\circ\Xi_{\mathbf{T}}$ is also an embedding, so its image

$$\mathbf{S}\prec\mathbf{T}:=\Xi_{\mathbf{S}}\,\mathbf{T}$$

is again a surreal substructure that we call the *imbrication* of $\mathbf{T}$ into $\mathbf{S}$. We say that $\mathbf{T}$ is a *left factor* (resp. *right factor*) of $\mathbf{S}$ if there is a surreal substructure $\mathbf{U}$ such that $\mathbf{S}=\mathbf{T}\prec\mathbf{U}$ (resp. $\mathbf{S}=\mathbf{U}\prec\mathbf{T}$).

By the associativity of the composition of functions, the imbrication of surreal substructures is associative. Right factors are determined by the two other substructures. More precisely, since $\Xi_{\mathbf{T}}$ is injective, the relation $\mathbf{S}=\mathbf{T}\prec\mathbf{U}=\Xi_{\mathbf{T}}\,\mathbf{U}$ yields $\mathbf{U}=\Xi_{\mathbf{T}}^{\mathrm{inv}}(\mathbf{S})$. The same does not hold for left factors:

$$(1\dotplus\mathbf{No})\dotplus(\omega\dotplus\mathbf{No})=\mathbf{No}\prec(\omega\dotplus\mathbf{No})=\omega\dotplus\mathbf{No}.$$

Proposition 9.2.1. *If* $\mathbf{S},\mathbf{T}$ *are surreal substructures, then* $\mathbf{T}$ *is a left factor of* $\mathbf{S}$ *if and only if* $\mathbf{S}\subseteq\mathbf{T}$.

Proof. If $\mathbf{S}=\mathbf{T}\prec\mathbf{U}$, then $\mathbf{S}=\Xi_{\mathbf{T}}\,\mathbf{S}\subseteq\mathbf{T}$. Assume that $\mathbf{S}\subseteq\mathbf{T}$ and let $\mathbf{U}=\Xi_{\mathbf{T}}^{\mathrm{inv}}(\mathbf{S})$. We have $\mathbf{U}=(\Xi_{\mathbf{T}}^{\mathrm{inv}}\upharpoonright\mathbf{S})\,\Xi_{\mathbf{S}}\,\mathbf{No}$ where $\Xi_{\mathbf{T}}^{\mathrm{inv}}\upharpoonright\mathbf{S}$ and $\Xi_{\mathbf{S}}$, are respectively embeddings $(\mathbf{S},<,\sqsubset)\longrightarrow(\mathbf{No},<,\sqsubset)$ and $(\mathbf{No},<,\sqsubset)\longrightarrow(\mathbf{S},<,\sqsubset)$ so $(\Xi_{\mathbf{T}}^{\mathrm{inv}}\upharpoonright\mathbf{S})\,\Xi_{\mathbf{S}}$ is an embedding $(\mathbf{No},<,\sqsubset)\longrightarrow(\mathbf{No},<,\sqsubset)$. Hence $\mathbf{U}$ is a surreal substructure with $\Xi_{\mathbf{T}}\,\mathbf{U}=\mathbf{S}$, which means that $\mathbf{T}\prec\mathbf{U}=\mathbf{S}$. □

9.2.2 Surreal substructures as trees

Through the identification $\mathbf{No}\simeq\{-1,1\}^{<\mathbf{On}}$, the class of surreal numbers can be represented by a full binary tree of uniform depth $\mathbf{On}$, as illustrated in Figure 9.2.1.

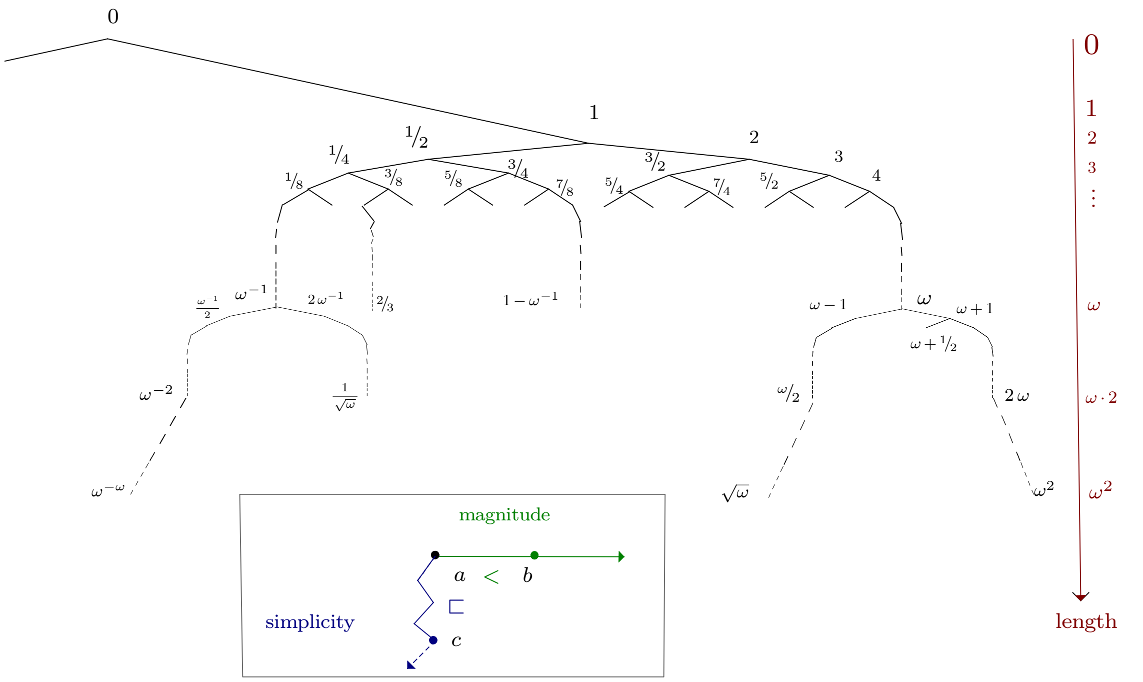


Figure 9.2.1. The class of positive surreal numbers as a tree. For clarity, only a few numbers up to the birth day ω^2 are represented. Negative numbers are obtained through symmetry w.r.t. the y-axis.

For each ordinal α, we let $\mathbf{No}(\alpha)$ denote the subtree of $\mathbf{No}$ of nodes of depth $<\alpha$, that is, the set of numbers x with $\mathrm{bd}(x)<\alpha$. This can be represented as the subtree obtained by cropping the picture at depth α. In order to characterize surreal substructures in tree-theoretic terms, we need to investigate chains for $\sqsubset$: given a subclass $\mathbf{X}\subseteq\mathbf{No}$, a $\sqsubset$-*chain* in $\mathbf{X}$ is a linearly ordered (and thus well-ordered) *subset* C of $(\mathbf{X},\sqsubset)$. If a $\sqsubset$-chain C in $(\mathbf{X},\sqsubset)$ admits a supremum in $(\mathbf{X},\sqsubset)$, we denote it $\sup_{\mathbf{X},\sqsubset}C$. Note that the empty set has a supremum in $(\mathbf{X},\sqsubset)$ if and only if $\mathbf{X}$ has a root, in which case $\sup_{\mathbf{X},\sqsubset}\varnothing=\mathbf{X}^{\bullet}$. We say that $y\in\mathbf{X}$ is the *left successor* of $x\in\mathbf{X}$ if $y<x$ and $z\sqsupseteq y$ for every $z<x$ in $\mathbf{X}$. Right successors are defined similarly.

Proposition 9.2.2. *Let* $\mathbf{S}$ *be a class of surreal numbers. Then the following assertions are equivalent:*

a) $\mathbf{S}$ *is a surreal substructure.*

b) *Every element of* $\mathbf{S}$ *has a left and a right successor in* $\mathbf{S}$ *and every* $\sqsubset$*-chain in* $\mathbf{S}$ *has a supremum in* $(\mathbf{S},\sqsubset)$.

Proof. Let $\mathbf{S}$ be a surreal substructure. In $\mathbf{No}$, any element x clearly admits a left successor $\{x_L\mid x\}$ and a right successor $\{x\mid x_R\}$, and every $\sqsubset$-chain clearly admits a supremum. Since these properties are preserved by the isomorphism $\Xi_{\mathbf{S}}$, we deduce *b*).

Assume now that b) holds. We derive a) by inductively defining an isomorphism $\Xi: (\mathbf{No}, <, \sqsubset) \longrightarrow (\mathbf{S}, <, \sqsubset)$. Applying b) to the empty chain, we note that the supremum of $\varnothing$ in $(\mathbf{S}, \sqsubset)$ is the minimum of $\mathbf{S}$ for $\sqsubset$. So $\mathbf{S}$ is rooted and we may define $\Xi 0 = \mathbf{S}^{\bullet}$. Let $0 < \alpha$ be an ordinal such that Ξ is defined and strictly $(<, \sqsubset)$-increasing on $\mathbf{No}(\alpha)$. We distinguish two cases:

- If α is limit, then let x be a surreal number with length α. Thus x is a limit number and $\Xi x_{\sqsubset}$ is a $\sqsubset$-chain in $\mathbf{S}$. We define $\Xi x = \sup_{\mathbf{X}, \sqsubset} \Xi x_{\sqsubset}$.
- Assume now that α is successor, let x be a number with length α, and write $x = u \dotplus \sigma$ where $\sigma \in \{-1, 1\}$. Let u_{-1} and u_1 be the left and right successors of Ξu. Then we define $\Xi x = u_{\sigma}$.

In both cases, this defines Ξ on $\mathbf{No}(\alpha + 1)$ and the extension is clearly strictly $\sqsubset$-increasing and strictly $<$-increasing on every set $x_{\sqsubseteq} := \{x\} \cup x_{\sqsubset}$ for $x \in \mathbf{No}(\alpha + 1)$.

It remains to be shown that Ξ is strictly $<$-increasing on $\mathbf{No}(\alpha + 1)$. Given $a < b$ in $\mathbf{No}(\alpha + 1)$, let $c \in \mathbf{No}(\alpha)$ be their $\sqsubset$-maximal common initial segment. We either have $a \leqslant c < b$ and thus $\Xi a \leqslant \Xi c < \Xi b$, or $a < c \leqslant b$ and thus $\Xi a < \Xi c \leqslant \Xi b$. So Ξ is strictly $<$-increasing on $\mathbf{No}(\alpha + 1)$.

By induction, the function Ξ is defined and $(<, \sqsubset)$-increasing on $\mathbf{No} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{No}(\alpha)$. Note that $(\mathbf{S}, \sqsubset)$ is well-founded since $(\mathbf{No}, \sqsubset)$ is well-founded and $\mathbf{S} \subseteq \mathbf{No}$. By induction over $y \in \mathbf{S}$, let us show that y lies in the range of Ξ. If y is the left or right successor of an element $v \in \mathbf{S}$, then the induction hypothesis implies the existence of some $u \in \mathbf{No}$ with $v = \Xi u$, and we get $y = \Xi(u \dotpm 1)$. Otherwise, we have $y = \sup_{\sqsubset} y_{\sqsubset}^{\mathbf{S}} = \Xi \sup_{\sqsubset} C$ where $C = \{x \in \mathbf{No} : \Xi x \sqsubset y\}$. We conclude that Ξ is an isomorphism. $\square$

Example 9.2.3. Consider the class **Inc** defined by $\Xi_{\mathbf{Inc}} 0 := 1$, $\Xi_{\mathbf{Inc}} (u \dotplus \sigma) = (\Xi_{\mathbf{Inc}} u) \dotplus \sigma \dotplus 1$, for all $u \in \mathbf{No}$ and $\sigma \in \{-1, 1\}$ and $\Xi_{\mathbf{Inc}} \sup_{\sqsubseteq} C = (\sup_{\sqsubseteq} \Xi_{\mathbf{Inc}} C) \dotplus 1$ for every non-empty $\sqsubset$-chain C without maximum in $(\mathbf{No}, \sqsubseteq)$. It is easy to check that we have $\operatorname{bd}(\Xi_{\mathbf{Inc}} z) > \operatorname{bd}(z)$ and $\operatorname{bd}(\Xi_{\mathbf{Inc}} z) \in \mathbf{On} \dotplus 1$ for every surreal number z.

Example 9.2.4. Let $\mathbf{S} = \mathbf{No}^{\geqslant} \setminus \{1\}$. Then $(\mathbf{S}, \sqsubset)$ is isomorphic to $(\mathbf{No}, \sqsubset)$, but $\mathbf{S}$ is not a surreal structure. In other words, the condition b) cannot be replaced by the weaker condition that $(\mathbf{S}, \sqsubset)$ and $(\mathbf{No}, \sqsubset)$ be isomorphic.

The characterization b) gives us some freedom in constructing a surreal substructure: one only has to provide a mechanism for choosing left and right successors of already constructed elements, as well as least upper bounds for already constructed branches (i.e. $\sqsubset$-chains). Intuitively speaking, this corresponds to a way to "draw" $\mathbf{S}$ as a full binary tree inside the binary tree that represents $\mathbf{No}$.

9.2.3 Convex subclasses

If $\mathbf{X} \subseteq \mathbf{Y}$ are subclasses of $\mathbf{No}$, recall that $\mathbf{X}$ is *convex in* $\mathbf{Y}$ if

$$\forall x, z \in \mathbf{X}, \forall y \in \mathbf{Y}, (x \leqslant y \leqslant z \Longrightarrow y \in \mathbf{X}),$$

and $\mathbf{X}$ is $\sqsubset$*-convex in* $\mathbf{Y}$ if

$$\forall x, z \in \mathbf{X}, \forall y \in \mathbf{Y}, (x \sqsubseteq y \sqsubseteq z \Longrightarrow y \in \mathbf{X}).$$

We simply say that $\mathbf{X}$ is convex (resp. $\sqsubset$-convex) if it is convex (resp. $\sqsubset$-convex) in $\mathbf{No}$. We let $\mathbf{Hull}_{\mathbf{Y}}(\mathbf{X})$ denote the convex hull of $\mathbf{X}$ in $\mathbf{Y}$, that is, for every number y, we have $y \in \mathbf{Hull}_{\mathbf{Y}}(\mathbf{X})$ if and only if $y \in \mathbf{Y}$ and there are elements x, z of $\mathbf{X}$ such that $x \leqslant y \leqslant z$. The convex hull of $\mathbf{X}$ in $\mathbf{Y}$ is the smallest convex subclass of $\mathbf{Y}$ containing $\mathbf{X}$.

Lemma 9.2.5. *Assume that* $\mathbf{S}$ *is a surreal substructure. Then every non-empty convex subclass of* $\mathbf{S}$ *is rooted.*

Proof. In view of Propositions 9.1.6 and 9.1.7, it suffices to prove the lemma for $\mathbf{S} = \mathbf{No}$. Let $\mathbf{C}$ be a non-empty convex subclass of $\mathbf{No}$. Assume for contradiction that $u, v \in \mathbf{C}$ are two simplest elements with $u < v$. Let α be the smallest ordinal such that $u[\alpha] < v[\alpha]$. Since $u \not\sqsubseteq v$ and $v \not\sqsubseteq u$, we must have $u[\alpha] = -1$ and $v[\alpha] = 1$. Now consider the number w whose sign sequence is $u \upharpoonright \alpha = v \upharpoonright \alpha$. Then $u < w < v$, whence $w \in \mathbf{C}$, but also $w \sqsubseteq u$; a contradiction. $\square$

Proposition 9.2.6. *Let* $\mathbf{S}$ *be a surreal substructure.*

a) *A convex subclass* $\mathbf{C}$ *of* $\mathbf{S}$ *is a surreal substructure if and only if it has no cofinal or coinitial subset.*

b) *For subsets* $L < R$ *of* $\mathbf{S}$*, the cut* $(L \mid R)_{\mathbf{S}}$ *is a surreal substructure.*

c) *If* $\mathbf{T} \subseteq \mathbf{S}$ *is a surreal substructure, then* $\mathbf{Hull}_{\mathbf{S}}(\mathbf{T})$ *is a surreal substructure.*

d) *If* $\mathbf{T}$ *is a surreal substructure,* $(L|R)_{\mathbf{S}}$ *is a cut in* $\mathbf{S}$ *and* $f: \mathbf{T} \longrightarrow \mathbf{S}$ *is strictly monotonic and surjective, then* $f^{-1}((L \mid R)_{\mathbf{S}})$ *is a surreal substructure.*

e) *The intersection of any set-sized decreasing family of surreal substructures that are convex in* $\mathbf{S}$ *is a surreal substructure.*

Proof. *a)* Assume that $\mathbf{C}$ has no cofinal or coinitial subset and let $L < R$ be subsets of $\mathbf{C}$.

- If both L and R are empty, then $L < c < R$ for any $c \in \mathbf{C}$. Notice that $\mathbf{C} \neq \varnothing$, since $\varnothing$ is not cofinal in $\mathbf{C}$.
- If $L = \varnothing$ and $R \neq \varnothing$, then there exists an $x \in \mathbf{C}$ with $x < R$, since R is not coinitial in $\mathbf{C}$. Let $y = \{x \mid R\}_{\mathbf{S}}$ and $r \in R$. Then $x < y < r$, so $y \in \mathbf{C}$, and $y \in (L \mid R)_{\mathbf{C}}$.
- Similarly, if $L \neq \varnothing$ and $R = \varnothing$, then $\{L \mid y\}_{\mathbf{S}} \in (L \mid R)_{\mathbf{C}}$ for some $y > L$ in $\mathbf{C}$.
- If $L \neq \varnothing$ and $R \neq \varnothing$, then $\{L \mid R\}_{\mathbf{S}} \in \mathbf{C}$, by convexity.

In each of the above cases, we have shown that $(L \mid R)_{\mathbf{C}}$ is a non-empty convex subclass of $\mathbf{S}$. By Lemma 9.2.5, it is rooted. By Proposition 9.1.7, it follows that $\mathbf{C}$ is a surreal substructure. Conversely, if $\mathbf{C}$ is a surreal substructure, then given a subset X of $\mathbf{C}$, we have

$$\mathbf{C} \ni \{\varnothing \mid X\}_{\mathbf{C}} < X < \{X \mid \varnothing\}_{\mathbf{C}} \in \mathbf{C},$$

so X is neither cofinal nor coinitial in $\mathbf{C}$.

b) This is a direct consequence of the previous point: the cut $(L \mid R)_{\mathbf{S}}$ is by definition a convex subclass of $\mathbf{S}$, and given a subset X of $(L \mid R)_{\mathbf{S}}$ we have

$$(L \mid R)_{\mathbf{S}} \ni \{L \mid X\}_{\mathbf{S}} < X < \{X \mid R\}_{\mathbf{S}} \in (L \mid R)_{\mathbf{S}}.$$

By Proposition 9.1.7, it follows that $(L \mid R)_{\mathbf{S}}$ is a surreal substructure.

c) Since $\mathbf{T}$ is a surreal substructure, it has no cofinal or coinitial subset. It follows that the same holds for $\mathbf{Hull}_{\mathbf{S}}(\mathbf{T})$, which is thus a surreal substructure.

d) We have $f^{-1}((L \mid R)_{\mathbf{S}}) = (f^{-1}(L) \mid f^{-1}(R))_{\mathbf{T}}$ is f is increasing and $f^{-1}((L \mid R)_{\mathbf{S}}) = (f^{-1}(R) \mid f^{-1}(L))_{\mathbf{T}}$ if f is decreasing. In both cases, $f^{-1}((L \mid R)_{\mathbf{S}})$ is a cut in $\mathbf{T}$, hence a surreal substructure by *c)*.

e) Let $(I, <)$ be a linearly ordered set and let $(\mathbf{C}_i)_{i \in I}$ be decreasing for $\subseteq$. Its intersection $\mathbf{C} := \bigcap_{i \in I} \mathbf{C}_i$ is convex. Let X be a subset of $\mathbf{C}$. For $i \in I$, we have $X \subseteq \mathbf{C}_i$ whence $l_i < X < r_i$ where $l_i = (\varnothing \mid X)^{\bullet}_{\mathbf{C}_i}$ and $r_i = (X \mid \varnothing)^{\bullet}_{\mathbf{C}_i}$. Writing $l = \{l_i : i \in I \mid X\}_{\mathbf{S}}$ and $r = \{X \mid r_i : i \in I\}_{\mathbf{S}}$, we have $l < X < r$. Moreover, for $i \in I$, we have $l_i < l < r < r_i$ so $l, r \in \mathbf{C}_i$ by convexity. This proves that $l, r \in \mathbf{C}$ and consequently that X is neither cofinal nor coinitial in $\mathbf{C}$. Therefore $\mathbf{C}$ is a surreal substructure by *a)*. □

Example 9.2.7. Cuts $(L \mid R)_{\mathbf{S}}$ where $L < R$ are subsets of $\mathbf{S}$ include $\sqsubset$-final substructures of $\mathbf{S}$ and non-empty open intervals of $\mathbf{S}$, which are therefore convex surreal substructures. Note that non-empty convex classes of $\mathbf{No}$ which are open in the order topology may fail to be surreal substructures. One counterexample is the class $\mathbf{No}^{\preccurlyeq} := \mathbf{Hull}(\mathbb{Z})$ of *finite surreal numbers*, since it admits the cofinal subset $\mathbb{N}$.

Example 9.2.8. Here are some further examples and counterexamples of convex surreal substructures that we will consider later on.

- The class $\mathbf{No}^{>} := (\{0\} \mid \varnothing)$ of strictly positive surreal numbers is a convex surreal substructure, and it is in fact the $\sqsubset$-final substructure $\mathbf{No}^{\sqsupseteq 1}$ of $\mathbf{No}$.
- Likewise, the class $\mathbf{No}^{>,\succ} := (\mathbb{N} \mid \varnothing) = \mathbf{No}^{\sqsupseteq \omega}$ of positive *infinite surreal numbers* is a convex surreal substructure.

- The class $\mathbf{No}^{\prec} := (\mathbb{R}^{<0} \mid \mathbb{R}^{>0})$ of *infinitesimals* forms a surreal substructure which can be split as the union of $\{0\}$ and the two $\sqsubset$-final substructures $\mathbf{No}^{\sqsupseteq -\omega^{-1}}$, $\mathbf{No}^{\sqsupseteq \omega^{-1}}$.
- Although every interval $(-n-1, n+1)$ for $n \in \mathbb{N}$ is a convex surreal substructure, their increasing union $\mathbf{No}^{\preccurlyeq}$ is not a surreal substructure.

Remark 9.2.9. For subsets $L < R$ of $\mathbf{S}$, the cut $(L \mid R)_{\mathbf{S}}$ may fail to be a $\sqsubset$-final substructure of $\mathbf{S}$. In fact, by Proposition 9.1.10(c), it is a $\sqsubset$-final substructure of $\mathbf{S}$ if and only if the canonical representation of $\{L \mid R\}_{\mathbf{S}}$ in $\mathbf{S}$ is cofinal with respect to (L, R), in which case we have $(L \mid R)_{\mathbf{S}} = \mathbf{S}^{\sqsupseteq \{L \mid R\}_{\mathbf{S}}}$.

Any convex subclass $\mathbf{C}$ of $\mathbf{S}$ is a generalized cut $\mathbf{C} = (\mathbf{L} \mid \mathbf{R})_{\mathbf{S}}$ in $\mathbf{S}$ where $\mathbf{L}$ is the class of strict lower bounds of $\mathbf{C}$ in $\mathbf{S}$ and $\mathbf{R}$ is the class of its strict upper bounds. However, those classes may not always be replaced by sets. In fact, the class $\mathbf{C}$ is a cut $\mathbf{C} = (L \mid R)_{\mathbf{S}}$ with subsets $L < R$ of $\mathbf{S}$ if and only if such sets can be found that are mutually cofinal with $(\mathbf{L}, \mathbf{R})$.

9.2.4 Cut equations

We already noted that the Conway bracket allows for elegant recursive definitions of functions on $\mathbf{No}$. Let us now study such definitions in more detail and examine how they generalize to arbitrary surreal substructures.

Definition 9.2.10. *Let* $\mathbf{S}, \mathbf{T}$ *be surreal substructures. Let* λ, ρ *be functions defined for cut representations in* $\mathbf{S}$ *and such that* $\lambda(L, R), \rho(L, R)$ *are subsets of* $\mathbf{T}$ *whenever* (L, R) *is a cut representation in* $\mathbf{S}$. *We say that a function* $F : \mathbf{S} \longrightarrow \mathbf{T}$ *has* **cut equation** $\{\lambda \mid \rho\}_{\mathbf{T}}$ *if for all* $x \in \mathbf{S}$, *we have*

$$\begin{aligned} \lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) &< \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \quad \textit{and} \\ F(x) &= \{\lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \mid \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}})\}_{\mathbf{T}}. \end{aligned}$$

We say that the cut equation is **extensive** *if it satisfies*

$$\forall x, y \in \mathbf{S}, (x \sqsubseteq y \Longrightarrow (\lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \subseteq \lambda(y_L^{\mathbf{S}}, y_R^{\mathbf{S}}) \wedge \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \subseteq \rho(y_L^{\mathbf{S}}, y_R^{\mathbf{S}}))).$$

Note. We will see in the proof of Proposition 9.2.14 below that extensive cut equations preserve simplicity.

Example 9.2.11. A simple example of a cut equation is (8.2.3): $\forall x \in \mathbf{No}, -x = \{-x_R \mid -x_L\}$. Here we have $\mathbf{S} = \mathbf{T} = \mathbf{No}$ and we can take $\lambda(x_L, x_R) = -x_R$ and $\rho(x_L, x_R) := -x_L$. Note that this cut equation is extensive.

Taking $\mathbf{S} = \mathbf{No}$ and $\mathbf{T} = \mathbf{No}^{>}$, $\lambda(x_L, x_R) = x_L \cap \mathbf{No}^{>}$ and $\rho(x_L, x_R) = x_R \cap \mathbf{No}^{>}$, we obtain the function F with $F(x) = 0$ for all $x \leqslant 0$ and $F(x) = x$ for all $x > 0$.

See Example 9.2.20 below for more examples.

Remark 9.2.12. Our notion of cut equation is not restrictive on the function, since any function F: $\mathbf{S} \longrightarrow \mathbf{T}$ has cut equation (λ, ρ) with $\lambda(L, R) := F(\{L \mid R\}_{\mathbf{S}})_L^{\mathbf{T}}$ and $\rho(L, R) := F(\{L \mid R\}_{\mathbf{S}})_R^{\mathbf{T}}$. Thus it should not be confused with the notions of *recursive definition* in [48] and *genetic definition* in [91].

Example 9.2.13. Given sets Λ, P of functions $\mathbf{S} \longrightarrow \mathbf{T}$, cut equations of the form (λ, ρ) with

$$\begin{aligned} \lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) &= \{\xi(l) : \xi \in \Lambda, l \in x_L^{\mathbf{S}}\} \\ \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) &= \{\psi(r) : \psi \in \mathrm{P}, r \in x_R^{\mathbf{S}}\} \end{aligned}$$

are extensive. We will write $\{\lambda(x_L^{\mathbf{S}}, x_R^{\mathbf{S}}) \mid \rho(x_L^{\mathbf{S}}, x_R^{\mathbf{S}})\}_{\mathbf{T}} = \{\Lambda(x_L^{\mathbf{S}}) \mid \mathrm{P}(x_R^{\mathbf{S}})\}_{\mathbf{T}}$ in this case. Note that it is common to consider well-defined cut equations of the form

$$F(x) = \{\Lambda(x_L^{\mathbf{S}}) \mid \mathrm{P}(x_R^{\mathbf{S}})\}_{\mathbf{T}},$$

where F itself belongs to Λ and P.

Proposition 9.2.14. *Let* $\mathbf{S},\mathbf{T}$ *be surreal substructures. Let* $F:\mathbf{S}\longrightarrow\mathbf{T}$ *be strictly* $<$*-increasing with extensive cut equation* $\{\lambda\mid\rho\}_{\mathbf{T}}$. *Then* $F(\mathbf{S})$ *is a surreal substructure, and we have* $F=\Xi^{\mathbf{S}}_{F(\mathbf{S})}$.

Proof. We claim that F is $\sqsubset$-increasing. Indeed, let $x,y\in\mathbf{S}$ with $x\sqsubseteq y$. We have $x_L^{\mathbf{S}}<y<x_R^{\mathbf{S}}$, so $x_L^{\mathbf{S}}\subseteq y_L^{\mathbf{S}}$ and $x_R^{\mathbf{S}}\subseteq y_R^{\mathbf{S}}$. We deduce by extensivity of (λ,ρ) that $\lambda(x_L^{\mathbf{S}},x_R^{\mathbf{S}})\subseteq\lambda(y_L^{\mathbf{S}},y_R^{\mathbf{S}})$ and $\rho(x_L^{\mathbf{S}},x_R^{\mathbf{S}})\subseteq\rho(y_L^{\mathbf{S}},y_R^{\mathbf{S}})$, and thus $\lambda(x_L^{\mathbf{S}},x_R^{\mathbf{S}})<F(y)<\rho(x_L^{\mathbf{S}},x_R^{\mathbf{S}})$. This implies that $F(x)\sqsubseteq F(y)$. Thus F is strictly $(<,\sqsubset)$-increasing. So the composition $F\circ\Xi_{\mathbf{S}}:\mathbf{No}\longrightarrow F(\mathbf{S})$ is strictly $(<,\sqsubset)$-increasing. The function $\Xi_{\mathbf{S}}:(\mathbf{No},<,\sqsubset)\longrightarrow(\mathbf{S},<,\sqsubset)$ is an embedding by Proposition 9.1.6, so F embeds $\mathbf{S}$ into $\mathbf{T}$. In particular, $F(\mathbf{S})$ is a surreal substructure. By Proposition 9.1.9, we conclude that $F=\Xi^{\mathbf{S}}_{F(\mathbf{S})}$. □

As an application, we get the following well-known result (see [18, Proposition 4.22]).

Proposition 9.2.15. *Let* φ *be a number, and let* $\mathbf{No}^{\prec\operatorname{supp}\varphi}$ *denote the class of numbers* x *with* $x\prec\operatorname{supp}\varphi$. *Then* $\mathbf{No}^{\prec\operatorname{supp}\varphi}$ *and* $\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}$ *are surreal substructures with*

$$\forall x\in\mathbf{No},\Xi_{\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}}x=\varphi+\Xi_{\mathbf{No}^{\prec\operatorname{supp}\varphi}}x.$$

Proof. We have $\mathbf{No}^{\prec\operatorname{supp}\varphi}=(-\mathbb{R}^{>}\operatorname{supp}\varphi\mid\mathbb{R}^{>}\operatorname{supp}\varphi)$. By Proposition 9.2.6(b), this is a surreal substructure. Recall that for $x\in\mathbf{No}$, we have $\varphi+x=\{\varphi_L+x,\varphi+x_L\mid\varphi+x_R,\varphi_R+x\}$. If $x\in\mathbf{No}^{\prec\operatorname{supp}\varphi}$, then we have $\varphi_L+x<\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}<\varphi_R+x$ so we may write

$$\begin{aligned}\varphi+x &= \{\varphi+x_L\mid\varphi+x_R\}_{\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}}\\ &= \{\varphi+x_L^{\mathbf{No}^{\prec\operatorname{supp}\varphi}}\mid\varphi+x_R^{\mathbf{No}^{\prec\operatorname{supp}\varphi}}\}_{\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}}.\end{aligned}$$

Seen as a cut equation in x, this is an extensive cut equation, so by Proposition 9.2.14, we see that $\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}$ is a surreal substructure and that $x\mapsto\varphi+x$ realizes the isomorphism $\mathbf{No}^{\prec\operatorname{supp}\varphi}\longrightarrow\varphi+\mathbf{No}^{\prec\operatorname{supp}\varphi}$. □

Definition 9.2.16. *Let* F *be a function* $\mathbf{S}\longrightarrow\mathbf{T}$ *with cut equation* (λ,ρ). *We say that* (λ,ρ) **is uniform** *at* $x\in\mathbf{S}$ *if we have*

$$\begin{aligned}\lambda(L,R) &< \rho(L,R)\quad\text{and}\\ F(x) &= \{\lambda(L,R)\mid\rho(L,R)\}\end{aligned}$$

whenever (L,R) *is a cut representation of* x *in* $\mathbf{S}$. *We say that* (λ,ρ) *is* **uniform** *if it is uniform at every* $x\in\mathbf{S}$.

Remark 9.2.17. Although Gonshor [55] does not define what he calls equations and uniform equations in a systematic way, we take Definition 9.2.16 to be a valid formalization of his use of the term.

Example 9.2.18. Let $a\in\mathbf{No}$. The following cut equation for the function $y\longmapsto a\dotplus y:\mathbf{No}\longrightarrow 1\dotplus\mathbf{No}$ obtained from (8.1.3)

$$\forall x\in\mathbf{No},a\dotplus y=\{a_L,a\dotplus y_L\mid a\dotplus y_R,a_R\},$$

is uniform. On the contrary, the following cut equation for $x\longmapsto x\dotplus 1$ is not uniform:

$$\forall x\in\mathbf{No},x\dotplus 1=\{x,x_L\mid x_R\}.$$

Indeed, we have $0=\{\varnothing\mid 1\}$ and $0\dotplus 1=1$, but $\{0,\varnothing\mid 1\}=\{0\mid 1\}=1/2$.

Example 9.2.19. Let $b\in\mathbf{No}^{>}$. By (8.1.2), the function $\mathbf{No}\longrightarrow b\dot{\times}\mathbf{No};y\mapsto b\dot{\times}y$ has the following cut equation

$$\forall y\in\mathbf{No},b\dot{\times}y=\{b\dot{\times}y_L\dotplus b_L,b\dot{\times}y_R\dotplus(-b_R)\mid b\dot{\times}y_L\dotplus b_R,b\dot{\times}y_R\dotplus(-b_L)\},$$

which is uniform. On the contrary, the cut equation for $x\mapsto x\dot{\times}1/2$ is not uniform:

$$\forall x\in\mathbf{No},x\dot{\times}1/2=\{x_L,x\dotplus(-x_R)\mid x_R,x\dotplus(-x_L)\}.$$

Indeed, if we were to apply this cut equation to the cut presentation $(\{1/2\},\varnothing)$ of 1, then we would have $1/2$ as a left option and $1\dotplus(-1/2)\leqslant 1/2$ as a right option, which cannot be.

Example 9.2.20. Most common definitions of unary functions $\mathbf{No} \longrightarrow \mathbf{No}$ have known simple cut equations, and many of them are uniform, in particular throughout the work of H. Gonshor in [55]. For instance, the classical cut equations (8.2.3) and (11.1.3) for the functions $a \mapsto -a$ and $a \mapsto \exp a$ are uniform, so for $a \in \mathbf{No}$ and for any cut representation (L, R) of a in $\mathbf{No}$, we have

$$\begin{aligned} -a &= \{-R \mid -L\}, \text{ and} \\ \exp a &= \left\{0,\ [a-l]_{\mathbb{N}} \exp l,\ [a-r]_{2\mathbb{N}+1} \exp r \;\middle|\; \frac{\exp r}{[a-r]_{2\mathbb{N}+1}}, \frac{\exp l}{[l-a]_{\mathbb{N}}}\right\} \qquad (l \in L,\ r \in R). \end{aligned}$$

Example 9.2.21. We will also need an extension of the notion of uniform cut equation to functions $f\colon \mathbf{No} \times \mathbf{No} \longrightarrow \mathbf{No}$. Specifically, by [55, Theorem 3.2], the classical cut equation for the sum of two numbers x, y is uniform in the sense that, given cut representations (L_x, R_x) and (L_y, R_y) of x, y in $\mathbf{No}$, we have

$$x + y = \{L_x + y, x + L_y \mid x + R_y, R_y + y\}. \tag{9.2.1}$$

Similarly for the multiplication, we have

$$x + y = \{x'\, y + x\, y' - x'\, y', x''\, y + x\, y'' - x''\, y'' \mid x'\, y + x\, y'' - x'\, y'', x''\, y + x\, y' - x''\, y'\},$$

where x', x'', y' and y'' range in L_x, R_x, L_y and R_y respectively.

Uniform cut equations have the interesting property that they can be composed.

Lemma 9.2.22. *Let $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2$ be surreal substructures. Let $F_1\colon \mathbf{S}_0 \longrightarrow \mathbf{S}_1$ and $F_2\colon \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ be functions with uniform cut equations*

$$\begin{aligned} F_1 &\equiv \{\lambda_1 \mid \rho_1\}_{\mathbf{S}_1} \\ F_2 &\equiv \{\lambda_2 \mid \rho_2\}_{\mathbf{S}_2}. \end{aligned}$$

Then $F_2 \circ F_1$ has the uniform cut equation $(\lambda_{12}, \rho_{12})$ where for every cut representation (L, R) in $\mathbf{S}_0$, we have $\lambda_{12}(L, R) = \lambda_2(\lambda_1(L, R), \rho_1(L, R))$ and $\rho_{12}(L, R) = \rho_2(\lambda_1(L, R), \rho_1(L, R))$.

Proof. Let $x \in \mathbf{S}_0$, let (L, R) be a cut representation of x in $\mathbf{S}_0$. By uniformity of the cut equation of F_1 at x, we have

$$F_1(x) = \{\lambda_1(L, R) \mid \rho_1(L, R)\}_{\mathbf{S}_1}.$$

By uniformity of the cut equation of F_2 at $F_1(x)$, we have

$$F_2(F_1(x)) \;=\; \{\lambda_2(\lambda_1(L, R), \rho_1(L, R)) \mid \rho_2(\lambda_1(L, R), \rho_1(L, R))\},$$

whence the result. □

Recall that a class $\mathbf{X} \subseteq \mathbf{No}$ is *cofinal* (resp. *coinitial*) *with respect to* a class $\mathbf{Y} \subseteq \mathbf{No}$ if every element of $\mathbf{Y}$ has an upper bound (resp. lower bound) in $\mathbf{X}$. If $\mathbf{X} \subseteq \mathbf{Y}$, then we simply say that $\mathbf{X}$ is cofinal (resp. coinitial) *in* $\mathbf{Y}$.

Lemma 9.2.23. *When $\mathbf{S}, \mathbf{T}$ are surreal substructures, the cut equation $\Xi_{\mathbf{T}}^{\mathbf{S}}\, x \equiv \{\Xi_{\mathbf{T}}^{\mathbf{S}}\, x_L^{\mathbf{S}} \mid \Xi_{\mathbf{T}}^{\mathbf{S}}\, x_R^{\mathbf{S}}\}_{\mathbf{T}}$ is uniform and extensive.*

Proof. Let us first prove uniformity in the case when $\mathbf{S} = \mathbf{No}$. Let $L < R$ be sets of surreal numbers and let $x = \{L \mid R\}$. Since $\Xi_{\mathbf{T}}$ is strictly increasing and ranges in $\mathbf{T}$, the number $y = \{\Xi_{\mathbf{T}}\, L \mid \Xi_{\mathbf{T}}\, R\}_{\mathbf{T}}$ is well defined and $\Xi_{\mathbf{T}}\, L < \Xi_{\mathbf{T}}\, x < \Xi_{\mathbf{T}}\, R$, which yields $y \sqsubseteq \Xi_{\mathbf{T}}\, x$. Moreover, the set L is cofinal in x_L whereas R is coinitial in x_R, so $\Xi_{\mathbf{T}}\, x_L < y < \Xi_{\mathbf{T}}\, x_R$. Hence $\Xi_{\mathbf{T}}\, x \sqsubseteq y$ and $\Xi_{\mathbf{T}}\, x = y$, which shows that the cut equation $\Xi_{\mathbf{T}}\, x \equiv \{\Xi_{\mathbf{T}}\, x_L \mid \Xi_{\mathbf{T}}\, x_R\}_{\mathbf{T}}$ is uniform.

Now consider the general case and let $\Xi_{\mathbf{S}}\, A = L < R = \Xi_{\mathbf{S}}\, B$ be subsets of $\mathbf{S}$. Setting $z := \{A \mid B\}$ and $x := \{L \mid R\}_{\mathbf{S}}$, we have $x = \Xi_{\mathbf{S}}\, z$ by uniformity of the cut equation for $\Xi_{\mathbf{S}}$. Furthermore,

$$\begin{aligned} \{\Xi_{\mathbf{T}}^{\mathbf{S}}\, L \mid \Xi_{\mathbf{T}}^{\mathbf{S}}\, R\}_{\mathbf{T}} &= \{\Xi_{\mathbf{T}}\, A \mid \Xi_{\mathbf{T}}\, B\}_{\mathbf{T}} \\ &= \Xi_{\mathbf{T}}\, z, \end{aligned}$$

by uniformity of the cut equation for $\Xi_{\mathbf{T}}$. Hence $\{\Xi_{\mathbf{T}}^{\mathbf{S}} L \mid \Xi_{\mathbf{T}}^{\mathbf{S}} R\}_{\mathbf{T}} = \Xi_{\mathbf{T}} \Xi_{\mathbf{S}}^{\mathrm{inv}} x = \Xi_{\mathbf{T}}^{\mathbf{S}} z$, which proves that $\Xi_{\mathbf{T}}^{\mathbf{S}} \equiv \{\Xi_{\mathbf{T}}^{\mathbf{S}} L \mid \Xi_{\mathbf{T}}^{\mathbf{S}} R\}_{\mathbf{T}}$ is uniform. This cut equation has the form $\Xi_{\mathbf{T}}^{\mathbf{S}} z = \{\Lambda(z_L^{\mathbf{S}}) \mid \mathrm{P}(z_R^{\mathbf{S}})\}_{\mathbf{T}}$ where $\Lambda = \mathrm{P} = \{\Xi_{\mathbf{T}}^{\mathbf{S}}\}$ are sets of functions, so it is extensive. □

The above proposition shows that surreal isomorphisms satisfy natural extensive cut equations. Inversely, Proposition 9.2.14 shows that extensive cut equations give rise to surreal isomorphisms. As an application, if we admit that the operation

$$\forall z \in \mathbf{No},\ \dot{\omega}^z := \{0, \mathbb{N}\, \dot{\omega}^{z_L} \mid 2^{-\mathbb{N}}\, \dot{\omega}^{z_R}\}$$

is well defined, then we see that it defines a surreal isomorphism. This is the parametrization of the class $\mathbf{Mo}$ of *monomials*, that is, Conway's ω-map. This cut equation is also uniform (see [55, corollary of Theorem 5.2]), and we can for instance compute, for every number x, the number

$$\begin{aligned} \dot{\omega}^{\dot{\omega}^z} &= \dot{\omega}^{\{0, \mathbb{N}\dot{\omega}^{z_L} \mid 2^{-\mathbb{N}}\dot{\omega}^{z_R}\}} \\ &= \{0, \mathbb{N}\, \dot{\omega}^0, \mathbb{N}\, \omega^{\mathbb{N}\dot{\omega}^{z_L}} \mid 2^{-\mathbb{N}}\, \dot{\omega}^{2^{-\mathbb{N}}\dot{\omega}^{z_R}}\} \\ &= \{\mathbb{N}, \omega^{\mathbb{N}\dot{\omega}^{z_L}} \mid \dot{\omega}^{2^{-\mathbb{N}}\dot{\omega}^{x_R}}\}. \end{aligned}$$

Whenever they exist, this shows the usefulness of extensive cut equations. Unfortunately, many common surreal functions such as the exponential do not admit extensive cut equations. The next proposition describes a more general type of cut equation that is sometimes useful.

Proposition 9.2.24. *Let $\mathbf{S}, \mathbf{T}$ be surreal substructures. Let Λ be a function from $\mathbf{S}$ to the class of subsets of $\mathbf{T}$ such that for $x, y \in \mathbf{S}$ with $x < y$, the set $\Lambda(y)$ is cofinal with respect to $\Lambda(x)$. For $x \in \mathbf{S}$, let $\boldsymbol{\Lambda}[x]$ denote the class of elements u of $\mathbf{S}$ such that $\Lambda(x)$ and $\Lambda(u)$ are mutually cofinal. Let $\{\lambda \mid \rho\}_{\mathbf{T}}$ be an extensive cut equation on $\mathbf{S}$. Let $F: \mathbf{S} \longrightarrow \mathbf{T}$ be strictly increasing with cut equation*

$$\forall x \in \mathbf{S}, F(x) = \{\Lambda(x), \lambda(x) \mid \rho(x)\}_{\mathbf{T}}$$

Then F induces an embedding $(\boldsymbol{\Lambda}[x], <, \sqsubset) \longrightarrow (\mathbf{T}, <, \sqsubset)$ for each element x of $\mathbf{S}$.

Proof. Let $x \in \mathbf{S}$. If $u, w \in \boldsymbol{\Lambda}[x]$ and $v \in \mathbf{S}$ satisfies $u \leqslant v \leqslant w$, then $\Lambda(v)$ is cofinal with respect to $\Lambda(u)$ and hence to $\Lambda(x)$, and $\Lambda(x)$ is cofinal with respect to $\Lambda(w)$ and hence to $\Lambda(v)$, so $v \in \boldsymbol{\Lambda}[x]$. Therefore $\boldsymbol{\Lambda}[x]$ is a non-empty convex subclass of $\mathbf{S}$. Note that for $u \in \boldsymbol{\Lambda}[x]$, we have

$$F(u) = \{\Lambda(x), \lambda(u) \mid \rho(u)\}_{\mathbf{T}}.$$

For numbers u, v lying in $\boldsymbol{\Lambda}[x]$ with $u \sqsubseteq v$, we have $\Lambda(x) \cup \lambda(u) \subseteq \Lambda(x) \cup \lambda(v) < F(v) < \rho(v) \supseteq \rho(u)$, which implies that $F(u) \sqsubseteq F(v)$. Since $\boldsymbol{\Lambda}[x]$ is a non-empty convex subclass of $\mathbf{S}$ and $\Xi_{\mathbf{S}}: \mathbf{No} \longrightarrow \mathbf{S}$ is increasing and bijective, the class $\mathbf{C} := \Xi_{\mathbf{S}}^{-1}(\boldsymbol{\Lambda}[x])$ is a non-empty convex subclass of $\mathbf{No}$ on which $F \circ \Xi_{\mathbf{S}}$ is strictly $(<, \sqsubset)$-increasing. By Lemma 9.1.5, the function $F \circ \Xi_{\mathbf{S}}$ induces an embedding $(\mathbf{C}, <, \sqsubset) \longrightarrow (\mathbf{T}, <, \sqsubset)$ and thus F induces an embedding $(\boldsymbol{\Lambda}[x], <, \sqsubset) \longrightarrow (\mathbf{T}, <, \sqsubset)$. □

Example 9.2.25. A typical example is the following cut equation of [18, Theorem 3.8(1)] for the exponential function on the class $\mathbf{Mo}^{\succ} := \{\mathfrak{m} \in \mathbf{Mo} : \mathbb{R} < \mathfrak{m}\}$ of infinite monomials:

$$\forall \mathfrak{m} \in \mathbf{Mo}, \exp \mathfrak{m} = \{\mathfrak{m}^{\mathbb{N}}, (\exp \mathfrak{m}_L^{\mathbf{Mo}})^{\mathbb{N}} \mid (\exp \mathfrak{m}_R^{\mathbf{Mo}})^{\mathbb{N}}\}.$$

Here we have $\Lambda(\mathfrak{m}) = \mathfrak{m}^{\mathbb{N}}$ and $\boldsymbol{\Lambda}[\mathfrak{m}] = \{\mathfrak{n} \in \mathbf{Mo}^{\succ} : \exists p, q \in \mathbb{N}, \mathfrak{m}^{1/p} \prec \mathfrak{n} \prec \mathfrak{m}^p\}$.

Chapter 10

Convex partitions

The reader can note that given a confluent hyperserial field $(\mathbb{T},\circ)$ of force $\boldsymbol{\nu}\leqslant\mathbf{On}$, such partitions of $\mathbb{T}^{>,\succ}$ into convex subclasses were considered in e. We are thinking of the collections of classes $\mathcal{E}_\alpha[s]$ and $\mathcal{L}_\alpha[s]$ for $s\in\mathbb{T}^{>,\succ}$ and $\alpha=\omega^\mu$, $\mu<\boldsymbol{\nu}$. This chapter introduces a convenient way to construct surreal substructures, using partitions of a given surreal substructure such as $\mathbf{No}^{>,\succ}$ into convex subclasses, so as to take advantage of this occurrence when defining and studying the hyperserial structure on $\mathbf{No}$.

10.1 Convex partitions

Throughout this section, $\mathbf{S}$ stands for a surreal substructure. A partition of $\mathbf{S}$ is a formula $\boldsymbol{\Pi}=\Pi(x,y)$ which defines an equivalence relation on $\mathbf{S}$, i.e. such that for all $x,y,z\in\mathbf{S}$, we have

$$\Pi(x,x),\qquad \Pi(x,y)\Longrightarrow\Pi(y,x),\qquad \text{and } (\Pi(x,y)\wedge\Pi(y,z)\Longrightarrow\Pi(x,z)).$$

Given such a partition $\boldsymbol{\Pi}$ and $x\in\mathbf{S}$, we write $\boldsymbol{\Pi}[x]$ for the equivalence class of x in $\mathbf{S}$, that is,

$$\boldsymbol{\Pi}[x]=\{y\in\mathbf{S}:\Pi(x,y)\}.$$

10.1.1 Convex partitions and surreal substructures

Definition 10.1.1. *Let $\boldsymbol{\Pi}$ be a partition of $\mathbf{S}$ for which each class $\boldsymbol{\Pi}[x]$ for $x\in\mathbf{S}$ is convex in $(\mathbf{S},<)$. We say that $\boldsymbol{\Pi}$ is a* **convex partition** *of $\mathbf{S}$. For $x\in\mathbf{S}$ recall that $\boldsymbol{\Pi}[x]$ is rooted (by Lemma 9.2.5). We say that $x\in\mathbf{S}$ is* **$\boldsymbol{\Pi}$-simple** *if $x=\boldsymbol{\Pi}[x]^\bullet$, and we let $\mathbf{Smp}_{\boldsymbol{\Pi}}$ denote the class of $\boldsymbol{\Pi}$-simple elements of $\mathbf{S}$. For $x,y\in\mathbf{S}$ we write:*

$$\begin{aligned} x=_{\boldsymbol{\Pi}}y &\text{ if } \boldsymbol{\Pi}[x]=\boldsymbol{\Pi}[y],\\ x<_{\boldsymbol{\Pi}}y &\text{ if } \boldsymbol{\Pi}[x]<\boldsymbol{\Pi}[y],\\ x\leqslant_{\boldsymbol{\Pi}}y &\text{ if } \boldsymbol{\Pi}[x]=\boldsymbol{\Pi}[y]\text{ or }\boldsymbol{\Pi}[x]<\boldsymbol{\Pi}[y].\end{aligned}$$

Remark 10.1.2. Convex partitions are sometime called *condensations* [90, Definition 4.1].

We can obtain $\mathbf{S}$ as $\mathbf{Smp}_{\boldsymbol{\Pi}_{\text{disc}}}$ through the *discrete partition* $\boldsymbol{\Pi}_{\text{disc}}$ with $\boldsymbol{\Pi}_{\text{disc}}[x]=\{x\}$ for all $x\in\mathbf{S}$. Let $\pi_{\boldsymbol{\Pi}}(x):=\boldsymbol{\Pi}[x]^\bullet\in\mathbf{S}$ for all $x\in\mathbf{S}$. The map $\pi_{\boldsymbol{\Pi}}:\mathbf{S}\longrightarrow\mathbf{Smp}_{\boldsymbol{\Pi}}$ is a surjective, increasing projection. We refer to it as the *$\boldsymbol{\Pi}$-simple projection*.

For the remainder of this subsection, let $\boldsymbol{\Pi}$ be a convex partition of $\mathbf{S}$. A *quasi-order* (or *preorder*) is a binary relation that is reflexive and transitive. The following lemma states basic facts on partitions of a linear order into convex subclasses.

Lemma 10.1.3. *The relation $\leqslant_{\boldsymbol{\Pi}}$ is a linear quasi-order and restricts to a linear order on $\mathbf{Smp}_{\boldsymbol{\Pi}}$. For $x,y\in\mathbf{S}$, we have $x\leqslant_{\boldsymbol{\Pi}}y$ if and only if $\pi_{\boldsymbol{\Pi}}(x)\leqslant\pi_{\boldsymbol{\Pi}}(y)$.*

Proof. It is well known that the partition $\mathbf{\Pi}$ corresponds to the equivalence relation $=_{\mathbf{\Pi}}$ on $\mathbf{S}$. The transitivity and irreflexivity of $<_{\mathbf{\Pi}}$ follow from that of $<$ on subclasses of $\mathbf{No}$. That its restriction to $\mathbf{Smp}_{\mathbf{\Pi}}$ is a linear order is a direct consequence of the definition of $\mathbf{Smp}_{\mathbf{\Pi}}$ and the equivalence stated above, which we now prove. If $\mathbf{\Pi}$ has only one member, then the result is trivial. Otherwise, let $x, y \in \mathbf{S}$ with $x <_{\mathbf{\Pi}} y$. We have $\pi_{\mathbf{\Pi}}(x) \in \mathbf{\Pi}[x] < \mathbf{\Pi}[y] \ni \pi_{\mathbf{\Pi}}(y)$ so $\pi_{\mathbf{\Pi}}(x) < \pi_{\mathbf{\Pi}}(y)$. Conversely, assume that $\pi_{\mathbf{\Pi}}(x) < \pi_{\mathbf{\Pi}}(y)$. Then $\mathbf{\Pi}[x] \neq \mathbf{\Pi}[y]$ which since $\mathbf{\Pi}$ is a partition implies that $\mathbf{\Pi}[x] \cap \mathbf{\Pi}[y] = \varnothing$. For $x' \in \mathbf{\Pi}[x]$, there may be no element z of $\mathbf{\Pi}[y]$ such that $z \leqslant x$ for this would imply $z \leqslant x \leqslant \pi_{\mathbf{\Pi}}(y)$ whence $x \in \mathbf{\Pi}[y]$ by convexity of this class: a contradiction. We thus have $\mathbf{\Pi}[x] < \mathbf{\Pi}[y]$, that is, $x <_{\mathbf{\Pi}} y$. By definition of $\pi_{\mathbf{\Pi}}$, the relation $x =_{\mathbf{\Pi}} y$ implies that $\pi_{\mathbf{\Pi}}(x) = \pi_{\mathbf{\Pi}}(y)$, whereas $\pi_{\mathbf{\Pi}}(x) = \pi_{\mathbf{\Pi}}(y)$ implies that $\mathbf{\Pi}[x] \cap \mathbf{\Pi}[y] \neq \varnothing$, so $\mathbf{\Pi}[x] = \mathbf{\Pi}[y]$, so $x =_{\mathbf{\Pi}} y$. □

For any subclass $\mathbf{X}$ of $\mathbf{S}$, we write $\mathbf{\Pi}[\mathbf{X}] := \bigcup_{x \in \mathbf{X}} \mathbf{\Pi}[x]$.

Lemma 10.1.4. *Let* $\mathbf{A}$,$\mathbf{B}$ *be subclasses of* $\mathbf{S}$. *Then the following statements are equivalent:*

a) $\mathbf{A} < \mathbf{\Pi}[\mathbf{B}]$.

b) $\mathbf{\Pi}[\mathbf{A}] < \mathbf{B}$.

c) $\mathbf{\Pi}[\mathbf{A}] < \mathbf{\Pi}[\mathbf{B}]$.

Proof. All inequalities are vacuously true if $\mathbf{A} = \varnothing$ or $\mathbf{B} = \varnothing$. Assume that $\mathbf{A}$ and $\mathbf{B}$ are non-empty and let $a \in \mathbf{A}$ and $b \in \mathbf{B}$. Assume for contradiction that $\mathbf{A} < \mathbf{\Pi}[\mathbf{B}]$, but $\mathbf{\Pi}[\mathbf{A}] \not< \mathbf{\Pi}[\mathbf{B}]$. Then there exist $a' \in \mathbf{\Pi}[a]$ and $b' \in \mathbf{\Pi}[b]$ with $a < b' \leqslant a'$. By convexity of $\mathbf{\Pi}[a]$, this yields $b' \in \mathbf{\Pi}[a]$, whence $a \in \mathbf{\Pi}[b]$. This contradiction shows that $\mathbf{A} < \mathbf{\Pi}[\mathbf{B}] \Longrightarrow \mathbf{\Pi}[\mathbf{A}] < \mathbf{\Pi}[\mathbf{B}]$. The inverse implication clearly holds. The equivalence $\mathbf{\Pi}[\mathbf{A}] < \mathbf{B} \Longleftrightarrow \mathbf{\Pi}[\mathbf{A}] < \mathbf{\Pi}[\mathbf{B}]$ holds for similar reasons. □

Lemma 10.1.5. *For* $x \in \mathbf{S}$, *the three following statements are equivalent:*

a) x *is* $\mathbf{\Pi}$*-simple.*

b) There is a cut representation (L, R) *of* x *in* $\mathbf{S}$ *such that* $\mathbf{\Pi}[L] < x < \mathbf{\Pi}[R]$.

c) $\mathbf{\Pi}[x_L^{\mathbf{S}}] < x < \mathbf{\Pi}[x_R^{\mathbf{S}}]$.

Proof. Since $(x_L^{\mathbf{S}}, x_R^{\mathbf{S}})$ is a cut representation of x in $\mathbf{S}$, the assertion *c)* implies *b)*.

Conversely, if (L, R) is a cut representation of x in $\mathbf{S}$ with $\mathbf{\Pi}[L] < x < \mathbf{\Pi}[R]$, then we have $L < \mathbf{\Pi}[x] < R$ by the previous lemma. By Proposition 9.1.10(b), the cut representation (L, R) is cofinal with respect to $(x_L^{\mathbf{S}}, x_R^{\mathbf{S}})$, so $x_L^{\mathbf{S}} < \mathbf{\Pi}[x] < x_R^{\mathbf{S}}$. Hence $\mathbf{\Pi}[x_L^{\mathbf{S}}] < x < \mathbf{\Pi}[x_R^{\mathbf{S}}]$, again by Lemma 10.1.4. This shows that *b)* implies *c)*.

Assume now that x is $\mathbf{\Pi}$-simple and let us prove *c)*. For $u \in x_L^{\mathbf{S}}$, we have $u \sqsubset x$, so $u \notin \mathbf{\Pi}[x]$, whence $u \neq_{\mathbf{\Pi}} x$. We do not have $\mathbf{\Pi}[x] < \mathbf{\Pi}[u]$ since $x \not< u$, so Lemma 10.1.3 yields $\mathbf{\Pi}[u] < \mathbf{\Pi}[x]$, and in particular $\mathbf{\Pi}[u] < x$. This proves that $\mathbf{\Pi}[x_L^{\mathbf{S}}] < x$, and similar arguments yield $x < \mathbf{\Pi}[x_R^{\mathbf{S}}]$.

Assume finally that *c)* holds and let us prove *a)*. We have $\mathbf{\Pi}[x]^{\bullet} \sqsubseteq x$ so $\mathbf{\Pi}[x]^{\bullet} \in x_L^{\mathbf{S}} \cup \{x\} \cup x_R^{\mathbf{S}}$. Now the class $\mathbf{\Pi}[\mathbf{\Pi}[x]^{\bullet}] = \mathbf{\Pi}[x]$ is neither strictly greater nor strictly lower than x, so our assumption imposes $\mathbf{\Pi}[x]^{\bullet} = x$. We conclude that x is $\mathbf{\Pi}$-simple. □

An order $<$ on a set S is said to be *dense* if for any $a, b \in S$ with $a < b$, there exists a $c \in S$ with $a < c < b$.

Proposition 10.1.6. *Assume that* $\mathbf{Smp}_{\mathbf{\Pi}}$ *is dense. Then* $\mathbf{\Pi}$ *is the unique convex partition of* $\mathbf{S}$ *such that* $\mathbf{Smp}_{\mathbf{\Pi}}$ *is the class of* $\mathbf{\Pi}$*-simple elements of* $\mathbf{S}$.

Proof. For $a \in \mathbf{Smp}_{\mathbf{\Pi}}$, let $\mathbf{A}_a$ denote the class of elements x of $\mathbf{S}$ such that no $\mathbf{\Pi}$-simple element lies strictly between a and x. The definition of the family $(\mathbf{A}_b)_{b \in \mathbf{Smp}_{\mathbf{\Pi}}}$ only depends on the class $\mathbf{Smp}_{\mathbf{\Pi}}$, and not specifically on $\mathbf{\Pi}$. For $a \in \mathbf{Smp}_{\mathbf{\Pi}}$, we have $\mathbf{\Pi}[a] \subseteq \mathbf{A}_a$.

Conversely, let $x \in \mathbf{A}_a$, and assume for contradiction that x lies outside of $\mathbf{\Pi}[a]$, say $a <_{\mathbf{\Pi}} x$. Then $a <_{\mathbf{\Pi}} \pi_{\mathbf{\Pi}}(x)$ and, $\mathbf{Smp}_{\mathbf{\Pi}}$ being dense, there exists a $\mathbf{\Pi}$-simple element b between a and $\pi_{\mathbf{\Pi}}(x)$. But $a <_{\mathbf{\Pi}} b <_{\mathbf{\Pi}} \pi_{\mathbf{\Pi}}(x)$ implies $a < b < x$, which contradicts the assumption that there is no simple element between a and x. We conclude that $\mathbf{\Pi}[a] = \mathbf{A}_a$, which entails in particular that the partition $\mathbf{\Pi}$ is uniquely determined by $\mathbf{Smp}_{\mathbf{\Pi}}$. □

If $\mathbf{Smp}_{\mathbf{\Pi}}$ is dense, then we call $\mathbf{\Pi}$ the *defining partition* of $\mathbf{Smp}_{\mathbf{\Pi}}$. Notice that this is in particular the case when $\mathbf{Smp}_{\mathbf{\Pi}}$ is a surreal substructure. We next consider a set-theoretic condition under which $\mathbf{Smp}_{\mathbf{\Pi}}$ is always a surreal substructure.

We say that $\mathbf{\Pi}$ is *thin* if each member of $\mathbf{\Pi}$ has a cofinal and coinitial subset, i.e. if each member is the convex hull in $\mathbf{S}$ of a subset of $\mathbf{S}$. For instance, the convex partition $\mathbf{\Pi}$ of $\mathbf{No}$ where

$$\mathbf{\Pi}[x] := \{y \in \mathbf{No} : \exists n \in \mathbb{N}, -n < x - y < n\},$$

is thin. Indeed each class $\mathbf{\Pi}[x]$ for $x \in \mathbf{No}$ admits the cofinal and coinitial subset $x + \mathbb{Z}$.

If $\mathbf{\Pi}$ is thin, then (see [11, Appendix]) we may pick a distinguished family $(\Pi[x])_{x \in \mathbf{S}}$ such that each $\Pi[x]$ for $x \in \mathbf{S}$ is a cofinal and coinitial *subset* of $\mathbf{\Pi}[x]$, with $\Pi[x] = \Pi[y] \Longleftrightarrow x =_{\mathbf{\Pi}} y$. Given sets $L, R \subseteq \mathbf{Smp}_{\mathbf{\Pi}}$ with $L < R$ and $(l, r) \in L \times R$, we have $\mathbf{\Pi}[l] < \mathbf{\Pi}[r]$ by Lemma 10.1.3, whence $\Pi[l] < \Pi[r]$. We deduce that the number

$$\{\Pi[l] : l \in L \mid \Pi[r] : r \in R\}_{\mathbf{S}}$$

exists, and we see that

$$\{\Pi[l] : l \in L \mid \Pi[r] : r \in R\}_{\mathbf{S}} = \{\mathbf{\Pi}[L] \mid \mathbf{\Pi}[R]\}_{\mathbf{S}}. \tag{10.1.1}$$

Theorem 10.1.7. *If $\mathbf{\Pi}$ is thin, then $\mathbf{Smp}_{\mathbf{\Pi}}$ is a surreal substructure. If (L, R) is a cut representation in $\mathbf{Smp}_{\mathbf{\Pi}}$, then we have*

$$\{L \mid R\}_{\mathbf{Smp}_{\mathbf{\Pi}}} = \{\mathbf{\Pi}[L] \mid \mathbf{\Pi}[R]\}_{\mathbf{S}}.$$

Proof. Let $L < R$ be subsets of $\mathbf{Smp}_{\mathbf{\Pi}}$. By (10.1.1), the number $x := \{\mathbf{\Pi}[L] \mid \mathbf{\Pi}[R]\}_{\mathbf{S}}$ is well defined. This number is $\mathbf{\Pi}$-simple by Lemma 10.1.5. Now let $y \in (L \mid R)_{\mathbf{S}}$ be $\mathbf{\Pi}$-simple. Given $l \in L$ and $r \in R$, the $\mathbf{\Pi}$-simplicity of l, r, and y implies that $\mathbf{\Pi}[l] < y < \mathbf{\Pi}[r]$. We deduce that $x \sqsubseteq y$, so $x = \{L \mid R\}_{\mathbf{Smp}_{\mathbf{\Pi}}}$. By Proposition 9.1.7, we conclude that the class $\mathbf{Smp}_{\mathbf{\Pi}}$ is a surreal substructure. □

When $\mathbf{\Pi}$ is thin, the structure $\mathbf{Smp}_{\mathbf{\Pi}}$ is in addition cofinal and coinitial in $\mathbf{S}$, since for $x \in \mathbf{S}$, we have $\mathbf{Smp}_{\mathbf{\Pi}} \ni \{\varnothing \mid \Pi[x]\}_{\mathbf{S}} \leqslant x \leqslant \{\Pi[x] \mid \varnothing\}_{\mathbf{S}} \in \mathbf{Smp}_{\mathbf{\Pi}}$. By the previous proposition, we may say that $\mathbf{Smp}_{\mathbf{\Pi}}$ is thin if its defining partition $\mathbf{\Pi}$ is thin. If $\mathbf{\Pi}$ is not thin, then $\mathbf{Smp}_{\mathbf{\Pi}}$ may fail to be a surreal substructure, but one can prove that there exists a unique $\sqsubset$-initial subclass $\mathbf{I}$ of $\mathbf{No}$ and a unique isomorphism between $\mathbf{Smp}_{\mathbf{\Pi}}$ and $\mathbf{I}$.

For instance, we can obtain the ring $\mathbf{Oz} := \mathbf{No}_{\succ} + \mathbb{Z}$ of *omnific integers* of [28, Chapter 5] as $\mathbf{Smp}_{\mathbf{\Pi}_{\mathbf{Oz}}}$ where for each number $z \in \mathbf{Oz}$, we set $\mathbf{\Pi}_{\mathbf{Oz}}[z] := [z, z+1)$. This is not a surreal substructure since the cut $(0 \mid 1)_{\mathbf{Oz}}$ is empty. Nevertheless, $\mathbf{Oz}$ is $\sqsubset$-initial in $\mathbf{No}$. Note that different partitions may yield the same class $\mathbf{Oz}$ (for instance replacing $\mathbf{\Pi}_{\mathbf{Oz}}[0]$ and $\mathbf{\Pi}_{\mathbf{Oz}}[1]$ with $[0, 1/2)$ and $[1/2, 2)$ respectively and leaving the other classes unchanged), in contrast to the case of dense partitions from Proposition 10.1.6.

Proposition 10.1.8. *Assume that $\mathbf{\Pi}$ is thin. Then we have the following uniform cut equation for $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}}$ and $x \in \mathbf{No}$:*

$$\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} x = \{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} x_L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} x_R]\}_{\mathbf{S}}.$$

Proof. The cut equation follows from Theorem 10.1.7 and the relation

$$\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} x = \{\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} x_L \mid \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} x_R\}_{\mathbf{Smp}_{\mathbf{\Pi}}}.$$

Now toward uniformity, consider a cut representation (L, R) of a number y. We have $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L <_{\mathbf{\Pi}} \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R$ so the number $\{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R]\}_{\mathbf{S}}$ is well defined. Since (L, R) is cofinal with respect to (y_L, y_R) and $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}}$ is strictly increasing, the number $\{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R]\}_{\mathbf{S}}$ lies in the cut $(\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y_L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y_R])_{\mathbf{S}}$, so

$$\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y \sqsubseteq \{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R]\}_{\mathbf{S}}.$$

Conversely, we have $L < y < R$, so $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L < \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y < \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R$. Since $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L \cup \{\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y\} \cup \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R \subseteq \mathbf{Smp}_{\mathbf{\Pi}}$, we have $\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L] < \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y < \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R]$, whence $\{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R]\}_{\mathbf{S}} \sqsubseteq \Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y$. We conclude that $\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} y = \{\mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} L] \mid \mathbf{\Pi}[\Xi_{\mathbf{Smp}_{\mathbf{\Pi}}} R]\}_{\mathbf{S}}$. □

10.1.2 Comparing thin convex partitions

For convex partitions $\mathbf{\Pi}, \mathbf{\Pi}'$ of $\mathbf{S}$, we write $\mathbf{\Pi} \underline{\angle} \mathbf{\Pi}'$ if we have $\mathbf{\Pi}[x] \subseteq \mathbf{\Pi}'[x]$ for every $x \in \mathbf{S}$, and say that $\mathbf{\Pi}$ is *finer* than $\mathbf{\Pi}'$. If $\mathbf{\Pi} \underline{\angle} \mathbf{\Pi}'$, then $\mathbf{Smp}_{\mathbf{\Pi}'} \subseteq \mathbf{Smp}_{\mathbf{\Pi}}$. We write $\mathbf{\Pi} \angle \mathbf{\Pi}'$ if $\mathbf{\Pi} \underline{\angle} \mathbf{\Pi}'$ but we do not have $\mathbf{\Pi}' \underline{\angle} \mathbf{\Pi}$. The (meta)relation $\angle$ is anti-reflexive and transitive, so we will talk of $\angle$-increasing families in the expected sense.

Recall that a *directed set* is a partial order $(J, <)$ such that for all $j, j' \in J$, there exists a $j'' \in J$ with $j, j' \leqslant j''$.

Proposition 10.1.9. *Let* $\mathbf{S}$ *be a surreal substructure. Let* $(J, <)$ *be a non-empty directed set. If* $(\mathbf{\Pi}_j)_{j \in J}$ *is an* $\angle$*-increasing family of thin convex partitions of* $\mathbf{S}$*, then the intersection* $\bigcap_{j \in J} \mathbf{Smp}_{\mathbf{\Pi}_j}$ *is a surreal substructure with defining thin partition* $\mathbf{\Pi}_J$ *given by*

$$\forall x \in \mathbf{S}, \quad \mathbf{\Pi}_J[x] \; = \; \bigcup_{j \in J} \mathbf{\Pi}_j[x].$$

Proof. Given $x \in \mathbf{S}$, the class $\mathbf{\Pi}_J[x] := \bigcup_{j \in J} \mathbf{\Pi}_j[x]$ is a non-empty convex subclass of $\mathbf{S}$ and $\bigcup_{x \in \mathbf{S}} \mathbf{\Pi}_J[x] = \mathbf{S}$. Let $x, y \in \mathbf{S}$ be such that $\mathbf{\Pi}_J[x] \cap \mathbf{\Pi}_J[y] \neq \varnothing$ and let $i \in J$. Since J is directed, there exists a $j \geqslant i$ in J such that $\mathbf{\Pi}_j[x] \cap \mathbf{\Pi}_j[y] \neq \varnothing$, whence $\mathbf{\Pi}_j[x] = \mathbf{\Pi}_j[y]$. In particular, $\mathbf{\Pi}_i[x] \subseteq \mathbf{\Pi}_J[y]$ and $\mathbf{\Pi}_i[y] \subseteq \mathbf{\Pi}_J[x]$. Since this is true for any $i \in J$, it follows that $\mathbf{\Pi}_J[x] = \mathbf{\Pi}_J[y]$, so $\mathbf{\Pi}_J$ defines a convex partition of $\mathbf{S}$.

For $x \in \mathbf{S}$, we have $\mathbf{\Pi}_J[x_L^{\mathbf{S}}] < x < \mathbf{\Pi}_J[x_R^{\mathbf{S}}]$ if and only if $\mathbf{\Pi}_j[x_L^{\mathbf{S}}] < x < \mathbf{\Pi}_j[x_R^{\mathbf{S}}]$ holds for all $j \in J$, so Lemma 10.1.5 implies $\bigcap_{j \in J} \mathbf{Smp}_{\mathbf{\Pi}_j} = \mathbf{Smp}_{\mathbf{\Pi}_J}$. Now for $x \in \mathbf{S}$, the set $\bigcup_{j \in J} \Pi_j x$ is cofinal and coinitial in $\mathbf{\Pi}_J[x]$, so $\mathbf{\Pi}_J$ is thin. Theorem 10.1.7 therefore implies that the class $\bigcap_{j \in J} \mathbf{Smp}_{\mathbf{\Pi}_j}$ is a surreal substructure. □

10.2 Function groups

In this section, we study one particularly important way in which convex partitions of surreal substructures arise, namely as convex hulls of orbits under a group action. We fix a surreal substructure $\mathbf{S}$.

10.2.1 Actions by strictly increasing bijections

Let X be a set, and consider a formula $\mathcal{X}(x_0, x_1, x_2)$ which defines a family of strictly increasing bijections $\mathbf{S} \longrightarrow \mathbf{S}$ indexed by X. That is, we have

$$\forall x_0, x_1, x_2, (\mathcal{X}(x_0, x_1, x_2) \Longrightarrow x_0 \in X \wedge x_1 \in \mathbf{S} \wedge x_2 \in \mathbf{S}),$$

and for each $x \in X$, the class

$$\mathcal{F}_x := \{(a, b) \in \mathbf{S}^2 : \mathcal{X}(x, a, b)\}$$

is a strictly increasing bijection $\mathbf{S} \longrightarrow \mathbf{S}$. We say that $\mathcal{X}$ is a *function set* acting on $\mathbf{S}$.

Consider any set-sized multiplicatively denoted group $(G, \times, 1)$. An action of G by strictly increasing bijections on $\mathbf{S}$ is an action $\mathcal{G}$ of G on $\mathbf{S}$ by strictly increasing bijections, where for all $g, h \in G$ and $x, z \in \mathbf{S}$, we have

$$\mathcal{G}(g h, x, z) \Longleftrightarrow (\exists y \in \mathbf{S}, \mathcal{G}(h, x, y) \wedge \mathcal{G}(g, y, z)).$$

So $\mathcal{F}_{gh} = \mathcal{F}_g \circ \mathcal{F}_h$ for all $g, h \in G$. Given such an action, we identify each element g of G with $\mathcal{F}_g$, and accordingly write $g x$ for the unique element of $\mathbf{S}$ for which $\mathcal{G}(g, x, g x)$ holds. We will also write $g \in \mathcal{G}$ as a shorthand for $g \in G$. We call $\mathcal{G}$ a *function group acting on* $\mathbf{S}$.

Now let $X, \mathcal{X}(x, y, z)$ be a function set acting on $\mathbf{S}$. Write $(X \times \{-1, 1\})^\star$ for the set of finite words on $X \times \{-1, 1\}$. By the principle of definition by induction, there is a formula $\mathcal{X}'(x, y, z)$ such that for all $w = ((x_1, \iota_1), \ldots, (x_k, \iota_k)) \in (X \times \{-1, 1\})^\star$, the formula $\mathcal{X}'(w, y, z)$ defines the function $\mathcal{F}_w := \mathcal{F}_{x_1}^{\circ \iota_1} \circ \cdots \circ \mathcal{F}_{x_k}^{\circ \iota_k}$. Quotienting $(X \times \{-1, 1\})^\star$ by the relation $w \leftrightharpoons w'$ if $\mathcal{F}_w(x) = \mathcal{F}_{w'}(x)$ for all $x \in \mathbf{S}$, we obtain a group $\langle X \rangle$ under concatenation of words, and we have an action of $\langle X \rangle$ on $\mathbf{S}$ given by

$$\langle \mathcal{X} \rangle(x, y, z) : (x \in \langle X \rangle \wedge \exists w \in x, (\mathcal{X}'(w, y, z))).$$

In that sense, we have a function group acting on $\mathbf{S}$, which is generated by functions $\mathcal{F}_x, x \in X$ and their inverses. Identifying each $x \in X$ with $\mathcal{F}_x$, we will simply denote this function group by $\langle \mathcal{F}_x : x \in X \rangle$.

10.2.2 Function groups and convex partitions

We fix a function group $\mathcal{G}$ acting on $\mathbf{S}$. For $\mathbf{X} \subseteq \mathbf{S}$, we write $\mathcal{G}\,\mathbf{X} := \{gx : g \in G\}$. If $\mathbf{X} = \{x\}$ is a singleton, then we simply write $\mathcal{G}x := \mathcal{G}\,\{x\}$.

Definition 10.2.1. *We define the* **halo** $\mathcal{G}[x]$ *of an element* $x \in \mathbf{S}$ *under the action of* $\mathcal{G}$ *by*

$$\mathcal{G}[x] \;=\; \{y \in \mathbf{S} : \exists g, h \in \mathcal{G}, (gx \leqslant y \leqslant hx)\} = \mathbf{Hull}_{\mathbf{S}}(\mathcal{G}x).$$

Proposition 10.2.2. *The classes* $\mathcal{G}[x]$ *for* $x \in \mathbf{S}$ *form a thin convex partition of* $\mathbf{S}$ *.*

Proof. Let $x \in \mathbf{S}$. For any $y \in \mathcal{G}[x]$, we have $\mathcal{G}[y] = \mathcal{G}[x]$. Indeed, we have $gx \leqslant y \leqslant hx$ for certain $g, h \in \mathcal{G}$. Given $z \in \mathcal{G}[y]$, we also have $g'\,y \leqslant z \leqslant h'\,y$ for certain $g', h' \in \mathcal{G}$, whence $(g'\,g)\,x \leqslant g'\,y \leqslant z \leqslant h'\,y \leqslant (h'\,h)\,x$, so that $z \in \mathcal{G}[x]$. We also have $h^{-1}\,y \leqslant x \leqslant g^{-1}\,y$, whence $x \in \mathcal{G}[y]$ and $z \in \mathcal{G}[y]$ for any $z \in \mathcal{G}[x]$. The class $\mathcal{G}[x]$ is convex by definition. For $a \in \mathbf{S}$, we know that $\mathcal{G}[a]$ contains a, so the $\mathcal{G}[a]$ for $a \in \mathbf{S}$ form a convex partition of $\mathbf{S}$. For $x \in \mathbf{S}$, the set $\mathcal{G}x$ is cofinal and coinitial in $\mathcal{G}[x]$, so this partition is thin. □

We write $\mathbf{\Pi}_{\mathcal{G}}$ for the partition from Proposition 10.2.2 and say that an element of $\mathbf{S}$ is $\mathcal{G}$-*simple* if it is $\mathbf{\Pi}_{\mathcal{G}}$-simple. We let $\mathbf{Smp}_{\mathcal{G}}$ denote the class of $\mathcal{G}$-simple elements. Proposition 10.2.2 implies that every property from Lemmas 10.1.3, 10.1.5 and 10.1.4 applies to the class of $\mathcal{G}$-simple elements. We call $\pi_{\mathcal{G}} := \pi_{\mathbf{\Pi}_{\mathcal{G}}}$ the $\mathcal{G}$-*simple projection* and write $<_{\mathcal{G}}$, $=_{\mathcal{G}}$, and $\leqslant_{\mathcal{G}}$ instead of $<_{\mathbf{\Pi}_{\mathcal{G}}}, =_{\mathbf{\Pi}_{\mathcal{G}}}$, and $\leqslant_{\mathbf{\Pi}_{\mathcal{G}}}$.

Proposition 10.2.3. $\mathbf{Smp}_{\mathcal{G}}$ *is a surreal substructure with the following uniform cut equation in* $\mathbf{No}$*:*

$$\forall x \in \mathbf{No}, \Xi_{\mathbf{Smp}_{\mathcal{G}}} x = \{\mathcal{G}\,\Xi_{\mathbf{Smp}_{\mathcal{G}}} x_L \mid \mathcal{G}\,\Xi_{\mathbf{Smp}_{\mathcal{G}}} x_R\}_{\mathbf{S}}.$$

Proof. This is a direct consequence of Proposition 10.2.2, Theorem 10.1.7 and Proposition 10.1.8, where we take $\mathcal{G}\,(\mathcal{G}[x]^{\bullet})$ to be the required cofinal and coinitial subset of $\mathcal{G}[x]$ for each $x \in \mathbf{S}$. □

Remark 10.2.4. Consider actions $\mathcal{X}$ and $\mathcal{Y}$ of sets X, Y respectively on $\mathbf{S}$ by strictly increasing bijections. We say that $\mathcal{X}$ is *pointwise cofinal with respect to* $\mathcal{Y}$ and we write $\mathcal{Y} \trianglelefteq \mathcal{X}$ if

$$\forall x \in \mathbf{S}, \forall f \in \langle \mathcal{Y} \rangle, \exists g \in \langle \mathcal{X} \rangle, (fx \leqslant gx).$$

This relation is transitive and reflexive. If $\mathcal{Y} \trianglelefteq \mathcal{X}$, then $\mathbf{\Pi}_{\langle \mathcal{X} \rangle} \trianglelefteq \mathbf{\Pi}_{\langle \mathcal{Y} \rangle}$, so $\mathbf{Smp}_{\langle \mathcal{Y} \rangle} \subseteq \mathbf{Smp}_{\langle \mathcal{X} \rangle}$. If $\mathcal{X} \trianglelefteq \mathcal{Y}$ and $\mathcal{Y} \trianglelefteq \mathcal{X}$, then we say that $\mathcal{X}$ and $\mathcal{Y}$ are *mutually pointwise cofinal* and we write $\mathcal{X} \lesseqgtr \mathcal{Y}$. In that case, we have $\mathbf{Smp}_{\langle \mathcal{X} \rangle} = \mathbf{Smp}_{\langle \mathcal{Y} \rangle}$. Finally if $\mathcal{Y} \trianglelefteq \mathcal{X}$ but we do not have $\mathcal{X} \trianglelefteq \mathcal{Y}$, then we write $\mathcal{X} \angle \mathcal{Y}$. The relation $\angle$ is anti-reflexive and transitive.

Let us now specialize Proposition 10.1.9 to group-induced convex partitions.

Proposition 10.2.5. *Let* $(J, <)$ *be a non-empty directed set. If* $(\mathcal{G}_j)_{j \in J}$ *is a* $\angle$*-increasing family of function groups acting on* $\mathbf{S}$*, then the function group* $\mathcal{G}_J = \langle \mathcal{G}_j : j \in J \rangle$ *generated by* $\bigcup_{j \in J} \mathcal{G}_j$ *satisfies*

$$\mathbf{Smp}_{\mathcal{G}_J} = \bigcap_{j \in J} \mathbf{Smp}_{\mathcal{G}_j}.$$

Proof. If $x \in \mathbf{S}$ is $\mathcal{G}_J$-simple, then for $j \in J$, we have $\mathcal{G}_j\,x_L^{\mathbf{S}} \subseteq \mathcal{G}_J\,x_L^{\mathbf{S}} < x < \mathcal{G}_J\,x_R^{\mathbf{S}} \supseteq \mathcal{G}_j\,x_R^{\mathbf{S}}$ so x is $\mathcal{G}_j$-simple. Conversely, assume $x \in \mathbf{S}$ is $\mathcal{G}_j$-simple for all $j \in J$. Then let $g = g_{j_1} \cdots g_{j_k} \in \mathcal{G}_J$ where for $1 \leqslant k \leqslant n$, we have $g_{j_k} \in \mathcal{G}_{j_k}$. Since $(J, <)$ is directed and $(\mathcal{G}_j)_{j \in J}$ is $\angle$-increasing, there exists an index $j \in J$ with $j_1, \ldots, j_n \leqslant j$ and an element $g_j \in \mathcal{G}_j$ such that for all $u \in \mathbf{S}$ we have $g_j^{-1}\,u \leqslant g_{j_i}\,u \leqslant g_j\,u$ for all $i \in \{1, \ldots, n\}$, and thus $g_j^{-n}\,u \leqslant gu \leqslant g_j^n\,u$. Since x is $\mathcal{G}_j$-simple, we have $g_j^n\,x_L^{\mathbf{S}} < x < g_j^{-n}\,x_R^{\mathbf{S}}$. This yields $gx_L^{\mathbf{S}} < x < gx_R^{\mathbf{S}}$, so x is $\mathcal{G}_J$-simple. This proves that $\bigcap_{j \in J} \mathbf{Smp}_{\mathcal{G}_j} = \mathbf{Smp}_{\mathcal{G}_J}$. □

10.2.3 Common group actions

We conclude our study of convex partitions with a closer examination of the action of various common types of function groups. We intentionally introduce these function groups without assigning specific domains; this will allow us to let them act on various surreal substructures.

Given $c \in \mathbf{No}$, we define the *translation by c* to be the map

$$T_c: x \longmapsto x + c.$$

We have a function group $\mathcal{T} := \{r \in \mathbb{R}\}$ acting in particular on $\mathbf{No}$ and $\mathbf{No}^{>,\succ}$. Halos for the action of $\mathcal{T}$ on $\mathbf{No}$ are called *finite halos* $\mathcal{T}[x]$ and $\mathcal{T}$-simple elements correspond to purely infinite numbers. The class $\mathbf{No}_{\succ}$ of purely infinite numbers is sometimes denoted $\mathbb{J}$; see [28, 55].

Given $s \in \mathbf{No}^{>}$, we define the *homothety by the factor s* to be the map

$$H_s: x \longmapsto s x.$$

We have a function group $\mathcal{H} := \{H_r : r \in \mathbb{R}^{>}\}$ acting in particular on $\mathbf{No}, \mathbf{No}^{>}$, and $\mathbf{No}^{>,\succ}$. Halos for the action of $\mathcal{H}$ on $\mathbf{No}^{>}$ are the positive parts of *Archimedean classes* and $\mathcal{H}$-simple elements are exactly monomials. We recall class of monomials $\mathbf{Mo} = \dot{\omega}^{\mathbf{No}}$ is parametrized by the ω-map $\Xi_{\mathbf{Mo}}$ and forms a multiplicative cross section that is isomorphic to the value group of $\mathbf{No}$ for the natural valuation. The relations $<_{\mathcal{H}}$, $\leqslant_{\mathcal{H}}$, $=_{\mathcal{H}}$ correspond to the asymptotic relations $\prec$, $\preccurlyeq$, and $\asymp$ from [63, 4]. Given $a \in \mathbf{No}^{\neq}$, the projection $\pi_{\mathcal{H}}(x)$ coincides with the *dominant monomial* $\mathfrak{d}_a$, when considering a as a generalized series in $\mathbb{R}[[\mathbf{Mo}]]$.

Given $s \in \mathbf{No}^{>}$, we define the *s-th power function* by

$$P_s: x \longmapsto x^s = \exp(s \log x).$$

Here exp and log are the exponential and logarithm functions of Chapter 11. We have a function group $\mathcal{P} := \{P_r : r \in \mathbb{R}^{>}\}$ acting in particular on $\mathbf{No}^{>}$ and $\mathbf{No}^{>,\succ}$. Halos $\mathcal{P}[x]$ for the action of $\mathcal{P}$ on $\mathbf{No}^{>,\succ}$ are sometimes called *multiplicative classes* and $\mathcal{P}$-simple elements *fundamental monomials*. The class $\mathbf{Smp}_{\mathcal{P}} = \exp(\mathbf{Mo}^{\succ}) = \dot{\omega}^{\dot{\omega}^{\mathbf{No}}} = \mathbf{Mo}^{\prec 2}$ of fundamental monomials is parametrized by the ω^{ω}-map: see [71, Proposition 2.5].

Chapter 11

Surreal exponentiation

In this chapter, we give a presentation of the theory of exponentiation on surreal numbers. Our presentation is mostly based on Gonshor's work [55, Chapter 10] as well as on [17]. The reader can find good surveys of surreal exponentiation in relation to Hardy fields and transseries in [78, 16].

11.1 Surreal exponentiation

Now that we can represent **No** as a field of well-based series, we need only define a logarithm on **No** in order to turn it into a transserial field. Following a method from [17], we will show how to define such a logarithm in the simplest way. It will turn out that this logarithm corresponds to that defined by Gonshor.

11.1.1 Transserial logarithm on No

We already know by Proposition 3.1.10 that it is sufficient to define a strictly increasing group morphism $\log \colon \mathbf{Mo} \longrightarrow \mathbb{R}[[\mathbf{Mo}^{\succ}]]$ such that $\log \mathfrak{m} \prec \mathfrak{m}$ for all $\mathfrak{m} \succ 1$. Berarducci, Kuhlmann, Mantova and Matusinski [17] found that such a definition could be reduced to defining log on $\mathbf{Mo} \prec \mathbf{Mo} = \dot{\omega}^{\mathbf{Mo}}$. Let us state their main arguments. Assume that a function $\log_* \colon \dot{\omega}^{\mathbf{Mo}} \longrightarrow \mathbf{Mo}^{\succ}$ is defined, strictly increasing and satisfies $\log_* \dot{\omega}^{\mathfrak{m}} \prec \dot{\omega}^{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathbf{Mo}$. Then we define a morphism $\log \colon \mathbf{Mo} \longrightarrow \mathbb{R}[[\mathbf{Mo}^{\succ}]]$ by setting

$$\log \dot{\omega}^{\sum z_{\mathfrak{n}} \mathfrak{n}} := \sum_{\mathfrak{n}} z_{\mathfrak{n}} \left(\log_* \dot{\omega}^{\mathfrak{n}}\right) \tag{11.1.1}$$

for all $z = \sum z_{\mathfrak{n}} \mathfrak{n} \in \mathbf{No}$, and this morphism has the desired properties. Recall that the unique extension of the logarithm to $\mathbf{No}^{>}$ is surjective if and only if $\log \colon \mathbf{Mo} \longrightarrow \mathbb{R}[[\mathbf{Mo}^{\succ}]]$ is surjective, hence by (11.1.1) if and only if $\log_* \colon \dot{\omega}^{\mathbf{Mo}} \longrightarrow \mathbf{Mo}^{\succ}$ is surjective. Recall that $\mathbf{Mo}^{\succ} = \dot{\omega}^{\mathbf{No}^{>}}$ and $\dot{\omega}^{\mathbf{Mo}} = \dot{\omega}^{\dot{\omega}^{\mathbf{No}}}$, so $\log_*$ induces a function $h \colon \mathbf{No} \longrightarrow \mathbf{No}^{>}$ with $\log_* \dot{\omega}^{\dot{\omega}^z} = \dot{\omega}^{h(z)}$ for all $z \in \mathbf{No}$. Moreover h is surjective if and only if $\log_*$ is surjective. We see that $\log_*$ is strictly increasing if and only if h is strictly increasing, and we have an equivalence

$$(\forall \mathfrak{m} \in \mathbf{Mo}, (\log_* \dot{\omega}^{\mathfrak{m}} \prec \dot{\omega}^{\mathfrak{m}})) \Longleftrightarrow (\forall z \in \mathbf{No}, (h(z) \prec \dot{\omega}^z)).$$

So defining a logarithm on **No** reduces to defining a strictly increasing bijection $h \colon \mathbf{No} \longrightarrow \mathbf{No}^{>}$ satisfying $h(z) \prec \dot{\omega}^z$ for all $z \in \mathbf{No}$. The simplest such function is given by the cut equation

$$\forall z \in \mathbf{No}, h(z) = \{h(z_L) \mid h(z_R), \mathbb{R}^{>} \dot{\omega}^z\}_{\mathbf{No}^{>}}. \tag{11.1.2}$$

11.1.2 Gonshor's exponential

Harry Gonshor defined the standard exponential function on **No** using an inductive cut equation. Given $n \in \mathbb{N}$ and $a \in \mathbf{No}$, we define

$$[a]_n := \sum_{k \leqslant n} \frac{a^k}{k!}.$$

If $a' \in a_L$ is such that $\exp(a')$ is already defined, then for $n \in \mathbb{N}$, we should have

$$\exp(a) = \exp(a') \exp(a - a') > \exp(a') [a - a']_n$$

and one expects that such inequalities give sharp approximations of $\exp a$. Following this line of thought, Gonshor defined

$$\exp a = \left\{0,\ [a-a']_{\mathbb{N}}\exp a',\ [a-a'']_{2\mathbb{N}+1}\exp a'' \;\middle|\; \frac{\exp a''}{[a-a'']_{2\mathbb{N}+1}},\ \frac{\exp a'}{[a'-a]_{\mathbb{N}}}\right\} \quad (a'\in a_L,\ a''\in a_R). \tag{11.1.3}$$

The reciprocal of exp, defined on $\mathbf{No}^{>}$, is denoted log. This also leads to a natural powering operation: given $a\in\mathbf{No}^{>}$ and $b\in\mathbf{No}$, we define $a^b=\exp(b\log(a))$. Given $r\in\mathbb{R}$, we have $\dot{\omega}^r=\omega^r$, but for more general elements $z\in\mathbf{No}$, one does not necessarily have $\dot{\omega}^z=\omega^z$. (see [16] for more details).

Gonshor showed that the exponential of an infinite monomial $\dot{\omega}^y$ for $y\in\mathbf{No}^{>}$ was a fundamental monomial $\dot{\omega}^{\dot{\omega}^{g(y)}}$ where $g\colon\mathbf{No}^{>}\longrightarrow\mathbf{No}$ is uniquely determined by the following inductive cut equation

$$g(y)=\{\operatorname{Ind}(y), g(y_L^{\mathbf{No}^{>}}) \mid g(y_R^{\mathbf{No}^{>}})\}$$

where $\operatorname{Ind}(y)$ is the unique number with $\mathfrak{d}_y=\dot{\omega}^{\operatorname{Ind}(y)}$, or equivalently $y\asymp\dot{\omega}^{\operatorname{Ind}(y)}$. It turns out that g is the functional inverse of the function h defined by (11.1.2) above: $g=h^{\operatorname{inv}}$. Therefore $\exp\upharpoonright\dot{\omega}^{\mathbf{No}^{>}}$ is the reciprocal of the function $\log_*$ above. Gonshor's results can be summarized as follows:

Theorem 11.1.1. [55, Corollaries 10.1 and 10.3 and Theorems 10.2, and 10.7–10.9] *The function* exp *defined in* (11.1.3) *is an isomorphism* $(\mathbf{No},+,0,<)\longrightarrow(\mathbf{No}^{>},\times,1,<)$ *which coincides with the exponential on* $\mathbb{R}$ *and satisfies*

$$\begin{aligned}\exp(\dot{\omega}^{\mathbf{No}^{>}}) &= \dot{\omega}^{\dot{\omega}^{\mathbf{No}}},\\ \exp(\mathbf{No}_{\succ}) &= \mathbf{Mo},\\ \forall\varepsilon\prec 1, \exp(\varepsilon) &= \sum_{k\in\mathbb{N}}\frac{1}{k!}\varepsilon^k.\end{aligned} \tag{11.1.4}$$

Moreover, we have $\exp(a)\geqslant 1+a$ for all $a\in\mathbf{No}$ as a consequence of Section 11.1.1. It follows that the reciprocal log of exp satisfies all the axioms in the definition of transserial fields. In other words $(\mathbf{No},\log)$ is a transserial field with a total exponential. It was later shown [96] that $(\mathbf{No},+,\times,<,\exp)$ is an elementary expansion of $\mathbb{R}$.

Since exp is total and $\mathbf{No}_{\succ}$ is closed under scalar multiplication by real numbers, the identity (11.1.4) gives a real powering operation

$$\begin{aligned}\mathbb{R}\times\mathbf{Mo} &\longrightarrow \mathbf{Mo}\\ (r,\mathfrak{m}) &\longmapsto \mathfrak{m}^r=\exp(r\log\mathfrak{m})\end{aligned}$$

on $\mathbf{Mo}$. Now consider the hyperserial skeleton $(\mathbf{No},\log\upharpoonright\mathbf{Mo}^{\succ})$. That exp is total also implies that the product axiom $\mathbf{P}_\mu$ for $\mu=1$ is satisfied. Thus $(\mathbf{No},\log\upharpoonright\mathbf{Mo}^{\succ})$ is a hyperserial skeleton of force $(1,1)$. It was shown by Berarducci and Mantova [18, Corollary 5.11] that $(\mathbf{No},\log\upharpoonright\mathbf{Mo}^{\succ})$ is confluent. Therefore

Proposition 11.1.2. $(\mathbf{No},\log\upharpoonright\mathbf{Mo}^{\succ})$ *is a confluent hyperserial skeleton of force* $(1,1)$.

We will next give more details on log-atomic surreal numbers.

11.1.3 Exponential groups

Recall that we write

$$\begin{aligned}\exp_n &:= \exp^{[n]}=\exp\circ\overset{n\times}{\cdots}\circ\exp\\ \log_n &:= \log^{[n]}=\log\circ\overset{n\times}{\cdots}\circ\log\end{aligned}$$

for all $n\in\mathbb{N}$. We define

$$\mathcal{E}^*:=\langle\exp\rangle \qquad\text{and}\qquad \mathcal{E}:=\langle\exp_n\circ H_r\circ\log_n : r\in\mathbb{R}^{>}, n\in\mathbb{N}\rangle.$$

Both $\mathcal{E}^*$ and $\mathcal{E}$ act in particular on $\mathbf{No}^{>,\succ}$.

Halos $\mathcal{E}[x]$ and $\mathcal{E}^*[x]$ for the actions of $\mathcal{E}$ and $\mathcal{E}^*$ on $\mathbf{No}^{>,\succ}$ are sometimes called *logarithmic-exponential* and *exponential classes* respectively. The class $\mathbf{Smp}_{\mathcal{E}}$ of $\mathcal{E}$-simple is parametrized by the λ-map: see [18, Section 5]. The class of $\mathcal{E}^*$-simple elements is denoted by $\mathbf{K}$ and parametrized by the κ-map: see [71, Section 3]. Given $z \in \mathbf{No}$, one traditionally writes $\lambda_z := \Xi_{\mathcal{E}} z$ and $\kappa_z := \Xi_{\mathcal{E}^*} z$.

It was shown by Berarducci and Mantova [18] that $\mathbf{Smp}_{\mathcal{E}}$ coincides with the class $\mathbf{Mo}_\omega$ of *log-atomic* numbers, which we recall consists of those infinite monomials $\mathfrak{m} \in \mathbf{Mo}^{\succ}$ such that $\log_n \mathfrak{m} \in \mathbf{Mo}$ for all $n \in \mathbb{N}$. Such numbers were essential for the definition of well-behaved formal derivations on $\mathbf{No}$. This was first achieved in [18], while building on analogue results in the context of transseries [92, 60].

The structure $\mathbf{K} = \mathbf{Smp}_{\mathcal{E}^*}$ of κ-numbers was introduced and studied in detail in [71], as an intermediate subclass between fundamental monomials and the log-atomic numbers. It turns out that the structure $\mathbf{K}$ is not big enough to describe all log-atomic numbers. Indeed, it was noticed in [78] that $\mathbf{K} = \mathbf{Mo}_\omega \prec \mathbf{No}_\succ$, as a corollary of [6, Proposition 2.5].

Proposition 11.1.3. [6, Proposition 2.5] *For all $z \in \mathbf{No}$, we have*

$$\exp(\lambda_z) \;=\; \lambda_{z+1}$$

Proof. We rely on the following uniform version of [18, Theorem 3.8(1)] from [6, Lemma 2.4]: if $\mathfrak{m} = \{L \mid R\}$ is a monomial, where $\mathbb{R}\, L \subseteq \mathbf{Hull}(L)$ and $\mathbb{R}\, R \subseteq \mathbf{Hull}(R)$, then

$$\exp(\mathfrak{m}) = \{\mathfrak{m}^{\mathbb{N}}, \exp(L) \mid \exp(R)\}.$$

In fact, we have $\mathcal{P} \subseteq \mathcal{E} < \{\exp\}$ on $\mathbf{No}^{>,\succ}$, so $\exp(\mathfrak{m}) > \mathcal{E}\,\mathfrak{m} \supseteq \mathfrak{m}^{\mathbb{N}}$, and

$$\exp(\mathfrak{m}) = \{\mathcal{E}\,\mathfrak{m}, \exp(L) \mid \exp(R)\}. \tag{11.1.5}$$

Now let z be a number with $\lambda_{u+1} = \exp(\lambda_u)$ for all $u \in z_{\sqsubset}$. Then $z + 1 = \{z, z_L + 1 \mid z_R + 1\}$. The uniformity of the cut equation for the λ-map thus yields

$$\begin{aligned}
\lambda_{z+1} &= \{\mathcal{E}\lambda_z, \mathcal{E}\lambda_{z_L+1} \mid \mathcal{E}\lambda_{z_R+1}\} \\
&= \{\mathcal{E}\lambda_z, \mathcal{E}\exp(\lambda_{z_L}) \mid \mathcal{E}\exp(\lambda_{z_R})\} \\
&= \{\mathcal{E}\lambda_z, \exp\circ\mathcal{E}\lambda_{z_L} \mid \exp\circ\mathcal{E}\lambda_{z_R}\} && \text{(since } \exp\circ\mathcal{E} = \mathcal{E}\circ\exp) \\
&= \exp\lambda_z && \text{(by (11.1.5))}
\end{aligned}$$

The result follows by induction. □

Corollary 11.1.4. [18] *$\mathbf{Smp}_{\mathcal{E}}$ coincides with the class $\mathbf{Mo}_\omega$ of log-atomic surreal numbers.*

Proof. We have $\log_n \lambda_z = \lambda_{z-n} \in \mathbf{Smp}_{\mathcal{E}}$ for all $n \in \mathbb{N}$, whence $\log_n \mathbf{Smp}_{\mathcal{E}} \subseteq \mathbf{Smp}_{\mathcal{E}} \subseteq \mathbf{Mo}$. This shows that every element of $\mathbf{Smp}_{\mathcal{E}}$ is log-atomic.

Conversely, let λ be a log-atomic number and assume $\lambda \notin \mathbf{Smp}_{\mathcal{E}}$. Note that $\pi_{\mathcal{E}}(\lambda)$ is log-atomic by our previous argument. Assume for instance that $\pi_{\mathcal{E}}(\lambda) < \lambda$. For $n \in \mathbb{N}$, we have $\log_n \pi_{\mathcal{E}}(\lambda) \neq \log_n \lambda$. Since both $\log_n \lambda$ and $\log_n \pi_{\mathcal{E}}(\lambda)$ are monomials, it follows that $\log_n \pi_{\mathcal{E}}(\lambda) \prec \log_n \lambda$. We deduce that $(\exp_n \circ \mathcal{H} \circ \log_n)(\pi_{\mathcal{E}}(\lambda)) < \lambda$, whence $\mathcal{E}\pi_{\mathcal{E}}(\lambda) < \lambda$, which contradicts the defining relation $\pi_{\mathcal{E}}(\lambda) =_{\mathcal{E}} \lambda$. Likewise, $\pi_{\mathcal{E}}(\lambda) > \lambda$ is impossible. We conclude that $\lambda = \pi_{\mathcal{E}}(\lambda) \in \mathbf{Smp}_{\mathcal{E}}$. □

Proposition 11.1.5. [78] *We have $\mathbf{K} = \mathbf{Mo}_\omega \prec \mathbf{No}_\succ$.*

Proof. Following Mantova-Matusinski, we have the following equivalences for any number $z \in \mathbf{No}$:

$$\begin{aligned}
z \in \mathbf{No}_\succ &\iff z_L + \mathbb{N} < z < z_R - \mathbb{N} \\
&\iff \exp_{\mathbb{N}}(\lambda_{z_L}) < \lambda_z < \log_{\mathbb{N}}(\lambda_{z_R}) \\
&\iff \exp_{\mathbb{N}}(\mathcal{E}\lambda_{z_L}) < \lambda_z < \log_{\mathbb{N}}(\mathcal{E}\lambda_{z_R}) \\
&\iff \mathcal{E}^*(\lambda_{z_L}) < \lambda_z < \mathcal{E}^*(\lambda_{z_R}) \\
&\iff \lambda_z \in \mathbf{K}.
\end{aligned}$$

□

11.2 Numbers as transseries

We conclude Part III by discussing the benefits and shortcomings of representing surreal numbers as transseries.

11.2.1 Iterated transserial expansions

In **No**, any non-zero number $a_0 = \sum_{\mathfrak{m}} (a_0)_{\mathfrak{m}} \mathfrak{m}$ can be expanded as we next explain. Pick one of its terms $(a_0)_{\mathfrak{m}_0} \mathfrak{m}_0$, write $r_0 := (a_0)_{\mathfrak{m}_0}$ and $a_1 := \log \mathfrak{m}_0 \in \mathbf{No}$, and let φ_0, δ_0 respectively denote the segments of the series a_0 lying above and below $r_0 \mathfrak{m}_0$. Systematically placing terms in series from greatest to least, we obtain the expression

$$a_0 = \varphi_0 + r_0 \, \mathrm{e}^{a_1} + \delta_0.$$

Repeating this process by picking a term $r_1 \mathfrak{m}_1 = r_1 \mathrm{e}^{a_2}$ in a_1, we get

$$a_0 = \varphi_0 + r_0 \, \mathrm{e}^{\varphi_1 + r_1 \mathrm{e}^{a_2} + \delta_1} + \delta_0.$$

Repeating this iteratively, we obtain a series of expressions

$$a_0 = \varphi_0 + r_0 \, \mathrm{e}^{\varphi_1 + r_1 \mathrm{e}^{\cdot^{\cdot^{\cdot \varphi_i + r_i \mathrm{e}^{\cdot^{\cdot^{\cdot}}} + \delta_i}}} + \delta_1} + \delta_0. \tag{11.2.1}$$

We thus form what is called an infinite path $P = (r_i \, \mathfrak{m}_i)_{i \in \mathbb{N}}$ in a_0 as in [92, 18]. One can then study conditions under which formal expressions such as (11.2.1) correspond to numbers.

11.2.2 Schmeling's axiom T4

It was shown by Berarducci and Mantova [18, Theorem 8.10] that **No** satisfies Schmeling's axiom **T4** [92, Definition 2.2.1]. This means that the sequences $(\varphi_i)_{i \in \mathbb{N}}$, $(r_i)_{i \in \mathbb{N}}$ and $(\delta_i)_{i \in \mathbb{N}}$ occurring in (11.2.1) must verify that there exist an $i_0 \in \mathbb{N}$ with $r_i \in \{-1, 1\}$ and $\delta_i = 0$ whenever $i \geqslant i_0$. In other words, the expansion process in (11.2.1) must eventually yield expressions of the form

$$a_{i_0} = \varphi_{i_0} \pm \mathrm{e}^{\varphi_{i_0+1} \pm \mathrm{e}^{\cdot^{\cdot^{\cdot \varphi_{i_0+j} \pm \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}}. \tag{11.2.2}$$

The axiom **T4** is in fact defined in the more general context of a transserial field $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ with its logarithm log. It was introduced as a way to avoid certain problems in defining structure on transserial fields. Indeed, the condition on the coefficients r_i's avoids certain problem in comparing numbers which have distinct infinite expansions such as (11.2.2) (see [92, Section 2.5]), whereas the condition on the tails δ_i's avoids the existence of certain ill-based families occurring as derivatives of such numbers (see [11, p. 49]).

11.2.3 Nested expansions

The converse problem is the existence of numbers with a given expansion of the form (11.2.2). In [11, Section 8], we give conditions under which such numbers exist and actually form a proper subclass of **No**. This is in particular the case for the sequences with $\varphi_i = \sqrt{\log_i \omega}$, $r_i = 1$ and $\delta_i = 0$ for all $i \in \mathbb{N}$. This yields many numbers with the expansion

$$a = \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot \sqrt{\log_i \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}}. \tag{11.2.3}$$

In the general expansion process, note that if any of the terms $r_i \, \mathfrak{m}_i = r_i \, \mathrm{e}^{a_{i+1}}$ that we choose is a real number, then we have $a_{i+1} = 0$, so the expansion stops. On the contrary, if $a_0 = \mathfrak{a}$ is log-atomic (for instance if $\mathfrak{a} = \omega$ or $\mathfrak{a} = \varepsilon_0$), then the only possible continuation of the path is with $\forall k > 0, a_k = \log_k \mathfrak{a}$. This corresponds to the sequences $\varphi_i = 0$, $r_i = 1$ and $\delta_i = 0$ for all $i \in \mathbb{N}$, i.e. to the expansion

$$\mathfrak{a} = \mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}}. \tag{11.2.4}$$

Such expansion gives no information on $\mathfrak{a}$ except that it is log-atomic. For that reason, it is not necessary or useful to expand $\mathfrak{a}$ further.

11.2.4 Expansions and derivations

Consider (11.2.1) and assume that a well-behaved partially defined derivative $D: b \mapsto b'$ (see also [18, Definition 9.1]), is defined for all the terms φ_i, δ_i occurring in (11.2.1). Taking the compatibility relation $\partial_s(\exp b) = (\partial_s(b)) \exp b$ into account, we see that this expression admits a "syntactic derivative" $\partial_s(a_0)$ given by

$$\begin{aligned} \partial_s(a_0) &= \varphi_0' + r_0\, \mathfrak{m}_0\, \partial_s(a_1) + \delta_0' \\ &= \varphi_0' + r_0\, \mathfrak{m}_0\, (\varphi_1' + r_1\, \mathfrak{m}_1\, \partial_s(a_2) + \delta_1') + \delta_0' \\ &\vdots \quad \cdots \\ &= \varphi_0' + r_0\, \mathfrak{m}_0\, \varphi_1' + r_0\, r_1\, \mathfrak{m}_0\, \mathfrak{m}_1\, \varphi_2' + \cdots + \delta_0' + r_0\, \mathfrak{m}_0\, \delta_1' + \cdots. \end{aligned}$$

This expression is in addition the simplest one which extends D and which may be extended into a transserial derivation. Since **No** satisfies **T4**, we may assume that we have $r_i \in \{-1, 1\}$ and $\delta_i = 0$ for all $i \in \mathbb{N}$. Crucially [18, Proposition 6.20], this turns the syntactic derivative $\partial_s(a_0)$ into a well-defined sum

$$\partial_s(a_0) = \varphi_0 \pm \mathfrak{m}_0\, \varphi_1' \pm \mathfrak{m}_0\, \mathfrak{m}_1\, \varphi_2' \pm \cdots.$$

One complication occurs when applying this method to the expansion (11.2.4). Indeed this systematically yields a syntactic derivative $\partial_s(\mathfrak{a}) = 0$, although $\mathfrak{a} \notin \mathbb{R}$. Therefore D should already be defined at each $\mathfrak{a} \in \mathbf{Mo}_\omega$. Berarducci and Mantova found minimal conditions on $D: \mathbf{Mo}_\omega \longrightarrow \mathbf{No}$; $\mathfrak{a} \mapsto \mathfrak{a}'$ for the resulting derivation ∂_s to be transserial. They constructed their derivation by relying on the "simplest" solution D.

The class $\mathbf{Mo}_\omega$ is proper class-sized and its apparent complexity seems to mirror that of **No** itself (see for instance [18, Propositions 5.15, 5.17 and Corollary 5.17] and [9, Theorem 29]). Defining a suited mapping D on $\mathbf{Mo}_\omega$ thus requires insight. We think that Berarducci and Mantova's choice does not fit into a coherent description of surreal numbers as (surreal-valued) functions, as is highlighted in [19, Theorem 8.4]. We expect that a different approach relying on the hyperserial structure on **No** will yield a suited derivation.

Part IV

Numbers as hyperseries

Seeing the forest for the tree

We have seen in Part III that starting from a complete binary tree $\{-1,1\}^{<\mathbf{On}} \simeq \mathbf{No}$, it was possible to define a structure of transseries field in a canonical way. In Part IV we will, in more detail, show that surreal numbers can be represented as formal hyperseries f "applied" at the number ω. Pending a definition of what a hyperseries is, this statement is informal, so in order to make sense of it, we will introduce the notion of hyperserial description. We will show that numbers can be represented purely in terms of hyperseries, i.e. as (non-rooted) trees or forests indexed by ordinal and real numbers.

A never-withering gardening metaphor

Let us describe our results by employing a gardening metaphor. In the first two Parts of the thesis, our main task was to construct large fields of hyperseries by adjoining formal monomials to more classical fields of transseries. We will now be working in the fixed ambient field **No**, seen as the full binary tree $(\mathbf{No}, <, \sqsubset)$ with its algebraic and transserial structure. In this barren land, we are to find a way to see a natural structure of hyperserial field, with the additional goal that this structure should be sufficient to represent all surreal numbers as hyperseries, and be amenable to derivations and composition laws. This implies sawing the seeds, making sure a steady growth is possible, cutting down unwanted sprouts, before we can collect the fruit of our work. The end goal of representing numbers as forests indexed by ordinals and real numbers is but a first step in the program of defining a derivation and a composition law on **No** that will be compatible with the hyperserial structure and with one another. Unfortunately, this program does not fit in the present thesis, and we will have to be content with this modest gardening task.

Atomic seed, hyperlogarithmic-hyperexponential flourish

The first stage will be to define a confluent hyperserial skeleton $(L_{\omega^\mu})_{\mu \in \mathbf{On}}$ of force $(\mathbf{On}, \mathbf{On})$ on **No**. In this task, we are guided by the simplicity heuristic and the axioms for hyperserial skeletons, which will give us a "simplest" way of defining those partial functions. This will occupy us for Chapter 12.

Having done that, we obtain by Theorem 7.2.10 a hyperserial field $(\mathbf{No}, \circ)$ of force $(\mathbf{On}, \mathbf{On})$. This gives us the basis for our hyperserial representation process: we have the simplest positive infinite number ω, real numbers, and hyperlogarithmic and hyperexponential operators $L_\alpha : a \mapsto L_\alpha(a)$ and $E_\alpha : a \mapsto E_\alpha(a)$ as tools to construct involved hyperseries with simpler ones. In fact we will see that ω is the only atomic number in $(\mathbf{No}, \circ)$, whence there is a unique embedding $\tilde{\mathbb{L}} \longrightarrow \mathbf{No}$ of force **On**, whose range is denoted $\tilde{\mathbb{L}} \circ \omega$. Crucially, surreal numbers are a proper extension of $\tilde{\mathbb{L}} \circ \omega$. The difference between $\tilde{\mathbb{L}} \circ \omega$ and **No** lies in the existence of nested numbers in **No**. Studying those numbers is the main task undertaken in the last two chapters of the thesis.

Hyperserial expansions

We saw in Section 11.2 that the representation of numbers as transseries was insufficient in describing log-atomic numbers. Indeed those expand as

$$\mathfrak{a} = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}, \tag{1}$$

and such expansion gives no information on how the derivative of $\mathfrak{a}$ should be defined, or on how $\mathfrak{a}$ should behave on a surreal valued function $b \mapsto \mathfrak{a} \circ b$ if a composition law $\circ : \mathbf{No} \times \mathbf{No}^{>, \succ} \longrightarrow \mathbf{No}$ is to be defined on surreal numbers.

A naive solution to the problem of expansions (1) would be to systematically rewrite (1) as $\mathfrak{a} = E_\alpha^\varphi$ for a certain additively indecomposable ordinal $\alpha \geqslant \omega$ and a certain α-truncated number φ. We can assume that $\mathfrak{a} \neq \omega$, for it is intended that the simple number ω should not be decomposed. Thus α can be chosen largest such that $\mathfrak{a} \in \mathbf{Mo}_\alpha$. We could extend this rule to general non log-atomic monomials $\mathfrak{m}$ by expanding them as

$$\mathfrak{m} = E_1^\psi \tag{2}$$

for $\psi = \log \mathfrak{m}$. Unfortunately, this representation method is not practical for reasons which we next explain.

Ideally, there should exist a measure $\varsigma : \mathbf{No} \longrightarrow \mathbf{On}$ of the "complexity" of surreal numbers seen as hyperseries, so that when expressing $\mathfrak{m} = E_\alpha^\varphi$, we would have $\varsigma(\varphi) < \varsigma(\mathfrak{m})$. Indeed it would then be possible to define, for instance, well-behaved derivatives $\partial(\mathfrak{m})$ and compositions $\mathfrak{m} \circ b$ for $b \in \mathbf{No}^{>, \succ}$ by induction on this complexity. We will see that the existence of (infinitely) nested numbers, which can be seen as an unavoidable consequence of the fundamental property (Proposition 8.1.1) forbids the existence of such a function ς.

The next best thing then is to find a way to expand monomials $\mathfrak{m}$ in such a way that the complexities $\varsigma(a_1), \ldots, \varsigma(a_n)$ of the surreal parameters $a_1, \ldots, a_n$ involved in the expansion are strictly smaller than $\varsigma(\mathfrak{m})$ unless $\mathfrak{m}$ turns out to be a nested number. We expect that this is indeed possible if we expand monomials using two surreal parameters ψ, u, as

$$\mathfrak{m} = \mathrm{e}^\psi \, (L_\beta \, E_\alpha^u)^\iota$$

where α, β are ordinals, $\iota \in \{-1, 1\}$, and the tuple $(\psi, \beta, \alpha, u, \iota)$ satisfies a list of technical conditions (see Definition 13.1.2). For $\mathfrak{n} = \mathrm{e}^\varphi \, (L_\gamma \, E_\rho^v)^\sigma$ as above, if $\psi = \varphi$ and $\iota = \sigma$ then we have

$$L_\beta \, E_\alpha^u \sqsubset L_\gamma \, E_\rho^v \Longrightarrow \mathfrak{m} \sqsubset \mathfrak{n}.$$

Under certain conditions (see Lemma 13.1.25), if $(\rho, \gamma) = (\alpha, 0)$, then we have

$$u \sqsubset v \Longrightarrow \mathfrak{m} \sqsubset \mathfrak{n}.$$

Thus simplicity relations between monomials written in this form can be read in simplicity relations between the parameters of the respective expansions, which is not the case for the "naive representation" (2).

As explained in Section 11.2, we have a related notion of path P as a sequence of terms. A path in $a \in \mathbf{No}^{\neq}$ is a possibly infinite sequence $P(0), P(1), \ldots$ of terms where $P(0) \in \operatorname{term} a$ and expanding each monomial $\mathfrak{d}_{P(i)}$ as $\mathfrak{d}_{P(i)} = \mathrm{e}^{\psi_{P,i+1}} \, (L_{\beta_{P,i}} \, E_{\alpha_{P,i}}^{u_{i+1}})^{\iota_{P,i}}$, the number $P(i+1)$ if it is defined is a term in $\psi_{P,i+1}$ or in $u_{P,i+1}$. Our first task toward studying nested numbers is to understand hyperserial expansions and paths.

Ad *infinitum*

Having waited **On**-many days, we come back to our garden and find it sprouting. Not only do we have trees with arbitrarily long finite branches that can already be found in $\tilde{\mathbb{L}} \circ \omega$, but we also possibly have infinite branches, that is, infinite paths $(P_i)_{i \in \mathbb{N}}$ in certain numbers a. The same problems mentioned in Section 11.2 for the transserial case occur in the hyperserial case. More precisely, the existence of arbitrary infinite paths would be problematic. Similarly to the axiom **T4**, surreal numbers satisfy a structure theorem, namely Theorem 13.2.7, that states that for all infinite paths, there must exist an $i_0 \in \mathbb{N}$ such that for all $i \geqslant i_0$, the coefficient $r_{P,i}$ of the term $P(i)$ is in $\{-1, 1\}$, the monomial $\mathfrak{d}_{P(i+1)}$ is the minimum of $\operatorname{supp} u_{P,i}$, and $\beta_{P,i} = 0$. Assume for a moment that $i_0 = 0$. Writing $(\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i) = (a_{\succ \mathfrak{d}_{P(i)}}, r_{P,i}, \psi_{P,i+1}, \iota_{P,i}, \alpha_{P,i})$ for all $i \in \mathbb{N}$, we have an infinite nested expansion

$$a = \varphi_0 + \!\!\!+ \, \varepsilon_0 \, \mathrm{e}^{\psi_0} \left(E_{\alpha_0}^{\varphi_1 + \!\!\!+ \, \varepsilon_1 \mathrm{e}^{\psi_1} \left(E_{\alpha_1}^{\cdot^{\cdot^{\cdot}}} \right)^{\iota_1}} \right)^{\iota_0} \tag{3}$$

for a. The main goal of Chapter 13 will be to trim the possible infinite branches in iterated hyperserial expansions of surreal numbers, by proving Theorem 13.2.7.

Reaping what you saw

The problem of existence of infinite paths in **No** is more subtle than it looks and raises several questions. Consider the expansion (3) above and the corresponding sequence $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$. We say that a is Σ-nested. On what condition on Σ can there be numbers which expand as (3), how many such numbers exist, and how can those numbers be distinguished?

There are three stages to addressing these questions. The first one is formal: the numbers $\varphi_i, \varepsilon_i, \psi_i, \iota_i$ and α_i should satisfy certain conditions (for instance that each ψ_i be purely large, see Definition 14.1.1) that reflect the fact that they occur in the process of iterated hyperserial expansions. Sequences which satisfy those conditions are called *coding sequences*.

On a second level, having an expansion such as (3) implies in particular some order theoretic constraints on the parameters. For instance the existence of a implies that there is a positive infinite number b with $\log b \prec \operatorname{supp} \psi_0$ and such that $\operatorname{supp} \varphi_0 \succ \mathrm{e}^{\psi_0} b^{\iota_0}$. This translates into inequalities involving the parameters φ_0, ψ_0 and ι_0 (see Section 14.1.2). Coding sequences which satisfy all such inequalities are said *admissible*.

It turns out that being admissible is not sufficient to guarantee the existence of corresponding nested numbers. This is why we introduce a further condition on (admissible) coding sequences that roughly states that if given $i \in \mathbb{N}$, a number c expands as

$$c = \varphi_{i+1} + \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} \left(E_{\alpha_{i+1}}^{\varphi_{i+2} + \varepsilon_{i+2} \mathrm{e}^{\psi_{i+2}} \left(E_{\alpha_{i+2}}^{\cdot^{\cdot^{\cdot}}} \right)^{\iota_{i+2}}} \right)^{\iota_{i+1}}, \tag{4}$$

then the number $d := \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} (E_{\alpha_i} c)^{\iota_i}$ expands as

$$d = \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i} \left(E_{\alpha_i}^{\varphi_{i+1} + \varepsilon_{i+1} \mathrm{e}^{\psi_{i+1}} \left(E_{\alpha_{i+1}}^{\cdot^{\cdot^{\cdot}}} \right)^{\iota_{i+1}}} \right)^{\iota_i}.$$

This is not a trivial condition, since the expansion (4) says nothing about φ_i, ψ_i, about the fact that c be α_i-truncated, and so on... Admissible sequences that satisfy this further condition are called *nested sequences*. Nested sequences are particularly well behaved. In particular, we will prove Theorem 14.2.4 which states that the class of corresponding Σ-nested numbers is a surreal substructure. That there are plenty of Σ-nested numbers provided Σ is a nested sequence is, we expect, a key feature in defining composition laws on surreal numbers.

It is sufficient to study nested sequences in order to represent surreal numbers as hyperseries because of a key result in Section-14.1.3 that states that if Σ is admissible, then for large enough $k \in \mathbb{N}$, the sequence $\Sigma_{\nearrow k} := (\varphi_{k+i}, \varepsilon_{k+i}, \psi_{k+i}, \iota_{k+i}, \alpha_{k+i})_{i \in \mathbb{N}}$ is nested (Theorem 14.1.15).

Mapping the territory

Having studied the existence and multiplicity of nested numbers, we can finally represent surreal numbers as hyperseries. In order to do this, we use labelled forests, where each label is a tuple $(r, \iota, \alpha, \beta)$ that corresponds to the coefficient $r \in \mathbb{R}^{\neq}$ of a monomial $\mathfrak{m}$ in the support of a number and the parameters in its hyperserial expansion $\mathfrak{m} = \mathrm{e}^{\psi} (L_\beta E_\alpha^u)^\iota$. That the numbers ψ and u vanish by giving rise to further branches in the forest is the gist of the formal representation.

Representing numbers as hyperseries is thus a matter of introducing the right graph theoretic structure that can be uniquely ascribed to a surreal number. We will show that such a correspondence exists by proving Theorem 15.3.1.

Chapter 12

The hyperserial field of surreal numbers

Let us define a hyperserial skeleton $(\mathbf{No}, (L_{\omega^\mu})_{\mu \in \mathbf{On}})$ on surreal numbers. The task ahead of us is straightforward: we have a list of axioms for hyperserial skeletons (see Section 4.2.2), and we must define partial hyperlogarithms L_{ω^μ}, $\mu \in \mathbf{On}$ that satisfy those axioms. Starting with Gonshor's logarithm log, the class $\mathbf{Mo}_\omega$ is already defined, and characterized by Berarducci and Mantova's work as a surreal substructure. Let us see how this can be used to give an inductive definition of L_ω on $\mathbf{Mo}_\omega$. This particular case was first dealt with in [15].

Let $\mathfrak{a} \in \mathbf{Mo}_\omega$ such that L_ω is defined on $\mathfrak{a}_\sqsubset^{\mathbf{Mo}_\omega}$, and let us see how $L_\omega(\mathfrak{a})$ can be defined. Let $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\omega}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\omega}$. For $n \in \mathbb{N}$, the monotonicity axiom $\mathbf{M}_\mu$ at $\mu = 1$ gives

$$L_\omega(\mathfrak{a}') + \frac{1}{\log_n L_\omega(\mathfrak{a}')} < L_\omega(\mathfrak{a}) < L_\omega(\mathfrak{a}'') - \frac{1}{\log_n L_\omega(\mathfrak{a}'')}.$$

The asymptotics axiom $\mathbf{A}_\mu$ at $\mu = 1$ gives $L_\omega(\mathfrak{a}) < \log_n \mathfrak{a}$. Therefore $L_\omega(\mathfrak{a})$ should lie in the cut

$$\left(\mathbb{R} \cup \left\{L_\omega(\mathfrak{a}') + \frac{1}{\log_n L_\omega(\mathfrak{a}')} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\omega}\right\} \mid \left\{L_\omega(\mathfrak{a}'') - \frac{1}{\log_n L_\omega(\mathfrak{a}'')} : \mathfrak{a}'' \in \mathfrak{a}_L^{\mathbf{Mo}_\omega}\right\} \cup \log_{\mathbb{N}} \mathfrak{a}\right). \tag{1}$$

We will see that defining $L_\omega(\mathfrak{a})$ to be the simplest element of this cut yields a function L_ω: $\mathbf{Mo}_\omega \longrightarrow \mathbf{No}^{>,\succ}$ which satisfies all the axioms for hyperserial skeletons at $\mu = 1$.

There are subtle obstacles in making this method work. The first one is that it is not easy to prove by induction that (1) is indeed always a cut. In fact, we will need to define L_ω in a slightly different way before we can prove that (1) is always a cut with root $L_\omega(\mathfrak{a})$. The second one is that this definition via cut presentations is suited to impose strict inequalities as constraints on L_ω, but less so to insure that the remaining axioms for hyperserial skeletons hold. The product axiom $\mathbf{P}_\mu$ for any ordinal μ follows from the fact that log: $\mathbf{No}^> \longrightarrow \mathbf{No}$ is surjective. So we are left with two axioms that cannot seemingly be translated into sets of inequalities. Fortunately, we will see that they follow from the above definition. In other words, the functional equation $\mathbf{FE}_\mu$ and the regularity axiom $\mathbf{R}_\mu$ are consequences of the choice of simplest value for $L_\omega(\mathfrak{a})$ satisfying the above constraints.

For general infinie additively indecomposable ordinals α, we have a similar definition. Indeed if $\mathfrak{a} \in \mathbf{No}^{>,\succ}$ is $L_{<\alpha}$-atomic, then $L_\alpha \mathfrak{a}$ can be defined using the fairly simple recursive formula

$$L_\alpha(\mathfrak{a}) \ := \ \{\mathbb{R}, L_\alpha(\mathfrak{a}') + (L_\gamma(\mathfrak{a}'))^{-1} \mid L_\alpha(\mathfrak{a}'') - (L_\gamma(\mathfrak{a}))^{-1}, L_\gamma(\mathfrak{a})\}, \tag{2}$$

where $\mathfrak{a}', \mathfrak{a}''$ range over $L_{<\alpha}$-atomic numbers with $\mathfrak{a}', \mathfrak{a}'' \sqsubset \mathfrak{a}$ and $\mathfrak{a}' < \mathfrak{a} < \mathfrak{a}''$ and γ ranges in α; see also (12.3.1).

We prove that this definition is warranted and that the resulting functions L_α satisfy the axioms of hyperserial skeletons from [14, Section 3]. Our proof proceeds by induction on α and also relies on the fact that the class $\mathbf{Mo}_\alpha$ of $L_{<\alpha}$-atomic numbers actually forms a surreal substructure of $\mathbf{No}$. Our main result is the following rewriting of Theorem **C**.

Theorem 1. *The definition* (2) *gives* $\mathbf{No}$ *the structure of a confluent hyperserial skeleton. Consequently, we may uniquely extend* L_{ω^μ} *to* $\mathbf{No}^{>,\succ}$ *in a way that gives* $\mathbf{No}$ *the structure of a confluent hyperserial field. Moreover, for each ordinal* μ*, the extended function* L_{ω^μ}: $\mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ *is bijective.*

12.1 Inductive setting

Here, we make precise our induction hypotheses for the proof of Theorem 1 throughout Chapter 12.

12.1.1 Exponential-logarithmic groups

For $c \in \mathbb{R}$ and $r \in \mathbb{R}^{>}$, we define

$$\begin{aligned} T_r &:= a \longmapsto a + c && \text{acting on } \mathbf{No} \text{ or } \mathbf{No}^{>,\succ}. \\ H_c &:= a \longmapsto r\,a && \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ}. \\ P_c &:= a \longmapsto a^r && \text{acting on } \mathbf{No}^{>} \text{ or } \mathbf{No}^{>,\succ}. \end{aligned}$$

Now consider

$$\begin{aligned} \mathcal{T} &:= \{T_c : c \in \mathbb{R}\}, \\ \mathcal{H} &:= \{H_r : r \in \mathbb{R}^{>}\}, \\ \mathcal{P} &:= \{P_r : r \in \mathbb{R}^{>}\}, \\ \mathcal{E}' &:= \langle E_n\, H_r\, L_n : n \in \mathbb{N}, r \in \mathbb{R}^{>} \rangle, \text{ and} \\ \mathcal{E}^{*} &:= \{E_n, L_n : n \in \mathbb{N}\}. \end{aligned}$$

Then we have the following list of correspondences $\mathcal{G} \longmapsto \mathbf{Smp}_{\mathcal{G}}$:

- The action of $\mathcal{T}$ on $\mathbf{No}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{No}_{\succ}$ (resp. $\mathbf{No}_{\succ}^{>}$), e.g. $\mathbf{Smp}_{\mathcal{T}} = \mathbf{No}_{\succ}$.
- The action of $\mathcal{H}$ on $\mathbf{No}^{>}$ (resp. $\mathbf{No}^{>,\succ}$) yields $\mathbf{Mo}$ (resp. $\mathbf{Mo}^{\succ}$).
- The action of $\mathcal{P}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{Mo}{\prec}\mathbf{Mo} = E_1\,\mathbf{Mo}^{\succ}$.
- The action of $\mathcal{E}'$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{Mo}_{\omega}$.
- The action of $\mathcal{E}^{*}$ on $\mathbf{No}^{>,\succ}$ yields $\mathbf{K} := \mathbf{Mo}_{\omega}{\prec}\mathbf{No}_{\succ}$ (which will coincide with $E_{\omega}\,\mathbf{No}_{\succ}^{>}$).

Generalizations of those function groups will allow us to define certain surreal substructures related to the hyperlogarithms and hyperexponentials on $\mathbf{No}$.

We have seen in Section 11.1.2 that $(\mathbf{No}, L_1)$ is a confluent hyperserial skeleton of force (ν, ν) for $\nu = 1$. The aim of this section is to extend this result to any ordinal ν. More precisely, we will define a sequence $(L_{\omega^{\mu}})_{\mu \in \mathbf{On}}$ of partial functions on $\mathbf{No}$ such that for each ordinal ν, the structure $(\mathbf{No}, (L_{\omega^{\mu}})_{\mu < \nu})$ is a confluent hyperserial skeleton of force (ν, ν), and L_1 coincides with Gonshor's logarithm.

12.1.2 Hyperexponential-hyperlogarithmic groups

Assume for the moment that we can define L_{γ} and E_{γ} as bijective strictly increasing functions on $\mathbf{No}^{>,\succ}$ for all ordinals γ. This is the case already for $\gamma < \omega$. Let us introduce several useful groups that act on $\mathbf{No}$, as well as several remarkable subclasses of $\mathbf{No}$.

Given an ordinal ν, we set

$$\alpha := \omega^{\nu},$$

and we consider the function groups

$$\begin{aligned} \mathcal{E}'_{\alpha} &= \langle E_{\gamma}\, H_r\, L_{\gamma} : \gamma < \alpha \wedge r \in \mathbb{R}^{>} \rangle \\ \mathcal{E}^{*}_{\alpha} &= \langle E_{\gamma}, P_r : \gamma < \alpha \wedge r \in \mathbb{R}^{>} \rangle. \end{aligned}$$

where E_{γ}, H_s, P_s and L_{γ} act on $\mathbf{No}^{>,\succ}$. We also define

$$\begin{aligned} \mathcal{L}'_{\alpha} &= L_{\alpha}\, \mathcal{E}'_{\alpha}\, E_{\alpha} \\ \mathcal{L}^{*}_{\alpha} &= L_{\alpha}\, \mathcal{E}^{*}_{\alpha}\, E_{\alpha}. \end{aligned}$$

We write $L_{<\lambda} := \{L_{\gamma} : \gamma < \lambda\}$ and $E_{<\lambda} := \{E_{\gamma} : \gamma < \lambda\}$ for each $\lambda \leqslant \alpha$. In the case when $\alpha = 1$, note that

$$\begin{aligned} \mathcal{E}'_1 &= \mathcal{H} \\ \mathcal{E}^{*}_1 &= \mathcal{P} \\ \mathcal{L}'_1 &= \mathcal{T} \\ \mathcal{L}^{*}_1 &= \mathcal{H}. \end{aligned}$$

By Proposition 10.2.3 and the fact the set-sized function groups $\mathcal{E}'_\alpha$, $\mathcal{E}^*_\alpha$, $\mathcal{L}'_\alpha$, and $\mathcal{L}^*_\alpha$ induce thin partitions of $\mathbf{No}^{>,\succ}$, we may define the following surreal substructures

$$\begin{aligned}\mathbf{Mo}'_\alpha &:= \mathbf{Smp}_{\mathcal{E}'_\alpha}\\ \mathbf{Mo}^*_\alpha &:= \mathbf{Smp}_{\mathcal{E}^*_\alpha}\\ \mathbf{Tr}'_\alpha &:= \mathbf{Smp}_{\mathcal{L}'_\alpha}\\ \mathbf{Tr}^*_\alpha &:= \mathbf{Smp}_{\mathcal{L}^*_\alpha}.\end{aligned}$$

Here we note that $\mathbf{Mo}'_1$ corresponds to the class $\mathbf{Mo}^{\succ}=\mathbf{Mo}_1$ of infinite monomials in $\mathbf{No}$ and $\pi'_{\mathcal{E}_1}$ maps positive infinite numbers to their dominant monomial. Similarly, $\mathbf{Tr}'_1$ coincides with $\mathbf{No}^{>}_{\succ}$ and $\pi_{\mathcal{L}'_1}$ maps $a\in\mathbf{No}^{>,\succ}$ to $a_\succ$. In Section 12.3, we will prove the following identities.

$$\begin{aligned}
\mathbf{Mo}'_\alpha &= \mathbf{Mo}_\alpha, && \text{[Proposition 12.2.16]}\\
\pi_{\mathcal{E}'_\alpha} &= \mathfrak{d}_\alpha, && \text{[Proposition 12.2.16]}\\
\mathbf{Tr}'_\alpha &= \mathbf{No}_{\succ,\alpha}=L_\alpha\,\mathbf{Mo}_\alpha, && \text{[Proposition 12.3.6]}\\
\pi_{\mathcal{L}'_\alpha} &= \sharp_\alpha, && \text{[Proposition 12.3.6]}\\
\mathbf{Tr}^*_\alpha &= \mathbf{Tr}'_\alpha \quad \text{if } \nu \text{ is a limit ordinal}, && \text{[Lemma 12.2.9]}\\
\mathbf{Tr}^*_\alpha &= \mathbf{No}^{>}_{\succ} \quad \text{if } \nu \text{ is a successor ordinal}, && \text{[Lemma 12.3.8]}\\
\forall r\in\mathbb{R}, \Xi_{\mathbf{No}_{\succ,\alpha}}\,T_r &= T_r\,\Xi_{\mathbf{No}_{\succ,\alpha}} \quad \text{if } \nu \text{ is a successor ordinal}, && \text{[Lemma 12.3.7]}\\
\forall r\in\mathbb{R}, \Xi_{\mathbf{Mo}_\alpha}\,T_r &= E_\alpha\,T_r\,L_\alpha\,\Xi_{\mathbf{Mo}_\alpha} \quad \text{if } \nu \text{ is a successor ordinal}, && \text{[Proposition 12.3.10]}\\
\mathbf{Mo}^*_\alpha &= \mathbf{Mo}_\alpha\prec\mathbf{No}_\succ \quad \text{if } \nu \text{ is a successor ordinal}, && \text{[Proposition 12.3.12]}\\
\mathbf{Mo}^*_\alpha &= E_\alpha\,\mathbf{Tr}^*_\alpha. && \text{[Proposition 12.3.13]}
\end{aligned}$$

The first and third identities imply in particular that the classes $\mathbf{Mo}_\alpha$ and $\mathbf{No}_{\succ,\alpha}$ of $L_{<\alpha}$-atomic and α-truncated numbers respectively are in fact surreal substructures, when regarding $\mathbf{No}$ as a hyperserial field.

12.1.3 Induction hypotheses

For the definition of the partial hyperlogarithm L_{ω^μ}, we will proceed by induction on μ. Let μ be an ordinal. Until the end of this section we make the following induction hypotheses:

Induction hypotheses

$\mathbf{I}_{1,\mu}$. For $\eta<\mu$, the partial hyperlogarithm L_{ω^η} is defined on $\mathbf{Mo}_{\omega^\eta}$; we have $L_1=\log\restriction\mathbf{Mo}^{\succ}$ and $(\mathbf{No},(L_{\omega^\eta})_{\eta<\mu})$ is a confluent hyperserial skeleton of force (μ,μ).

$\mathbf{I}_{2,\mu}$. For $r,s\in\mathbb{R}$ with $1<s$ and for $\gamma,\rho<\omega^\mu$ with $\gamma<\rho$, we have

$$\forall a\in\mathbf{No}^{>,\succ},\quad E_\gamma(r\,L_\gamma\,a)<E_\rho(s\,L_\rho\,a).$$

$\mathbf{I}_{3,\mu}$. For $\eta\leqslant\mu$, the class $\mathbf{Mo}'_{\omega^\eta}$ is that of $L_{<\omega^\eta}$-atomic surreal numbers, i.e. $\mathbf{Mo}'_{\omega^\eta}=\mathbf{Mo}_{\omega^\eta}$.

These induction hypotheses require a few additional explanations. Assuming that $\mathbf{I}_{1,\mu}$ holds, the partial functions L_{ω^η} with $\eta<\mu$ extend into strictly increasing bijections $L_{\omega^\eta}\colon\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}^{>,\succ}$, by the results from Chapter 4. Using Cantor normal forms, this allows us to define a strictly increasing bijection $L_\gamma\colon\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}^{>,\succ}$ for any $\gamma<\mu$ and we denote by E_γ its functional inverse. In particular, this ensures that the hypotheses $\mathbf{I}_{2,\mu}$ and $\mathbf{I}_{3,\mu}$ make sense.

Remark 12.1.1. In addition to the above induction hypotheses, we will implicitly assume that our hyperlogarithms L_{ω^η} for $\eta<\mu$ are always defined by (12.2.1) below. In particular, our construction of L_{ω^μ} is *not* relative to any potential construction of the preceding hyperlogarithms L_{ω^η} with $\eta<\mu$ that would satisfy the induction hypotheses $\mathbf{I}_{1,\mu}$, $\mathbf{I}_{2,\mu}$, and $\mathbf{I}_{3,\mu}$. Instead, we define *one* specific family of functions $(L_{\omega^\eta})_{\eta\in\mathbf{On}}$ that satisfy our requirements, as well as the additional identities listed in subsection 12.1.2.

Proposition 12.1.2. *The axioms* $\mathbf{I}_{1,1}$, $\mathbf{I}_{2,1}$ *and* $\mathbf{I}_{3,1}$ *hold for* $(\mathbf{No},L_1)$.

Proof. Section 11.1.2 shows that $\mathbf{I}_{1,1}$ holds. Consider $r, s \in \mathbb{R}^{>}$ with $s > 1$. On $\mathbf{No}^{>,\succ}$, we have $T_{\log r} < H_s$, hence $H_r = E_1 T_{\log r} L_1 < E_1 H_s L_1$. It follows that we have $E_n H_r L_n < E_{n+1} H_s L_{n+1}$ on $\mathbf{No}^{>,\succ}$ for all $n \in \mathbb{N}$. This implies that $\mathbf{I}_{2,1}$ holds. Finally, $\mathbf{I}_{3,1}$ is valid because of the relation $\mathbf{Mo}_\omega = \mathbf{Smp}_{\mathcal{E}}$. □

Proposition 12.1.3. *Let ν be a limit ordinal and assume that $\mathbf{I}_{1,\mu}$, $\mathbf{I}_{2,\mu}$, and $\mathbf{I}_{3,\mu}$ hold for all $\mu < \nu$. Then $\mathbf{I}_{1,\nu}$, $\mathbf{I}_{2,\nu}$, and $\mathbf{I}_{3,\nu}$ hold.*

Proof. The statement $\mathbf{I}_{2,\nu}$ follows immediately by induction. Toward $\mathbf{I}_{3,\nu}$, note that we have $\mathbf{Mo}_\alpha = \bigcap_{\eta<\nu} \mathbf{Mo}_{\omega^\eta} = \bigcap_{\eta<\nu} \mathbf{Mo}'_{\omega^\eta}$ by $\mathbf{I}_{1,\eta}$ (and thus $\mathbf{DD}_\eta$) and $\mathbf{I}_{3,\eta}$ for all $\eta < \nu$. By Proposition 10.2.5, we have $\mathbf{Mo}'_\alpha = \bigcap_{\eta<\nu} \mathbf{Mo}'_{\omega^\eta} = \mathbf{Mo}_\alpha$. So $\mathbf{I}_{3,\nu}$ holds.

By $\mathbf{I}_{1,\eta}$ for all $\eta < \nu$, we need only justify that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is ν-confluent to deduce that $\mathbf{I}_{1,\nu}$ holds. For $a \in \mathbf{No}^{>,\succ}$, by $\mathbf{I}_{2,\nu}$, there are an $\mathfrak{a} \in \mathbf{Mo}'_\alpha = \mathbf{Mo}_\alpha$ and a $\beta := \omega^\eta < \alpha$ with $E_\beta(1/2\, L_\beta a) \leqslant \mathfrak{a} \leqslant E_\beta(2\, L_\beta a)$. We deduce that $L_\beta a \asymp L_\beta \mathfrak{a}$, thus $\mathfrak{a} \in \mathcal{E}_\beta[a]$. This concludes the proof. □

From now on, we assume that $\mathbf{I}_{1,\mu}$, $\mathbf{I}_{2,\mu}$, and $\mathbf{I}_{3,\mu}$ are satisfied for $\mu \geqslant 1$ and we define

$$\begin{aligned} \nu &:= \mu + 1 \\ \alpha &:= \omega^\nu \\ \beta &:= \omega^\mu. \end{aligned}$$

The following subsection is dedicated to the definition of L_β and the proof of the inductive hypotheses $\mathbf{I}_{1,\nu}$, $\mathbf{I}_{2,\nu}$, and $\mathbf{I}_{3,\nu}$ for ν. In combination with Propositions 12.1.2 and 12.1.3, this will complete our induction and the proof of Theorem 1.

12.2 Defining the hyperlogarithm

12.2.1 Definition, monotonicity and regularity

Recall that we have $\mathbf{Mo}'_\beta = \mathbf{Mo}_\beta$ by $\mathbf{I}_{3,\mu}$. In particular $\mathbf{Mo}_\beta$ is a surreal substructure. Consider $\eta < \nu$. The skeleton $(\mathbf{No}, (L_{\omega^\iota})_{\iota<\eta})$ is a confluent hyperserial skeleton of force (η, η) by $\mathbf{I}_{1,\mu}$. So for $a \in \mathbf{No}^{>,\succ}$, Proposition 4.3.5 and $\mathbf{I}_{2,\mu}$ yield $\mathcal{E}_{\omega^\eta}[a] = \mathcal{E}'_{\omega^\eta}[a]$.

In view of $\mathbf{A}_{\boldsymbol{\mu}}$ and $\mathbf{M}_{\boldsymbol{\mu}}$, the simplest way to define L_β is *via* the cut equation:

$$\forall \mathfrak{a} \in \mathbf{Mo}_\beta, \quad L_\beta\, \mathfrak{a} := \left\{ \mathbb{R}, L_\beta\, \mathfrak{a}' + \frac{1}{L_{<\beta}\, \mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta} \,\middle|\, L_\beta\, \mathfrak{a}_R^{\mathbf{Mo}_\beta} - \frac{1}{L_{<\beta}\, \mathfrak{a}}, L_{<\beta}\, \mathfrak{a} \right\}. \tag{12.2.1}$$

The reader can compare this cut equation to that found by Gonshor for the logarithm [55, Definition p.161]. Note the asymmetry between left and right options $L_\beta\, \mathfrak{a}' + (L_{<\beta}\, \mathfrak{a}')^{-1}$ and $L_\beta\, \mathfrak{a}'' - (L_{<\beta}\, \mathfrak{a})^{-1}$ (instead of $L_\beta\, \mathfrak{a}'' - (L_{<\beta}\, \mathfrak{a}'')^{-1}$) for generic $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$. In Corollary 12.3.4 below, we will derive a more symmetric but equivalent cut equation for L_β, as promised in the introduction. For now, we prove that (12.2.1) is warranted and that $\mathbf{A}_{\boldsymbol{\mu}}$, $\mathbf{M}_{\boldsymbol{\mu}}$, and $\mathbf{R}_{\boldsymbol{\mu}}$ hold.

Proposition 12.2.1. *The function L_β is well-defined on $\mathbf{Mo}_\beta$ and, for $\mathfrak{a} \in \mathbf{Mo}_\beta$, we have*

$\mathbf{H}_\mathfrak{a}$: $\left(\forall \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}, L_\beta\, \mathfrak{a}' + \frac{1}{L_{<\beta}\, \mathfrak{a}'} < L_\beta\, \mathfrak{a} - \frac{1}{L_{<\beta}\, \mathfrak{a}}\right)$ *and* $\left(\forall \mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}, L_\beta\, \mathfrak{a} + \frac{1}{L_{<\beta}\, \mathfrak{a}} < L_\beta\, \mathfrak{a}'' - \frac{1}{L_{<\beta}\, \mathfrak{a}''}\right)$.

Proof. We prove this by induction on $(\mathbf{Mo}_\beta, \sqsubset)$. Let $\mathfrak{a} \in \mathbf{Mo}_\beta$ such that $\mathbf{H}_\mathfrak{b}$ holds for all $\mathfrak{b} \in \mathfrak{a}_\sqsubset^{\mathbf{Mo}_\beta}$. Let $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$. We have $\mathfrak{a}' \in (\mathfrak{a}'')_L^{\mathbf{Mo}_\beta}$ or $\mathfrak{a}'' \in (\mathfrak{a}')_R^{\mathbf{Mo}_\beta}$, so $\mathbf{H}_{\mathfrak{a}'}$ or $\mathbf{H}_{\mathfrak{a}''}$ yields

$$L_\beta\, \mathfrak{a}' + \frac{1}{L_{<\beta}\, \mathfrak{a}'} \;<\; L_\beta\, \mathfrak{a}'' - \frac{1}{L_{<\beta}\, \mathfrak{a}''}.$$

For $\gamma < \beta$, we have $\ell_{\gamma+1} \prec \frac{1}{2} \ell_\gamma$ and $\frac{1}{L_\gamma \mathfrak{a}'} \succ \frac{1}{L_\gamma \mathfrak{a}''}, \frac{1}{L_\gamma \mathfrak{a}}$, whence

$$\frac{1}{L_{\gamma+1} \mathfrak{a}'} \succ \frac{2}{L_\gamma \mathfrak{a}'} > \frac{1}{L_\gamma \mathfrak{a}'} + \frac{1}{L_\gamma \mathfrak{a}''} + \frac{1}{L_\gamma \mathfrak{a}},$$

for all $\gamma < \beta$. Hence,

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}}.$$

We clearly have $L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}} \asymp L_\beta \mathfrak{a}'' > \mathbb{R}$. Finally,

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} \asymp L_\beta \mathfrak{a}' \prec L_{<\beta} \mathfrak{a}',$$

so $L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_{<\beta} \mathfrak{a}$. This shows that $L_\beta \mathfrak{a}$ is defined and

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}}.$$

Since $\mathfrak{a}' < \mathfrak{a} < \mathfrak{a}''$, it follows that

$$L_\beta \mathfrak{a}' + \frac{1}{L_{<\beta} \mathfrak{a}'} < L_\beta \mathfrak{a} \pm \frac{1}{L_{<\beta} \mathfrak{a}} < L_\beta \mathfrak{a}'' - \frac{1}{L_{<\beta} \mathfrak{a}}.$$

By induction, this proves $\mathbf{H}_\mathfrak{a}$ for all $\mathfrak{a} \in \mathbf{Mo}_\beta$. □

Proposition 12.2.2. *The axiom* $\mathbf{M}_\mu$ *holds.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \mathbf{Mo}_\beta$ with $\mathfrak{a} \prec \mathfrak{b}$. Since $\mathbf{Mo}_\beta$ is a surreal substructure, there is a $\mathfrak{c} \in \mathbf{Mo}_\beta$ with $\mathfrak{c} \sqsubseteq \mathfrak{a}, \mathfrak{b}$ and $\mathfrak{a} \leqslant \mathfrak{c} \leqslant \mathfrak{b}$. If $\mathfrak{a} < \mathfrak{c}$, then we have $L_\beta \mathfrak{a} + (L_{<\beta} \mathfrak{a})^{-1} < L_\beta \mathfrak{c} - (L_{<\beta} \mathfrak{c})^{-1}$ by $\mathbf{H}_\mathfrak{a}$. If $\mathfrak{c} < \mathfrak{b}$, then we have $L_\beta \mathfrak{c} + (L_{<\beta} \mathfrak{c})^{-1} < L_\beta \mathfrak{b} - (L_{<\beta} \mathfrak{b})^{-1}$ by $\mathbf{H}_\mathfrak{b}$. We cannot have both $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{c} = \mathfrak{b}$, so this proves that $L_\beta \mathfrak{a} + (L_{<\beta} \mathfrak{a})^{-1} < L_\beta \mathfrak{b} - (L_{<\beta} \mathfrak{b})^{-1}$. Therefore $\mathbf{M}_\mu$ holds. □

Proposition 12.2.3. *The axiom* $\mathbf{A}_\mu$ *holds.*

Proof. The rightmost options in (12.2.1) directly yield $\mathbf{A}_\mu$. □

Proposition 12.2.4. *The axiom* $\mathbf{R}_\mu$ *holds.*

Proof. Let $\mathfrak{a} \in \mathbf{Mo}_\beta$ and write $\varphi := L_\beta \mathfrak{a}$. Let $\mathfrak{m} \in \operatorname{supp} \varphi$ with $\mathfrak{m} \prec 1$. We have $\varphi < L_{<\beta} \mathfrak{a}$ and $\varphi_{\succ \mathfrak{m}} \asymp \varphi$ so $\varphi_{\succ \mathfrak{m}} < L_{<\beta} \mathfrak{a}$. Moreover $\varphi_{\succ \mathfrak{m}}$ is positive infinite. By [55, Theorem 5.12] (see also [18, Proposition 2.8]), the number $\varphi_{\succ \mathfrak{m}}$ is strictly simpler than φ, so $\varphi_{\succ \mathfrak{m}}$ does not lie in the cut which defines $L_\beta \mathfrak{a}$ in (12.2.1). Therefore, there is an $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ or an $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$ and an ordinal $\gamma < \beta$ with $\varphi_{\succ \mathfrak{m}} \leqslant L_\beta \mathfrak{a}' + (L_\gamma \mathfrak{a}')^{-1}$ or $\varphi_{\succ \mathfrak{m}} \geqslant L_\beta \mathfrak{a}'' - (L_\gamma \mathfrak{a})^{-1}$. Consider the first case. We have $L_\beta \mathfrak{a}' + (L_{<\beta} \mathfrak{a}')^{-1} < \varphi \leqslant \varphi_{\succ \mathfrak{m}} + \varphi_\mathfrak{m} \mathfrak{m} + \delta$ for a certain $\delta \prec \mathfrak{m}$. So $\varphi_\mathfrak{m} > 0$ and

$$\frac{1}{L_{<\beta} \mathfrak{a}'} < \frac{1}{L_\gamma \mathfrak{a}'} + \varphi_\mathfrak{m} \mathfrak{m}.$$

For $\rho < \beta$ with $\gamma < \rho$, we have $(L_\rho \mathfrak{a}')^{-1} \succ (L_\gamma \mathfrak{a}')^{-1}$ so $(L_\rho \mathfrak{a}')^{-1} - (L_\gamma \mathfrak{a}')^{-1} \asymp (L_\rho \mathfrak{a}')^{-1}$. We deduce that $(L_\rho \mathfrak{a}')^{-1} \preccurlyeq \mathfrak{m}$ for all such ρ. It follows that $(L_\rho \mathfrak{a})^{-1} \preccurlyeq \mathfrak{m}$ for all $\rho < \beta$. In the second case, we directly get $\mathfrak{m} \succ (L_\gamma \mathfrak{a})^{-1}$. This proves that we always have $\mathfrak{m} \succ (L_{<\beta} \mathfrak{a})^{-1}$. In other words $\operatorname{supp} \varphi \succ (L_{<\beta} \mathfrak{a})^{-1}$, whence $\mathbf{R}_\mu$ holds. □

Remark 12.2.5. In $(\mathbf{No}, <, \sqsubset)$, given numbers a, b with $a \leqslant b$, the $\sqsubset$-maximal number c with $c \sqsubseteq a$, b is given by $c = \{a_L \mid b_R\}$, and it satisfies $a \leqslant c \leqslant b$. It follows by definition of surreal substructures that for any surreal substructure $\mathbf{S}$ and for any $u, v \in \mathbf{S}$ there is a $\sqsubset$-maximal element

$$w = \{u_L^\mathbf{S} \mid u_R^\mathbf{S}\}_\mathbf{S} \in \mathbf{S}$$

with $w \sqsubseteq u, v$, and we have $u \leqslant w \leqslant v$.

Proposition 12.2.6. *If* μ *is a successor ordinal, then the cut equation* (12.2.1) *is uniform.*

Proof. Let $(\mathfrak{L}_{\mathfrak{a}}, \mathfrak{R}_{\mathfrak{a}})$ be a cut representation in $\mathbf{Mo}_\beta$ and write $\mathfrak{a} := \{\mathfrak{L}_{\mathfrak{a}} \mid \mathfrak{R}_{\mathfrak{a}}\}_{\mathbf{Mo}_\beta}$. For $\mathfrak{l} \in \mathfrak{L}_{\mathfrak{a}}$, we have $L_\beta \mathfrak{l} < L_\beta \mathfrak{a} \prec L_{<\beta} \mathfrak{a}$ so $L_\beta \mathfrak{l} < L_{<\beta} \mathfrak{a}$. For $\mathfrak{r} \in \mathfrak{R}_{\mathfrak{a}}$, we have $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{r}$ by $\mathbf{R}_\mu$. Since $\mathfrak{l} \prec \mathfrak{a}$, it follows that $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{r} - (L_{<\beta} \mathfrak{a})^{-1}$. We may thus define the number

$$\varphi \; := \; \left\{\mathbb{R}, L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_{\mathfrak{a}} \;\middle|\; L_\beta \mathfrak{R}_{\mathfrak{a}} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a}\right\}.$$

In order to show that (12.2.1) is uniform, we need to prove that $L_\beta \mathfrak{a} = \varphi$, for any choice of the cut representation $(\mathfrak{L}_{\mathfrak{a}}, \mathfrak{R}_{\mathfrak{a}})$. We will do so by proving that $L_\beta \mathfrak{a} \sqsubseteq \varphi$ and $\varphi \sqsubseteq L_\beta \mathfrak{a}$.

Recall that $(\mathfrak{L}_{\mathfrak{a}}, \mathfrak{R}_{\mathfrak{a}})$ is cofinal with respect to $(\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid \mathfrak{a}_R^{\mathbf{Mo}_\beta})$ and that L_β is strictly increasing. Consequently, we have

$$\varphi \; < \; L_\beta \mathfrak{a}_R^{\mathbf{Mo}_\beta} - (L_{<\beta} \mathfrak{a})^{-1}.$$

Given $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$, there is an $\mathfrak{l} \in \mathfrak{L}_{\mathfrak{a}}$ with $\mathfrak{a}' \leqslant \mathfrak{l}$. By $\mathbf{M}_\mu$, we have $L_\beta \mathfrak{a}' + (L_\gamma \mathfrak{a}')^{-1} \leqslant L_\beta \mathfrak{l} + (L_\gamma \mathfrak{l})^{-1}$ for all $\gamma < \beta$, so $\varphi > \{L_\beta \mathfrak{a}' + (L_{<\beta} \mathfrak{a}')^{-1} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}\}$. This proves that φ lies in the cut defining $L_\beta \mathfrak{a}$ as per (12.2.1), whence $L_\beta \mathfrak{a} \sqsubseteq \varphi$.

Conversely, in order to prove that $\varphi \sqsubseteq L_\beta \mathfrak{a}$, it suffices to show that $L_\beta \mathfrak{a}$ lies in the cut

$$\left(L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_{\mathfrak{a}} \;\middle|\; L_\beta \mathfrak{R}_{\mathfrak{a}} - \frac{1}{L_{<\beta} \mathfrak{a}}\right).$$

Let $\mathfrak{l} \in \mathfrak{L}_{\mathfrak{a}}$ and let $\mathfrak{b} \in \mathbf{Mo}_\beta$ be $\sqsubseteq$-maximal with $\mathfrak{b} \sqsubseteq \mathfrak{l}, \mathfrak{a}$. We have $\mathfrak{l} \leqslant \mathfrak{b} \leqslant \mathfrak{a}$, whence $L_\beta \mathfrak{b} \leqslant L_\beta \mathfrak{a}$, by $\mathbf{M}_\mu$. If $\mathfrak{b} \sqsubset \mathfrak{l}$, then $\mathfrak{b} \in \mathfrak{l}_R^{\mathbf{Mo}_\beta}$, so $\mathbf{H}_{\mathfrak{l}}$ yields $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{b}$ and $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{a}$. Otherwise $\mathfrak{l} = \mathfrak{b} \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$, so $\mathbf{H}_{\mathfrak{a}}$ yields $L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} < L_\beta \mathfrak{a}$. This proves that $\{L_\beta \mathfrak{l} + (L_{<\beta} \mathfrak{l})^{-1} : \mathfrak{l} \in \mathfrak{L}_{\mathfrak{a}}\} < L_\beta \mathfrak{a}$.

Let $\mathfrak{r} \in \mathfrak{R}_{\mathfrak{a}}$ and consider by Remark 12.2.5 the $\sqsubset$-maximal $\mathfrak{c} \in \mathbf{Mo}_\beta$ with $\mathfrak{c} \sqsubseteq \mathfrak{r}, \mathfrak{a}$. As above, if $\mathfrak{c} \sqsubset \mathfrak{a}$, then $\mathfrak{c} \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$ so $\mathbf{H}_{\mathfrak{a}}$ yields $L_\beta \mathfrak{a} < L_\beta \mathfrak{c} - (L_{<\beta} \mathfrak{a})^{-1}$, whence $L_\beta \mathfrak{a} < L_\beta \mathfrak{r} - (L_{<\beta} \mathfrak{a})^{-1}$. Otherwise $\mathfrak{a} = \mathfrak{c} \in \mathfrak{r}_L^{\mathbf{Mo}_\beta}$ so $\mathbf{H}_{\mathfrak{r}}$ yields $L_\beta \mathfrak{r} > L_\beta \mathfrak{a} + (L_{<\beta} \mathfrak{a})^{-1}$. Hence $L_\beta \mathfrak{a} < L_\beta \mathfrak{R}_{\mathfrak{a}} - (L_{<\beta} \mathfrak{a})^{-1}$ and we conclude by induction. □

12.2.2 Functional equation

In this subsection we derive $\mathbf{FE}_\mu$, under the assumption that μ is a successor ordinal. We start with the following inequality.

Lemma 12.2.7. *If $\mu > 1$, then we have $E_{<\beta_{/\omega}} < E_{\beta_{/\omega}} H_2 L_{\beta_{/\omega}}$ on $\mathbf{No}^{>,\succ}$.*

Proof. For $\gamma < \beta_{/\omega}$, there are $\eta < \mu_-$ and $n < \omega$ with $\gamma < \omega^\eta n$. We have

$$E_\gamma \; < \; E_{\omega^\eta n} \; = \; E_{\omega^{\eta+1}} T_n L_{\omega^{\eta+1}} \; < \; E_{\omega^{\eta+1}} H_2 L_{\omega^{\eta+1}}$$

on $\mathbf{No}^{>,\succ}$ by the functional equation. Note that $\eta + 1 \leqslant \mu_- < \mu$, so $\mathbf{I}_{2,\mu}$ yields

$$E_{\omega^{\eta+1}} H_2 L_{\omega^{\eta+1}} \; \leqslant \; E_{\beta_{/\omega}} H_2 L_{\beta_{/\omega}},$$

whence $E_\gamma < E_{\beta_{/\omega}} H_2 L_{\beta_{/\omega}}$. □

Let $\mathfrak{a} \in \mathbf{Mo}_\beta$. Since $\mathbf{Mo}_\beta$ is a surreal substructure, we may consider the $L_{<\beta}$-atomic number

$$\mathfrak{b} \; := \; \{L_{\beta_{/\omega}} \mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta_{/\omega}} \mathfrak{a}_R^{\mathbf{Mo}_\beta}, \mathfrak{a}\}_{\mathbf{Mo}_\beta}.$$

We claim that $\mathfrak{b} = L_{\beta_{/\omega}} \mathfrak{a}$. Assume that $\mu = 1$ and write $\mathfrak{a} = \Xi_{\mathbf{Mo}_\omega} a$. We have

$$\begin{aligned}
\log \mathfrak{a} &= \Xi_{\mathbf{Mo}_\omega}(a-1) && \text{(by [6, Proposition 2.5])}\\
&= \Xi_{\mathbf{Mo}_\omega}\{a_L - 1 \mid a_R - 1, a\} && \text{(by (8.2.2))}\\
&= \{\Xi_{\mathbf{Mo}_\omega}(a_L - 1) \mid \Xi_{\mathbf{Mo}_\omega}(a_R - 1), \Xi_{\mathbf{Mo}_\omega} a\}_{\mathbf{Mo}_\omega} &&\\
&= \{\log \Xi_{\mathbf{Mo}_\omega} a_L \mid \log \Xi_{\mathbf{Mo}_\omega} a_R, \Xi_{\mathbf{Mo}_\omega} a\}_{\mathbf{Mo}_\omega} && \text{(by [6, Proposition 2.5])}\\
&= \{\log \mathfrak{a}_L^{\mathbf{Mo}_\omega} \mid \log \mathfrak{a}_R^{\mathbf{Mo}_\omega}, \mathfrak{a}\}_{\mathbf{Mo}_\omega} &&\\
&= \mathfrak{b}.
\end{aligned}$$

Assume now that $\mu > 1$. The function $L_{\beta/\omega}$ is strictly increasing with $L_{\beta/\omega} < \mathrm{Id}_{\mathbf{No}^{>,\succ}}$. Therefore

$$L_{\beta/\omega}\,\mathfrak{a} \;\in\; (L_{\beta/\omega}\,\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta/\omega}\,\mathfrak{a}_R^{\mathbf{Mo}_\beta}, \mathfrak{a})_{\mathbf{Mo}_\beta},$$

so $\mathfrak{b} \sqsubseteq L_{\beta/\omega}\,\mathfrak{a}$. Since $\mathfrak{a} \in \mathbf{Mo}_\beta$, the cut equation (12.2.1) for μ_- yields

$$L_{\beta/\omega}\,\mathfrak{a} \;=\; \{\mathbb{R}, L_{\beta/\omega}\,\mathfrak{a}' + (L_{<\beta}\,\mathfrak{a}')^{-1} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_{\beta/\omega}} \mid L_{\beta/\omega}\,\mathfrak{a}_R^{\mathbf{Mo}_{\beta/\omega}} - (L_{<\beta}\,\mathfrak{a})^{-1}, L_{<\beta/\omega}\,\mathfrak{a}\}. \tag{12.2.2}$$

Given $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_{\beta/\omega}}$, we have $\mathfrak{d}_\beta(\mathfrak{a}') \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}' \in \mathcal{E}_\beta[\mathfrak{d}_\beta(\mathfrak{a}')]$. We deduce that

$$L_{\beta/\omega}\,\mathfrak{a}' \;\in\; L_{\beta/\omega}\,\mathcal{E}_\beta[\mathfrak{d}_\beta(\mathfrak{a}')] \;=\; \mathcal{E}_\beta[L_{\beta/\omega}\,\mathfrak{d}_\beta(\mathfrak{a}')].$$

Moreover, by definition, we have

$$\mathfrak{b} \;>\; \mathcal{E}'_\beta[L_{\beta/\omega}\,\mathfrak{d}_\beta(\mathfrak{a}')] \;=\; \mathcal{E}_\beta[L_{\beta/\omega}\,\mathfrak{d}_\beta(\mathfrak{a}')],$$

so $\mathfrak{b} \succ L_{\beta/\omega}\,\mathfrak{a}'$. Symmetric arguments yield $\mathfrak{b} \prec L_{\beta/\omega}\,\mathfrak{a}_R^{\mathbf{Mo}_{\beta/\omega}}$. Lemma 12.2.7 implies that $L_{<\beta/\omega}\,\mathfrak{a} \subseteq \mathcal{E}_\beta[\mathfrak{a}]$, whence $\mathfrak{d}_\beta(L_{<\beta/\omega}\,\mathfrak{a}) = \{\mathfrak{a}\}$. We get $\mathfrak{b} < \mathcal{E}_\beta\,\mathfrak{d}_\beta(L_{<\beta/\omega}\,\mathfrak{a})$, whence $\mathfrak{b} < L_{<\beta/\omega}\,\mathfrak{a}$. Thus $\mathfrak{b}$ lies in the cut defining $L_{\beta/\omega}\,\mathfrak{a}$ in (12.2.2), so $L_{\beta/\omega}\,\mathfrak{a} \sqsubseteq \mathfrak{b}$. This proves our claim that

$$\forall \mathfrak{a} \in \mathbf{Mo}_\beta, \quad L_{\beta/\omega}\,\mathfrak{a} = \{L_{\beta/\omega}\,\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta/\omega}\,\mathfrak{a}_R^{\mathbf{Mo}_\beta}, \mathfrak{a}\}_{\mathbf{Mo}_\beta}. \tag{12.2.3}$$

We now derive $\mathbf{FE}_\mu$.

Proposition 12.2.8. *For $\mathfrak{a} \in \mathbf{Mo}_\beta$, we have $L_\beta\,L_{\beta/\omega}\,\mathfrak{a} = L_\beta\,\mathfrak{a} - 1$.*

Proof. We prove this by induction on $(\mathbf{Mo}_\beta, \sqsubset)$. Let $\mathfrak{a} \in \mathbf{Mo}_\beta$ be such that the result holds on $\mathfrak{a}_\sqsubset^{\mathbf{Mo}_\beta}$. By (12.2.3), we have

$$L_{\beta/\omega}\,\mathfrak{a} \;=\; \{L_{\beta/\omega}\,\mathfrak{a}_L^{\mathbf{Mo}_\beta} \mid L_{\beta/\omega}\,\mathfrak{a}_R^{\mathbf{Mo}_\beta}, \mathfrak{a}\}_{\mathbf{Mo}_\beta}.$$

Let $\mathfrak{a}'$ and $\mathfrak{a}''$ range in $\mathfrak{a}_L^{\mathbf{Mo}_\beta}$ and $\mathfrak{a}_R^{\mathbf{Mo}_\beta}$ respectively. Proposition 12.2.6 and our induction hypothesis yield:

$$\begin{aligned} L_\beta\,L_{\beta/\omega}\,\mathfrak{a} &= \left\{\mathbb{R}, L_\beta\,L_{\beta/\omega}\,\mathfrak{a}' + \frac{1}{L_{<\beta}\,L_{\beta/\omega}\,\mathfrak{a}'} \;\middle|\; L_\beta\,L_{\beta/\omega}\,\mathfrak{a}'' - \frac{1}{L_{<\beta}\,L_{\beta/\omega}\,\mathfrak{a}}, L_\beta\,\mathfrak{a} - \frac{1}{L_{<\beta}\,\mathfrak{a}}, L_{<\beta}\,L_{\beta/\omega}\,\mathfrak{a}\right\} \\ &= \left\{\mathbb{R}, L_\beta\,\mathfrak{a}' - 1 + \frac{1}{L_{<\beta}\,\mathfrak{a}'} \;\middle|\; L_\beta\,\mathfrak{a}'' - 1 - \frac{1}{L_{<\beta}\,\mathfrak{a}}, L_\beta\,\mathfrak{a} - \frac{1}{L_{<\beta}\,\mathfrak{a}}, L_{<\beta}\,\mathfrak{a}\right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L_\beta\,\mathfrak{a} - 1 &= \left\{\mathbb{R} - 1, L_\beta\,\mathfrak{a}' + \frac{1}{L_{<\beta}\,\mathfrak{a}'} - 1 \;\middle|\; L_\beta\,\mathfrak{a}'' - \frac{1}{L_{<\beta}\,\mathfrak{a}} - 1, L_{<\beta}\,\mathfrak{a} - 1, L_\beta\,\mathfrak{a}\right\} \\ &= \left\{\mathbb{R}, L_\beta\,\mathfrak{a}' + \frac{1}{L_{<\beta}\,\mathfrak{a}'} - 1 \;\middle|\; L_\beta\,\mathfrak{a}'' - \frac{1}{L_{<\beta}\,\mathfrak{a}} - 1, L_\beta\,\mathfrak{a}, L_{<\beta}\,\mathfrak{a}\right\}. \end{aligned}$$

In order to conclude that $L_\beta\,L_{\beta/\omega}\,\mathfrak{a} = L_\beta\,\mathfrak{a} - 1$, it remains to show that $L_\beta\,\mathfrak{a} - 1 < L_\beta\,\mathfrak{a} - (L_{<\beta}\,\mathfrak{a})^{-1}$ and that $L_\beta\,L_{\beta/\omega}\,\mathfrak{a} < L_\beta\,\mathfrak{a}$. The first inequality holds because $(L_{<\beta}\,\mathfrak{a})^{-1}$ is a set of infinitesimal numbers. An easy induction shows that $L_{\beta/\omega}\,a < a$ for all $a \in \mathbf{No}^{>,\succ}$. The second inequality follows, because L_β is strictly increasing on $\mathbf{Mo}_\beta$. This completes our inductive proof. □

Combining our results so far, we have proved that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is a hyperserial skeleton of force ν.

12.2.3 Confluence

We next prove that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is ν-confluent.

Lemma 12.2.9. *If μ is a non-zero limit ordinal, then the function groups $\mathcal{E}'_\beta$ and $\mathcal{E}^*_\beta$ are mutually pointwise cofinal. In particular, we have $\mathbf{Mo}_\beta = \mathbf{Mo}^*_\beta$ and $\mathbf{Tr}_\beta = \mathbf{Tr}^*_\beta$.*

Proof. For $\gamma \in (0, \beta)$ and $r \in \mathbb{R}^{>}$, we have $E_\gamma H_r L_\gamma < E_\gamma$ since $H_r < E_\gamma$. We have

$$\{L_\rho, E_\rho : \rho \in (0, \beta)\} \not\leqslant \mathcal{E}_\beta^*,$$

whereas $\mathbf{I}_{2,\mu}$ yields

$$\{E_\rho H_r L_\rho : \rho \in (0, \beta)\} \not\leqslant \mathcal{E}'_\beta.$$

Therefore $\mathcal{E}'_\beta \trianglelefteq \mathcal{E}_\beta^*$. For $\rho < \beta$, there is $\eta < \mu$ with $\rho < \omega^\eta$. The functional equation gives

$$E_\rho \;<\; E_{\omega^\eta} \;=\; E_{\omega^{\eta+1}} T_1 L_{\omega^{\eta+1}} \;<\; E_{\omega^{\eta+1}} H_2 L_{\omega^{\eta+1}},$$

which proves the inequality $\mathcal{E}_\beta^* \trianglelefteq \mathcal{E}'_\beta$. □

Lemma 12.2.10. *For each $a \in \mathbf{No}^{>,\succ}$, any $\sqsubset$-minimal element of $\mathcal{E}_\alpha[a]$ is $L_{<\alpha}$-atomic.*

Proof. Let $\mathfrak{A}$ denote the class of numbers $\mathfrak{a} \in \mathbf{No}^{>,\succ}$ that are $\sqsubset$-minimal in $\mathcal{E}_\alpha[\mathfrak{a}]$. Any such $\sqsubset$-minimal number $\mathfrak{a}$ is also $\sqsubset$-minimal in $\mathcal{E}'_\beta[\mathfrak{a}] = \mathcal{E}_\beta[\mathfrak{a}] \subseteq \mathcal{E}_\alpha[\mathfrak{a}]$, hence $L_{<\beta}$-atomic. Thus L_β is defined on $\mathfrak{A}$. It is enough to prove that $\mathfrak{A}$ is closed under L_β in order to obtain that $\mathfrak{A} \subseteq \mathbf{Mo}_\alpha$.

Consider $\mathfrak{a} \in \mathfrak{A}$, and recall that we have

$$L_\beta \,\mathfrak{a} \;=\; \left\{ \mathbb{R}, L_\beta\, \mathfrak{a}' + \frac{1}{L_{<\beta}\, \mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta} \;\middle|\; L_\beta\, \mathfrak{a}_R^{\mathbf{Mo}_\beta} - \frac{1}{L_{<\beta}\, \mathfrak{a}}, L_{<\beta}\, \mathfrak{a} \right\}. \tag{12.2.4}$$

Assume for contradiction that $L_\beta\, \mathfrak{a}$ is not $\sqsubset$-minimal in $\mathcal{E}_\alpha[L_\beta\, \mathfrak{a}]$. So there is a $\mathfrak{b} \in \mathcal{E}_\alpha[L_\beta\, \mathfrak{a}]$ with $\mathfrak{b} \sqsubset L_\beta\, \mathfrak{a}$. This implies that $\mathfrak{b}$ lies outside the cut defining $L_\beta\, \mathfrak{a}$, so $\mathfrak{b}$ is larger than a right option of (12.2.4) or smaller than a left option of (12.2.4).

Assume first that $\mathfrak{b} < L_\beta\, \mathfrak{a}$. So there is an $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\beta}$ with $\mathfrak{b} \preccurlyeq L_\beta\, \mathfrak{a}'$. We have $\mathfrak{d}_\alpha(L_\beta\, \mathfrak{a}) = \mathfrak{d}_\alpha(\mathfrak{b})$ so there is an $n \in \mathbb{N}$ with

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{b}) \;\asymp\; (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(L_\beta\, \mathfrak{a}).$$

The function $L_\beta \circ \mathfrak{d}_\beta$ is nondecreasing, so $(L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}$ is nondecreasing as well. So $(L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathfrak{a}') \preccurlyeq (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathfrak{a})$. But

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathfrak{a}') = (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(L_\beta\, \mathfrak{a}') \succcurlyeq (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{b}) \asymp (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(L_\beta\, \mathfrak{a}).$$

Thus

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathfrak{a}') \;\asymp\; (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathfrak{a}).$$

This contradicts the $\sqsubset$-minimality of $\mathfrak{a}$.

Now consider the other case when $\mathfrak{b} > L_\beta\, \mathfrak{a}$. In particular, $\mathfrak{b}$ must be larger than a right option of (12.2.4). Symmetric arguments to those above imply that we cannot have $\mathfrak{b} \succcurlyeq L_\beta\, \mathfrak{a}''$ for some $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\beta}$. So there must exist a $\gamma < \beta$ with $\mathfrak{b} \geqslant L_\gamma\, \mathfrak{a}$. If μ is a limit ordinal, then $\gamma < \mu_-$ so Lemma 12.2.9 yields $\mathfrak{d}_\beta(L_\gamma\, \mathfrak{a}) = \mathfrak{a}$, whence $\mathfrak{d}_\beta(\mathfrak{b}) \succcurlyeq \mathfrak{a}$. If μ is a successor ordinal, then there is a $k \in \mathbb{N}$ with $\gamma \leqslant \beta_{/\omega} k$, so

$$\mathfrak{d}_\beta(\mathfrak{b}) \;\geqslant\; \mathfrak{d}_\beta(L_{(\beta_{/\omega})k}\, \mathfrak{a}) \;=\; L_{(\beta_{/\omega})k}\, \mathfrak{a}$$

and Proposition 12.2.8 yields $L_\beta \mathfrak{d}_\beta(\mathfrak{b}) \succcurlyeq L_\beta\, \mathfrak{a} - k \succcurlyeq L_\beta\, \mathfrak{a}$. In both cases, we thus have $L_\beta \mathfrak{d}_\beta(\mathfrak{b}) \succcurlyeq L_\beta\, \mathfrak{a}$. For any integer $n > 1$, we deduce that

$$(L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{b}) \;\geqslant\; (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(\mathfrak{a}) \;>\; (L_\beta \circ \mathfrak{d}_\beta)^{\circ(n+1)}(\mathfrak{a}) \;=\; (L_\beta \circ \mathfrak{d}_\beta)^{\circ n}(L_\beta\, \mathfrak{a}).$$

This contradicts the fact that $\mathfrak{b}$ lies in $\mathcal{E}_\alpha[L_\beta\, \mathfrak{a}]$.

We have shown that the cases $\mathfrak{b} < L_\beta\, \mathfrak{a}$ and $\mathfrak{b} > L_\beta\, \mathfrak{a}$ both lead to a contradiction. Consequently, $L_\beta\, \mathfrak{a}$ is $\sqsubset$-minimal in $\mathcal{E}_\alpha[L_\beta\, \mathfrak{a}]$ and we conclude that $L_\beta \mathfrak{A} \subseteq \mathfrak{A}$, as claimed. □

Corollary 12.2.11. *$(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is ν-confluent.*

Proof. We already know that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\mu})$ is μ-confluent by $\mathbf{I}_{1,\mu}$. Recall that $(\mathbf{No}, \sqsubset)$ is well-founded, so each class $\mathcal{E}_\alpha[a]$ for $a \in \mathbf{No}^{>,\succ}$ contains a $\sqsubset$-minimal element. Lemma 12.2.10 therefore implies that $\mathbf{No}$ is ν-confluent. □

We now know that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is a confluent hyperserial skeleton of force ν. Moreover, the class $\mathbf{No}_{\succ,\beta}$ is that of $\lhd$-minima and thus $\sqsubset$-minima in the convex classes

$$\mathcal{L}_\beta[a] \;=\; \{b \in a + \mathbf{No}^{\prec} : b = a \vee (\exists \gamma < \beta, a < \ell_\beta^{\uparrow\gamma} \circ |a-b|^{-1})\},$$

for $a \in \mathbf{No}^{>,\succ}$. In other words, we have $\mathbf{No}_{\succ,\beta} = \mathbf{Smp}_{\mathcal{L}_\beta}$. In order to conclude that $\mathbf{No}_{\succ,\beta}$ is a surreal substructure, we still need to prove that the convex partition $\mathcal{L}_\beta$ is thin. This will be done at the end of section 12.2.4 below.

Proposition 12.2.12. *The cut equation* (12.2.1) *is uniform.*

Proof. Let $(\mathfrak{L}_\mathfrak{a}, \mathfrak{R}_\mathfrak{a})$ be a cut representation in $\mathbf{Mo}_\beta$ and write $\mathfrak{a} := \{\mathfrak{L}_\mathfrak{a} \mid \mathfrak{R}_\mathfrak{a}\}_{\mathbf{Mo}_\beta}$. We have

$$\mathcal{L}_\beta[L_\beta \mathfrak{L}_\mathfrak{a}] \;<\; \mathcal{L}_\beta[L_\beta \mathfrak{a}] \;<\; \mathcal{L}_\beta[L_\beta \mathfrak{R}_\mathfrak{a}].$$

By Proposition 5.3.8 and since the sets $\{\ell_{[\gamma,\beta)} : \gamma < \beta\}$ and $\{\ell_\gamma : \gamma < \beta\}$ are mutually coinitial, this shows that

$$L_\beta \mathfrak{a} \;\in\; \left(\mathbb{R}, L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a} \;\middle|\; L_\beta \mathfrak{R}_\mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a}\right).$$

In particular, the number

$$\varphi \;:=\; \left\{\mathbb{R}, L_\beta \mathfrak{l} + \frac{1}{L_{<\beta} \mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_\mathfrak{a} \;\middle|\; L_\beta \mathfrak{R}_\mathfrak{a} - \frac{1}{L_{<\beta} \mathfrak{a}}, L_{<\beta} \mathfrak{a}\right\}$$

is well-defined, with $\varphi \sqsubseteq L_\beta \mathfrak{a}$. As in the proof of Proposition 12.2.6, we have $L_\beta \mathfrak{a} \sqsubseteq \varphi$, whence $\varphi = L_\beta \mathfrak{a}$. We conclude that the cut equation (12.2.1) is uniform. □

12.2.4 Hyperexponentials

We have shown that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ is a hyperserial skeleton of force (ν, μ). In order to prove that $(\mathbf{No}, (L_{\omega^\eta})_{\eta<\nu})$ has force (ν, ν), it remains to show that every β-truncated number φ has a hyperexponential $E_\beta \varphi$. This is the purpose of this subsection.

Proposition 12.2.13. *We have $L_\beta \mathbf{Mo}_\beta = \mathbf{No}_{\succ,\beta}$, and E_β has the following cut equation on $\mathbf{No}_{\succ,\beta}$:*

$$\forall \varphi \in \mathbf{No}_{\succ,\beta}, \quad E_\beta \varphi = \left\{E_{<\beta} \varphi,\, E_{<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right), \mathcal{E}_\beta' E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \;\middle|\; \mathcal{E}_\beta' E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}\right\}. \tag{12.2.5}$$

Proof. We prove the result by induction on $(\mathbf{No}_{\succ,\beta}, \sqsubseteq)$. Let $\varphi \in \mathbf{No}_{\succ,\beta}$ such that E_β is defined on $\varphi_\sqsubset^{\mathbf{No}_{\succ,\beta}}$ with the given equation. We will first show that the number

$$\mathfrak{a} \;:=\; \left\{E_{<\beta} \varphi, E_{<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right), \mathcal{E}_\beta' E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \;\middle|\; \mathcal{E}_\beta' E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}\right\} \tag{12.2.6}$$

is well-defined. We will then prove that $L_\beta \mathfrak{a} = \varphi$.

Let $\varphi' \in \varphi_L^{\mathbf{No}_{\succ,\beta}}$ and $\varphi'' \in \varphi_R^{\mathbf{No}_{\succ,\beta}}$. If $\varphi' \in (\varphi'')_L^{\mathbf{No}_{\succ,\beta}}$, then $E_\beta \varphi'' > \mathcal{E}_\beta' E_\beta \varphi'$ by the definition of $E_\beta \varphi''$. So $\mathcal{E}_\beta' E_\beta \varphi' < \mathcal{E}_\beta' E_\beta \varphi''$. Otherwise, we have $\varphi'' \in (\varphi')_R^{\mathbf{No}_{\succ,\beta}}$, whence $\mathcal{E}_\beta' E_\beta \varphi'' > E_\beta \varphi'$ by definition of $E_\beta \varphi'$, so $\mathcal{E}_\beta' E_\beta \varphi' < \mathcal{E}_\beta' E_\beta \varphi''$. So we always have

$$\mathcal{E}_\beta' E_\beta \varphi_L^{\mathbf{No}_{\succ,\beta}} \;<\; \mathcal{E}_\beta' E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}.$$

We also have $E_{<\beta} \varphi'' < E_\beta \varphi''$, so $E_{<\beta} \varphi < \mathcal{E}_\beta' E_\beta \varphi''$. This proves that $E_{<\beta} \varphi < \mathcal{E}_\beta' E_\beta \varphi_R^{\mathbf{No}_{\succ,\beta}}$. It remains to show that

$$E_{<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right) \;<\; \mathcal{E}_\beta' E_\beta (\varphi_R^{\mathbf{No}_{\succ,\beta}}).$$

Note that $\varphi_R^{\mathbf{No}_{\succ,\beta}} > \mathcal{L}_\beta[\varphi]$, so by the definition of $\mathcal{L}_\beta[\varphi]$, we have

$$L_\beta^{\uparrow\gamma}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}} - \varphi}\right) \;<\; \varphi \;<\; \varphi_R^{\mathbf{No}_{\succ,\beta}} \tag{12.2.7}$$

for all $\gamma<\beta$. Hence $E_{<\beta}((\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi)^{-1})<E_\beta\varphi_R^{\mathbf{No}_{\succ,\beta}}$, which completes the proof that $\mathfrak{a}$ is well-defined.

Let us now prove that $L_\beta\mathfrak{a}=\varphi$. Since $\mathcal{E}_\beta\trianglelefteq\langle E_{<\beta}\rangle$, the definition (12.2.6) and the identity $\mathbf{Mo}_\beta=\mathbf{Mo}'_\beta$ thatgive $\mathfrak{a}\in\mathbf{Mo}_\beta$ by Lemma 10.1.5. First assume that μ is a limit ordinal. Lemma 12.2.9 yields $\langle E_{<\beta}\rangle\ntrianglelefteq\mathcal{E}_\beta$, so we may write

$$\mathfrak{a} \;=\; \left\{\mathfrak{d}_\beta(\varphi),\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right),E_\beta\varphi_L^{\mathbf{No}_{\succ,\beta}}\;\middle|\;E_\beta\varphi_R^{\mathbf{No}_{\succ,\beta}}\right\}_{\mathbf{Mo}_\beta}.$$

By Proposition 5.3.8, for $b\in\mathbf{No}^{>,\succ}$ the classes that $\mathcal{L}_\beta[L_\beta b]$ and $L_\beta b\pm(L_{<\beta}b)^{-1}$ are mutually cofinal and coinitial. Moreover, we have $L_\beta E_\beta\psi=\psi$ for all $\psi\in\varphi_\sqsubset^{\mathbf{No}_{\succ,\beta}}$, by our hypothesis on φ. Hence, Propositions 12.2.12 and 5.3.8 give

$$L_\beta\mathfrak{a} \;=\; \left\{\mathbb{R},\mathcal{L}_\beta[L_\beta\mathfrak{d}_\beta(\varphi)],\mathcal{L}_\beta\left[L_\beta\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right)\right],\mathcal{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}]\;\middle|\;\varphi_R^{\mathbf{No}_{\succ,\beta}}-\frac{1}{L_{<\beta}\mathfrak{a}},L_{<\beta}\mathfrak{a}\right\}.$$

Note that $L_\beta\mathfrak{a}\in(\varphi_L^{\mathbf{No}_{\succ,\beta}}\mid\varphi_R^{\mathbf{No}_{\succ,\beta}})_{\mathbf{No}_{\succ,\beta}}$, so $\varphi\sqsubseteq L_\beta\mathfrak{a}$. Now $L_\beta\mathfrak{d}_\beta(\varphi)\in\mathcal{L}_\beta[L_\beta\varphi]<\varphi$. We also have

$$L_\beta\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right)\;\in\;L_\beta\mathcal{E}'_\beta\left[\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right],$$

where

$$\begin{aligned}L_\beta\mathcal{E}'_\beta\left[\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right] &= L_\beta\mathcal{E}^*_\beta\left[\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right] && \text{(by Lemma 12.2.9)}\\ &\ntrianglelefteq L_\beta^{\uparrow<\beta}\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right)\\ &< \varphi. && \text{(by (12.2.7))}\end{aligned}$$

So $L_\beta\mathfrak{d}_\beta(\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi)^{-1}<\varphi$. Since $\varphi\in\mathbf{No}_{\succ,\alpha}$, Lemma 10.1.5 gives $\mathcal{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}]<\varphi$. Finally, we have by definition that $\mathfrak{a}>E_{<\beta}((\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi)^{-1})$, so $\varphi_R^{\mathbf{No}_{\succ,\beta}}-(L_{<\beta}\mathfrak{a})^{-1}>\varphi$. This proves that $L_\beta\mathfrak{a}\sqsubseteq\varphi$, so $L_\beta\mathfrak{a}=\varphi$.

Assume now that μ is a successor ordinal. For all $b\in\mathbf{No}^{>,\succ}$, the sets $E_{<\beta}\varphi$, $E_{<\beta}\mathfrak{d}_\beta(\varphi)$, and $E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta(\varphi)$ are mutually cofinal. So we can rewrite (12.2.6) as

$$\begin{aligned}\mathfrak{a} &= \left\{E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta(\varphi),E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right),\mathcal{E}'_\beta E_\beta\varphi_L^{\mathbf{No}_{\succ,\beta}}\;\middle|\;\mathcal{E}'_\beta E_\beta\varphi_R^{\mathbf{No}_{\succ,\beta}}\right\}\\ &= \left\{E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta(\varphi),E_{\beta_{/\omega}\mathbb{N}}\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right),E_\beta\varphi_L^{\mathbf{No}_{\succ,\beta}}\;\middle|\;E_\beta\varphi_R^{\mathbf{No}_{\succ,\beta}}\right\}_{\mathbf{Mo}_\beta}.\end{aligned}$$

As in the limit case, Proposition 12.2.12 yields

$$L_\beta\mathfrak{a} \;=\; \left\{\mathbb{R},\mathcal{L}_\beta[L_\beta^{\uparrow<\beta}\mathfrak{d}_\beta(\varphi)],\mathcal{L}_\beta\left[L_\beta^{\uparrow<\beta}\mathfrak{d}_\beta\left(\frac{1}{\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi}\right)\right],\mathcal{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}]\;\middle|\;\varphi_R^{\mathbf{No}_{\succ,\beta}}-\frac{1}{L_{<\beta}\mathfrak{a}},L_{<\beta}\mathfrak{a}\right\}.$$

Let $\gamma<\beta$. There is an $n\in\mathbb{N}$ with $\gamma<\beta_{/\omega}n$. Since $L_\beta\varphi<\varphi-(n+1)$, we have

$$\varphi\;>\;L_\beta^{\uparrow\beta_{/\omega}(n+1)}\mathfrak{d}_\beta(\varphi)\;\geqslant\;L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta(\varphi)+1.$$

In particular $\varphi>\mathcal{L}_\beta[L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta(\varphi)]$. We saw in (12.2.7) that $L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta((\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi)^{-1})<\varphi$, whence $\mathcal{L}_\beta[L_\beta^{\uparrow\gamma}\mathfrak{d}_\beta((\varphi_R^{\mathbf{No}_{\succ,\beta}}-\varphi)^{-1})]<\varphi$. We also obtain the inequalities

$$\mathcal{L}_\beta[\varphi_L^{\mathbf{No}_{\succ,\beta}}]\;<\;\varphi\;<\;\varphi_R^{\mathbf{No}_{\succ,\beta}}-(L_{<\beta}\mathfrak{a})^{-1}$$

in a similar way as in the limit case.

We conclude that $\varphi=L_\beta\mathfrak{a}$ holds in general. It follows by induction that the formula for E_β is valid. In particular $L_\beta\colon\mathbf{Mo}_\beta\longrightarrow\mathbf{No}_{\succ,\beta}$ is surjective. □

With Proposition 12.2.13, we have completed the proof of $\mathbf{I}_{1,\nu}$. By Proposition 4.3.5 we have

$$\mathcal{E}_{\beta\omega}[a]=\mathcal{E}'_{\beta\omega}[a] \tag{12.2.8}$$

for all $a\in\mathbf{No}^{>,\succ}$. Given $a\in\mathbf{No}_{\succ,\beta}$, we also deduce from Proposition 5.3.8 that the set $a\pm(L_{<\beta}E_\beta a)^{-1}$ is cofinal and coinitial in $\mathcal{L}_\beta[a]$. The convex partition defined by $\mathcal{L}_\beta$ is thus thin. By Theorem 10.1.7, the class $\mathbf{No}_{\succ,\beta}$ is a surreal substructure with uniform cut equation

$$\forall a\in\mathbf{No},\quad \Xi_{\mathbf{No}_{\succ,\beta}}a=\{\mathbb{R},\mathcal{L}_\beta[\Xi_{\mathbf{No}_{\succ,\beta}}a_L]\mid\mathcal{L}_\beta[\Xi_{\mathbf{No}_{\succ,\beta}}a_R]\}\tag{12.2.9}$$

For $a\in\mathbf{No}$, we have $\mathcal{L}_\beta[\Xi_{\mathbf{No}_{\succ,\beta}}a]<\Xi_{\mathbf{No}_{\succ,\beta}}a_R$, so $\Xi_{\mathbf{No}_{\succ,\beta}}a<\Xi_{\mathbf{No}_{\succ,\beta}}a_R-(L_{<\beta}E_\beta\Xi_{\mathbf{No}_{\succ,\beta}}a)^{-1}$. We deduce that the following cut equation is equivalent to (12.2.9):

$$\Xi_{\mathbf{No}_{\succ,\beta}}a\;=\;\left\{\mathbb{R},\Xi_{\mathbf{No}_{\succ,\beta}}a'+\frac{1}{L_{<\beta}E_\beta\Xi_{\mathbf{No}_{\succ,\beta}}a'}:a'\in a_L\;\middle|\;\Xi_{\mathbf{No}_{\succ,\beta}}a_R-\frac{1}{L_{<\beta}E_\beta\Xi_{\mathbf{No}_{\succ,\beta}}a}\right\}.\tag{12.2.10}$$

12.2.5 End of the inductive proof

We now prove $\mathbf{I}_{2,\nu}$, $\mathbf{I}_{3,\nu}$ and Theorem **C**.

Lemma 12.2.14. *If μ is a limit ordinal, then we have $E_\beta T_1 L_\beta>E_{<\beta}$ on $\mathbf{No}^{>,\succ}$.*

Proof. Let $a\in\mathbf{No}^{>,\succ}$. We have $\sharp_\beta(L_\beta a+1)>\sharp_\beta(L_\beta a)$, so Corollary 5.3.12 yields

$$\mathfrak{d}_\beta(E_\beta(L_\beta a+1))\;=\;E_\beta(\sharp_\beta(L_\beta a+1))\;\succ\;E_\beta(\sharp_\beta(L_\beta a))\;=\;\mathfrak{d}_\beta(a).$$

We deduce that $E_\beta(L_\beta a+1)>\mathcal{E}_\beta a$ so $E_\beta(L_\beta a+1)>E_{<\beta}a$ by Lemma 12.2.9. □

Proposition 12.2.15. *For $r,s\in\mathbb{R}$ with $s>1$ and $\gamma<\rho<\alpha$, we have $E_\gamma H_r L_\gamma<E_\rho H_s L_\rho$ on $\mathbf{No}^{>,\succ}$, i.e. $\mathbf{I}_{2,\nu}$ holds.*

Proof. Throughout this proof, we consider inequalities and equalities of functions on $\mathbf{No}^{>,\succ}$. Write $\gamma=\beta m+\iota$ and $\rho=\beta n+\theta$ where $m,n<\omega$ and $\iota,\theta<\beta$. We have

$$\begin{aligned}E_\gamma H_r L_\gamma\;&=\;E_{\beta m}E_\iota H_r L_\iota L_{\beta m}\quad\text{and}\\ E_\rho H_r L_\rho\;&=\;E_{\beta n}E_\theta H_s L_\theta L_{\beta n}.\end{aligned}$$

If $m=n$, then $\iota<\theta$, so $\mathbf{I}_{2,\mu}$ yields $E_\iota H_r L_\iota<E_\theta H_s L_\theta$, whence $E_\gamma H_r L_\gamma<E_\rho H_s L_\rho$. Assume that $m<n$. If μ_- is a successor ordinal, then there is $p<\omega$ with $\iota<\beta_{/\omega}p$. By $\mathbf{I}_{2,\mu}$, we have $E_\theta H_s L_\theta\geqslant H_s>T_p$. So $E_\beta(E_\theta H_s L_\theta)L_\beta>E_\beta T_p L_\beta=E_{\beta_{/\omega}p}$. We conclude by noting that $E_{\beta_{/\omega}p}>E_\iota>E_\iota H_r L_\iota$. If μ_- is a limit ordinal, then $E_\theta H_s L_\theta>T_1$ so $E_\beta(E_\theta H_s L_\theta)L_\beta>E_\iota>E_\iota H_r L_\iota$ by Lemma 12.2.14. It follows that for $k\in\mathbb{N}^>$, we have $E_{\beta(k+1)}E_\theta H_s L_\theta L_{\beta(k+1)}>E_{\beta k}E_\iota H_r L_\iota L_{\beta k}$. An easy induction on k yields the result. □

Proposition 12.2.16. *$\mathbf{Mo}'_\alpha$ is the class of $L_{<\alpha}$-atomic numbers, i.e. $\mathbf{I}_{3,\nu}$ holds.*

Proof. Let $a\in\mathbf{No}^{>,\succ}$. By Lemma 12.2.10, the simplest element of $\mathcal{E}_\alpha[a]$ is $L_{<\alpha}$-atomic. Since $\mathcal{E}_\alpha[a]=\mathcal{E}'_\alpha[a]$ (see (12.2.8) and recall that $\beta\omega=\alpha$), we deduce that $\mathbf{Mo}'_\alpha\subseteq\mathbf{Mo}_\alpha$.

Conversely, given $\mathfrak{a}\in\mathbf{Mo}_\alpha$, we have $\mathfrak{b}:=\pi_{\mathcal{E}'_\alpha}(\mathfrak{a})\in\mathbf{Mo}'_\alpha\subseteq\mathbf{Mo}_\alpha$. Now $\mathfrak{b}\in\mathcal{E}'_\alpha[\mathfrak{a}]$, so by $\mathbf{I}_{2,\nu}$, there are $r,s\in\mathbb{R}^>$ and $\gamma<\alpha$ with $E_\gamma(r\,L_\gamma\,\mathfrak{a})<\mathfrak{b}<E_\gamma(s\,L_\gamma\,\mathfrak{a})$ (here we use the fact that $\mathcal{E}'_\alpha$ is generated by the linearly ordered subset $\{E_\gamma(r\,L_\gamma\,\mathfrak{a}):\gamma<\alpha\wedge r\in\mathbb{R}^>\}$). Hence, $L_\gamma\,\mathfrak{b}\asymp L_\gamma\,\mathfrak{a}$, $L_\gamma\,\mathfrak{b}=L_\gamma\,\mathfrak{a}$ and $\mathfrak{b}=\mathfrak{a}$. We conclude that $\mathfrak{a}\in\mathbf{Mo}'_\alpha$. □

In particular, the class $\mathbf{Mo}_\alpha$ is a surreal substructure. We have proved $\mathbf{I}_{1,\nu},\mathbf{I}_{2,\nu}$, and $\mathbf{I}_{3,\nu}$, so we obtain the following by induction:

Theorem 12.2.17. *The field $(\mathbf{No},(L_{\omega^\eta})_{\eta\in\mathbf{On}})$ is a confluent hyperserial skeleton of force $(\mathbf{On},\mathbf{On})$.*

Combining this with Theorem **A**, we obtain Theorem **C**. Let us finally show that $(\mathbf{No},\circ)$ contains only one atomic, or $L_{<\mathbf{On}}$-atomic element.

Proposition 12.2.18. *The number ω is the only $L_{<\mathbf{On}}$-atomic element in $\mathbf{No}$. For all $a\in\mathbf{No}^{>,\succ}$, there is $\gamma\in\mathbf{On}$ with $L_\gamma a\asymp L_\gamma\omega$.*

Proof. The number ω lies in $\mathbf{Mo}_{\omega^\mu}$ for all $\mu \in \mathbf{On}$, so it is $L_{<\mathbf{On}}$-atomic. For $\nu \in \mathbf{On}$, the number $E_{\omega^\nu}\omega = \{E_{<\omega^\nu}\omega \mid \varnothing\}$ is an ordinal. As a sign sequence, the number $L_{\omega^\nu}\omega = \{\varnothing \mid L_{<\omega^\nu}\omega\}_{\mathbf{No}^{>,\succ}}$ is ω followed by a string containing only minuses [6, Lemma 2.6]. Since the sequences $(E_{\omega^\nu}\omega)_{\nu\in\mathbf{On}}$ and $(L_{\omega^\nu}\omega)_{\nu\in\mathbf{On}}$ are strictly increasing and strictly decreasing respectively, the classes $\{E_{\omega^\nu}\omega : \nu \in \mathbf{On}\}$ and $\{L_{\omega^\nu}\omega : \nu \in \mathbf{On}\}$ are respectively cofinal and coinitial in $\mathbf{No}^{>,\succ} = \{a \in \mathbf{No} : \omega \sqsubseteq a\}$. Thus for $a \in \mathbf{No}^{>,\succ}$, there is a $\nu \in \mathbf{On}$ with $E_{\omega^\nu}\omega > a > L_{\omega^\nu}\omega$, whence $L_{\omega^{\nu+1}}\omega \asymp L_{\omega^{\nu+1}} a$. □

12.3 Remarkable identities

In this section, we focus on those properties of the functions E_α and L_α defined in Chapter 12 that pertain to the simplicity relation on $\mathbf{No}$. Consequently, those properties only make sense within $\mathbf{No}$, as opposed to many properties derived in Chapter 12 which can be stated in general hyperserial fields. In particular, we will characterize certain classes defined in the hyperserial context as surreal substructures defined using methods in Chapter 10. In what follows, ν is a non-zero ordinal and $\alpha := \omega^\nu$.

12.3.1 Simplified cut equations for L_α and E_α

Given $\varphi \in \mathbf{No}^{>,\succ}$, let $E_{\lhd\alpha} := \{E_{(\alpha/\omega)n}\varphi : n \in \mathbb{N}\}$ if ν is a successor ordinal and $E_{\lhd\alpha}\varphi := \{\varphi\}$ if ν is a limit ordinal. In this subsection, we will derive the following simplified cut equations for L_α on $\mathbf{Mo}_\alpha$ and E_α on $\mathbf{No}_{\succ,\alpha}$:

$$\forall \mathfrak{a} \in \mathbf{Mo}_\alpha,\ L_\alpha \mathfrak{a} = \{L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha}\mathfrak{a}\}_{\mathbf{No}_{\succ,\alpha}} \tag{12.3.1}$$

$$= \left\{\mathbb{R},\ L_\alpha \mathfrak{a}' + \frac{1}{L_{<\alpha}\mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \,\middle|\, L_\alpha \mathfrak{a}'' - \frac{1}{L_{<\alpha}\mathfrak{a}''},\ L_{<\alpha}\mathfrak{a} : \mathfrak{a}'' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha}\right\}, \tag{12.3.2}$$

$$\forall \varphi \in \mathbf{No}_{\succ,\alpha},\ E_\alpha \varphi = \{E_{\lhd\alpha}\mathfrak{d}_\alpha(\varphi), E_\alpha \varphi_L^{\mathbf{No}_{\succ,\alpha}} \mid E_\alpha \varphi_R^{\mathbf{No}_{\succ,\alpha}}\}_{\mathbf{Mo}_\alpha} \tag{12.3.3}$$

$$= \{E_{<\alpha}\varphi, \mathcal{E}_\alpha E_\alpha \varphi_L^{\mathbf{No}_{\succ,\alpha}} \mid \mathcal{E}_\alpha E_\alpha \varphi_R^{\mathbf{No}_{\succ,\alpha}}\}. \tag{12.3.4}$$

For all $a \in \mathbf{No}^{>,\succ}$, the set $E_{\lhd\alpha}\mathfrak{d}_\alpha(a)$ contains only $L_{<\alpha}$-atomic numbers, so (12.3.3) is indeed a cut equation of the form $\{\rho \mid \lambda\}_{\mathbf{Mo}_\alpha}$.

Remark 12.3.1. The changes with respect to (12.2.1) and (12.2.5) lie in the occurrence of $\mathfrak{a}''$ instead of $\mathfrak{a}$ in (12.3.2) and the (related) absence of the left option $E_{<\alpha}((\varphi_R^{\mathbf{No}_{\succ,\alpha}} - \varphi)^{-1})$ in (12.3.4). So (12.3.2) and (12.3.4) give lighter sets of conditions than those in (12.2.1) and (12.2.5) to define L_α and E_α. This seemingly meagre simplification will be crucial in further work. Indeed, combined with Proposition 9.2.24, this allows one to determine large classes of numbers a, b with $a \sqsubseteq b \Longrightarrow E_\alpha a \sqsubseteq E_\alpha b$.

First note that the cut equations (12.3.1) and (12.3.3) if they hold are uniform (see [15, Remark 1]). Moreover, we claim that (12.3.1, 12.3.2) are equivalent and that (12.3.3, 12.3.4) are equivalent. Indeed, recall that for a thin convex partition $\mathbf{\Pi}$ of a surreal substructure $\mathbf{S}$ and any cut representation (L, R) in $\mathbf{Smp}_{\mathbf{\Pi}}$, one has

$$\{L \mid R\}_{\mathbf{Smp}_{\mathbf{\Pi}}} = \{\mathbf{\Pi}[L] \mid \mathbf{\Pi}[R]\}_{\mathbf{S}}.$$

For $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha}$ and $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\alpha}$ the classes $L_\alpha \mathfrak{a}' + (L_{<\alpha}\mathfrak{a}')^{-1}$ and $\mathcal{L}_\alpha[L_\alpha \mathfrak{a}']$ are mutually cofinal by Proposition 5.3.8. Similarly, $L_\alpha \mathfrak{a}'' - (L_{<\alpha}\mathfrak{a}'')^{-1}$ and $\mathcal{L}_\alpha[L_\alpha \mathfrak{a}'']$ are mutually coinitial. By Lemma 12.2.9, the classes $E_{<\alpha}\varphi$ and $\mathcal{E}_\alpha[E_{\lhd\alpha}\mathfrak{d}_\alpha(\varphi)]$ are mutually cofinal. So it is enough to prove that (12.3.1) and (12.3.3) are valid cut equations for L_α and E_α respectively.

Lemma 12.3.2. *If ν is a successor ordinal, then the identities* (12.3.1) *and* (12.3.3) *hold.*

Proof. Let $\mathfrak{a} \in \mathbf{Mo}_\alpha$ and set

$$\varphi := \{L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha}\mathfrak{a}\}_{\mathbf{No}_{\succ,\alpha}}$$

$$= \left\{\mathbb{R}, L_\alpha \mathfrak{a}' + \frac{1}{L_{<\alpha}\mathfrak{a}'} : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \,\middle|\, L_\alpha \mathfrak{a}'' - \frac{1}{L_{<\alpha}\mathfrak{a}''}, L_{<\alpha}\mathfrak{a} : \mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\alpha}\right\}.$$

We have $\mathcal{L}_\alpha[L_\alpha\,\mathfrak{a}_L^{\mathbf{Mo}_\alpha}] < \varphi < L_{<\alpha}\,\mathfrak{a}$ so in view of (12.2.1), it is enough to prove that $\varphi < L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha} - (L_{<\alpha}\,\mathfrak{a})^{-1}$ to conclude that $\varphi = L_\alpha\,\mathfrak{a}$. Let $\mathfrak{a}'' \in \mathfrak{a}_R^{\mathbf{Mo}_\alpha}$. If $\mathfrak{a}'' \in \mathcal{E}_\alpha^*[\mathfrak{a}]$, then the inequality $\varphi < L_\alpha\,\mathfrak{a}''$ entails $\varphi < \mathcal{L}_\alpha[L_\alpha\,\mathfrak{a}'']$ whence $\varphi < L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a}'')^{-1}$ and $\varphi < L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a})^{-1}$. Otherwise, we have $\mathfrak{a} < L_{<\alpha}\,\mathfrak{a}''$, so $L_\alpha\,\mathfrak{a} < L_\alpha\,\mathfrak{a}'' - 2$, and $L_\alpha\,\mathfrak{a}'' - (L_{<\alpha}\,\mathfrak{a})^{-1} > L_\alpha\,\mathfrak{a} + 1$. It is enough to prove that $L_\alpha\,\mathfrak{a} + 1 \geqslant \varphi$. Recall that

$$L_\alpha\,\mathfrak{a} + 1 \;=\; \left\{L_\alpha\,\mathfrak{a}, L_\alpha\,\mathfrak{a}' + \frac{1}{L_{<\alpha}\,\mathfrak{a}'} + 1 : \mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha} - \frac{1}{L_{<\alpha}\,\mathfrak{a}} + 1, L_{<\alpha}\,\mathfrak{a}\right\}$$

by (8.2.2). We see that $L_\alpha\,\mathfrak{a}' + (L_{<\alpha}\,\mathfrak{a}')^{-1} < L_\alpha\,\mathfrak{a} + 1$ for all $\mathfrak{a}' \in \mathfrak{a}_L^{\mathbf{Mo}_\alpha}$. We have $1 - (L_{<\alpha}\,\mathfrak{a})^{-1} \succ (L_{<\alpha}\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha})^{-1}$ so $L_\alpha\,\mathfrak{a}_R^{\mathbf{Mo}_\alpha} - (L_{<\alpha}\,\mathfrak{a})^{-1} + 1 > \varphi$. Thus $\varphi \leqslant L_\alpha\,\mathfrak{a} + 1$. So (12.3.1) holds.

Now let $\psi \in \mathbf{No}_{\succ,\alpha}$ and set

$$\mathfrak{b} \;:=\; \{E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi), E_\alpha\,\psi_L^{\mathbf{No}_{\succ,\alpha}} \mid E_\alpha\,\psi_R^{\mathbf{No}_{\succ,\alpha}}\}_{\mathbf{Mo}_\alpha}.$$

By uniformity of (12.3.1), we have

$$L_\alpha\,\mathfrak{b} \;=\; \{L_\alpha\,E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi), \psi_L^{\mathbf{No}_{\succ,\alpha}} \mid \psi_R^{\mathbf{No}_{\succ,\alpha}}, L_{<\alpha}\,\mathfrak{b}\}_{\mathbf{No}_{\succ,\alpha}},$$

whence $L_\alpha\,\mathfrak{b} \sqsupseteq \{\psi_L^{\mathbf{No}_{\succ,\alpha}} \mid \psi_R^{\mathbf{No}_{\succ,\alpha}}\}_{\mathbf{No}_{\succ,\alpha}} = \psi$. Conversely, $\mathfrak{b} > E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi)$ and $\mathfrak{b} > E_{<\alpha}\,\psi$, so $\psi < L_{<\alpha}\,\mathfrak{b}$. We have $L_\alpha\,E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi) = L_\alpha\,\mathfrak{d}_\alpha(\psi) + \mathbb{N}$. Since $L_\alpha\,\mathfrak{d}_\alpha(\psi) < L_{\alpha_{/\omega}}\,\mathfrak{d}_\alpha(\psi) \prec \psi$, this yields $L_\alpha\,E_{\alpha_{/\omega}\mathbb{N}}\,\mathfrak{d}_\alpha(\psi) < \psi$. This proves that ψ lies in the cut defining $L_\alpha\,\mathfrak{b}$. We conclude that $\psi = L_\alpha\,\mathfrak{b}$, hence (12.3.3) holds. □

We now assume that ν is a limit ordinal. For $z \in \mathbf{No}$, define

$$\begin{aligned} F(z) \;&:=\; \{\mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{\succ,\alpha}}\,z), F(z_L) \mid F(z_R)\}_{\mathbf{Mo}_\alpha}, \quad \text{and} \\ \Xi z \;&:=\; \{\mathbb{R}, \Xi z' + (L_{<\alpha}\,F(z'))^{-1} : z' \in z_L \mid \Xi z_R - (L_{<\alpha}\,F(z))^{-1}\}. \end{aligned}$$

Lemma 12.3.3. *For all $z \in \mathbf{No}$, we have*

$$F(z) \;\; \textit{is defined} \tag{12.3.5}$$

$$\Xi z \;\; \textit{is defined} \tag{12.3.6}$$

$$\Xi z \;=\; \Xi_{\mathbf{No}_{\succ,\alpha}}\,z \tag{12.3.7}$$

$$F(z) \;=\; E_\alpha\,\Xi z \tag{12.3.8}$$

Proof. We prove the result by induction on $(\mathbf{No}, \sqsubset)$. Let $z \in \mathbf{No}$ be such that (12.3.5), (12.3.6), (12.3.7) and (12.3.8) hold for all $y \in \mathbf{No}$ with $y \sqsubset z$.

For $z'' \in z_R$ and $z' \in z_L$, we have $\mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{\succ,\alpha}}\,z) \leqslant \mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{\succ,\alpha}}\,z'') < F(z'')$. We have $F(z') < F(z'')$ by definition of $F(z'')$ if $z' \in (z'')_L$ and by definition of $F(z')$ if $z'' \in (z')_R$. This proves that $F(z)$ is defined.

Let $z' \in z_L$ and $z'' \in z_R$. If $z' \in (z'')_L$, then we have $\Xi z'' > \Xi z' + (L_{<\alpha}\,F(z'))^{-1}$ by definition of $\Xi z''$. Since $F(z') < F(z)$ and $F(z), F(z') \in \mathbf{Mo}_\alpha$, we have $L_\gamma\,F(z') \prec L_\gamma\,F(z)$ for all $\gamma < \alpha$. We deduce that $\Xi z'' - (L_{<\alpha}\,F(z))^{-1} > \Xi z' + (L_{<\alpha}\,F(z'))^{-1}$. If $z'' \in (z')_L$, then $\Xi z' < \Xi z'' - (L_{<\alpha}\,F(z'))^{-1}$ by definition of $\Xi z'$. Since $F(z') < F(z)$, we obtain $\Xi z'' - (L_{<\alpha}\,F(z))^{-1} > \Xi z' + (L_{<\alpha}\,F(z'))^{-1}$. This proves that Ξz is defined.

Since (12.3.7) and (12.3.8) hold on $z_\sqsubset$, we have

$$\Xi z \;=\; \{\mathbb{R}, \Xi_{\mathbf{No}_{\succ,\alpha}}\,z' + (L_{<\alpha}\,E_\alpha\,\Xi_{\mathbf{No}_{\succ,\alpha}}\,z')^{-1} : z' \in z_L \mid \Xi_{\mathbf{No}_{\succ,\alpha}}\,z_R - (L_{<\alpha}\,E_\alpha\,\Xi_{\mathbf{No}_{\succ,\alpha}}\,z)^{-1}\}$$

By (12.2.10), this yields $\Xi z = \Xi_{\mathbf{No}_{\succ,\alpha}}\,z$, so (12.3.7) holds for z.

From (12.3.7), we get $\mathfrak{d}_\alpha(\Xi_{\mathbf{No}_{\succ,\alpha}}\,z) = \mathfrak{d}_\alpha(\Xi z)$. By Proposition 12.2.12 and our assumption that (12.3.8) holds on $z_\sqsubset$, we have

$$\begin{aligned} L_\alpha\,F(z) \;&=\; \{\mathbb{R}, \mathcal{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)], \mathcal{L}_\alpha[L_\alpha\,F(z_L)] \mid L_\alpha\,F(z_R) - (L_{<\alpha}\,F(z))^{-1}, L_{<\alpha}\,F(z)\} \\ &=\; \{\mathbb{R}, \mathcal{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)], \mathcal{L}_\alpha[\Xi z_L] \mid \Xi z_R - (L_{<\alpha}\,F(z))^{-1}, L_{<\alpha}\,F(z)\}. \end{aligned}$$

Recall that $\Xi z = \{\mathbb{R}, \mathcal{L}_\alpha[\Xi z_L] \mid \Xi z_R - (L_{<\alpha}\,F(z))^{-1}\}$. Therefore it suffices to show that Ξz lies in the cut $(\mathcal{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)] \mid L_{<\alpha}\,F(z))$ to conclude that $L_\alpha\,F(z) = \Xi z$ and thus that $F(z) = E_\alpha\,\Xi z$. Now $L_\alpha\,\mathfrak{d}_\alpha(\Xi z) < \mathcal{E}_\alpha^*[\Xi z]$ so $L_\alpha\,\mathfrak{d}_\alpha(\Xi z) \prec \Xi z$ and $\mathcal{L}_\alpha[L_\alpha\,\mathfrak{d}_\alpha(\Xi z)] < \Xi z$. We have $F(z) > \mathfrak{d}_\alpha(\Xi z)$, where $F(z) \in \mathbf{Mo}_\alpha$. Since ν is a limit ordinal, Lemma 12.2.9 implies that $F(z) > E_{<\alpha}\Xi z$, so $\Xi z < L_{<\alpha}\,F(z)$. This completes the proof that $F(z) = E_\alpha\,\Xi z$. □

Corollary 12.3.4. *The identities* (12.3.1), (12.3.2), (12.3.3), *and* (12.3.4) *all hold.*

Proof. It is enough to prove (12.3.1) and (12.3.3). The identity (12.3.3) follows from (12.3.7) and (12.3.8). In order to obtain (12.3.1), we consider $\mathfrak{a} \in \mathbf{Mo}_\alpha$, set $\psi := \{L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha} \mathfrak{a}\}_{\mathbf{No}_{\succ,\alpha}}$, and we show that $\mathfrak{a} = E_\alpha \psi$. Since (12.3.3) is uniform, we have

$$\begin{aligned} E_\alpha \psi &= \{\mathfrak{d}_\alpha(\psi), E_\alpha L_\alpha \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid E_\alpha L_\alpha \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, E_\alpha L_{<\alpha} \mathfrak{a}\}_{\mathbf{Mo}_\alpha} \\ &= \{\mathfrak{d}_\alpha(\psi), \mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, E_\alpha L_{<\alpha} \mathfrak{a}\}_{\mathbf{Mo}_\alpha}. \end{aligned}$$

We have $\mathfrak{d}_\alpha(\psi) < \mathfrak{a}$ because $\psi < L_{<\alpha} \mathfrak{a}$, and $E_\alpha L_{<\alpha} \mathfrak{a} > \mathfrak{a}$ because $E_\alpha > E_{<\alpha}$ on $\mathbf{No}^{>,\succ}$. Since $\mathfrak{a} = \{\mathfrak{a}_L^{\mathbf{Mo}_\alpha} \mid \mathfrak{a}_R^{\mathbf{Mo}_\alpha}\}_{\mathbf{Mo}_\alpha}$, we deduce that $E_\alpha \psi = \mathfrak{a}$. □

Remark 12.3.5. The simplified cut equations for E_α, L_α can be viewed as alternative definitions for those functions, since they hold inductively on their domain of definition. It is unclear how to develop our theory directly upon these alternative definitions. In particular, does there exists a direct way to see that the cut equation (12.3.2) is warranted, and that the corresponding function satisfies $\mathbf{R}_\mu$ and $\mathbf{M}_\mu$?

12.3.2 Identities involving $\mathbf{Tr}_\alpha$ and $\mathbf{Tr}_\alpha^*$.

Proposition 12.3.6. *Defining* $\mathbf{Tr}_\alpha := \mathbf{Smp}_{\mathcal{L}'_\alpha}$ *as in Section 12.1.2, we have* $\mathbf{Tr}_\alpha = \mathbf{No}_{\succ,\alpha}$.

Proof. Let $\varphi \in \mathbf{No}_{\succ,\alpha}$. We have $E_\alpha \mathcal{L}_\alpha[\varphi] = \mathcal{E}_\alpha[E_\alpha \varphi]$ by [14, Proposition 7.22]. Recall that $\mathcal{E}_\alpha[a] = \mathcal{E}'_\alpha[a]$ for all $a \in \mathbf{No}^{>,\succ}$. Now $\mathcal{E}'_\alpha \circ E_\alpha = E_\alpha \circ \mathcal{L}'_\alpha$ by definition of $\mathcal{L}'_\alpha$, so $E_\alpha \mathcal{L}_\alpha[\varphi] = E_\alpha \mathcal{L}'_\alpha[\varphi]$ and $\mathcal{L}_\alpha[\varphi] = \mathcal{L}'_\alpha[\varphi]$. By definition of $\mathbf{Tr}_\alpha$, we conclude that $\mathbf{Tr}_\alpha = \mathbf{Smp}_{\mathcal{L}_\alpha} = \mathbf{No}_{\succ,\alpha}$. □

Assume that ν is a successor ordinal. Then we have $\mathbf{No}_{\succ,\alpha} = \mathbf{No}_{\succ,\alpha} + \mathbb{R}$ by Lemma 5.3.2, so the functions $T_r \Xi_{\mathbf{No}_{\succ,\alpha}}$ and $\Xi_{\mathbf{No}_{\succ,\alpha}} T_r$ are both strictly increasing bijections from $\mathbf{No}$ onto $\mathbf{No}_{\succ,\alpha}$.

Lemma 12.3.7. *Assume that* ν *is a successor ordinal. Then for* $r \in \mathbb{R}$*, we have* $T_r \Xi_{\mathbf{No}_{\succ,\alpha}} = \Xi_{\mathbf{No}_{\succ,\alpha}} T_r$ *on* $\mathbf{No}$.

Proof. Let us abbreviate $\Xi := \Xi_{\mathbf{No}_{\succ,\alpha}}$. We prove the lemma by induction on $(\mathbf{No}, \sqsubseteq) \times (\mathbb{R}, \sqsubseteq)$. Let $(z, r) \in \mathbf{No} \times \mathbb{R}$ with

$$\Xi y + s = \Xi(y + s)$$

whenever $(y, s) \in \mathbf{No} \times \mathbb{R}$ is strictly simpler than (z, r). We let z', z'', r', r'' denote generic elements of z_L, z_R, r_L, r_R and we note that $r', r'' \in \mathbb{R}$. By (12.2.9), we have

$$\begin{aligned} \Xi(z + r) &= \left\{ \Xi(z' + r) + \frac{1}{L_{<\alpha} E_\alpha \Xi(z' + r)},\ \Xi(z + r') + \frac{1}{L_{<\alpha} E_\alpha \Xi(z + r')} \right| \\ &\qquad \left. \Xi(z + r'') - \frac{1}{L_{<\alpha} E_\alpha \Xi(z + r'')},\ \Xi(z'' + r) - \frac{1}{L_{<\alpha} E_\alpha \Xi(z'' + r)} \right\}_{\mathbf{No}^{>,\succ}} \\ &= \left\{ T_r \Xi z' + \frac{1}{L_{<\alpha} E_\alpha T_r \Xi z'},\ T_{r'} \Xi z + \frac{1}{L_{<\alpha} E_\alpha T_{r'} \Xi z} \right| \\ &\qquad \left. T_{r''} \Xi z - \frac{1}{L_{<\alpha} E_\alpha T_{r''} \Xi z},\ T_r \Xi z'' - \frac{1}{L_{<\alpha} E_\alpha T_r \Xi z''} \right\}_{\mathbf{No}^{>,\succ}}. \end{aligned}$$

Recall that ν is a successor ordinal. In view of the functional equation, the sets $L_{<\alpha} E_\alpha \mathcal{T} a$ and $L_{<\alpha} E_\alpha a$ are mutually cofinal and coinitial. Moreover $T_s(z + b) = T_s z + b$ for all $s \in \mathbb{R}$ and $b \in \mathbf{No}$, so

$$\begin{aligned} \Xi(z + r) &= \left\{ T_r\!\left(\Xi z' + \frac{1}{L_{<\alpha} E_\alpha \Xi z'} \right),\ T_{r'}\!\left(\Xi z + \frac{1}{L_{<\alpha} E_\alpha \Xi z} \right) \right| \\ &\qquad \left. T_{r''}\!\left(\Xi z - \frac{1}{L_{<\alpha} E_\alpha \Xi z} \right),\ T_r\!\left(\Xi z'' - \frac{1}{L_{<\alpha} E_\alpha \Xi z''} \right) \right\}_{\mathbf{No}^{>,\succ}}. \end{aligned}$$

By (8.2.2), we have

$$T_r \,\Xi z \;=\; \left\{ T_r\!\left(\Xi z' + \frac{1}{L_{<\alpha}\, E_\alpha\, \Xi z'}\right),\, T_{r'}\, \Xi z \;\middle|\; T_{r''}\, \Xi z,\, T_r\!\left(\Xi z'' - \frac{1}{L_{<\alpha}\, E_\alpha\, \Xi z''}\right) \right\}_{\mathbf{No}^{>,\succ}}.$$

The numbers $T_r \,\Xi z, T_{r'}\, \Xi z$ and $T_{r''}\, \Xi z$ are α-truncated so $T_r \,\Xi z$ lies in the cut

$$\left(\bigcup_{r'} T_{r'}\!\left(\Xi z + \frac{1}{L_{<\alpha}\, E_\alpha\, \Xi z}\right) \;\middle|\; \bigcup_{r''} T_{r''}\!\left(\Xi z - \frac{1}{L_{<\alpha}\, E_\alpha\, \Xi z}\right) \right)_{\mathbf{No}^{>,\succ}}.$$

We deduce that $T_r \,\Xi z = \Xi T_r\, z$. The result follows by induction. □

Lemma 12.3.8. *If ν is a successor ordinal, then we have $\mathcal{T} \lneqq \mathcal{L}_\alpha^*$ on $\mathbf{No}^{>,\succ}$. Consequently, $\mathbf{Tr}_\alpha^* = \mathbf{No}_\succ^>$.*

Proof. The set $E_{<\alpha}$ is pointwise cofinal in $\mathcal{E}_\alpha^*$. So $L_\alpha\, E_{<\alpha}\, E_\alpha$ is pointwise cofinal in $\mathcal{L}_\alpha^*$. For $\gamma < \alpha$, there is $n \in \mathbb{N}$ such that $\gamma \leqslant \alpha_{/\omega}\, n$. We have

$$L_\alpha\, E_\gamma\, E_\alpha \;\leqslant\; L_\alpha\, E_{\alpha_{/\omega} n}\, E_\alpha \;=\; (L_\alpha\, E_{\alpha_{/\omega}}\, E_\alpha)^{\circ n} \;=\; (L_\alpha\, E_\alpha\, T_1)^{\circ n} \;=\; T_1^{\circ n} \;=\; T_n \;\in\; \mathcal{T}.$$

We deduce that $\mathcal{T} \lneqq \mathcal{L}_\alpha^*$ on $\mathbf{No}^{>,\succ}$, whence $\mathbf{Tr}_\alpha^* = \mathbf{Smp}_{\mathcal{T}} = \mathbf{No}_\succ^>$. □

12.3.3 Identities involving $\mathbf{Mo}_\alpha$ and $\mathbf{Mo}_\alpha^*$.

Lemma 12.3.9. *If ν is a successor ordinal, then for $z \in \mathbf{No}$ we have*

$$\Xi_{\mathbf{Mo}_\alpha}(z-1) \;=\; L_{\alpha_{/\omega}}\, \Xi_{\mathbf{Mo}_\alpha}\, z.$$

Proof. This can be seen as a converse to the proof of the identity (12.2.3). We proceed by induction on $(\mathbf{No}, \sqsubseteq)$. Let z be such that the relation holds on $z_\sqsubset$. By (12.2.3), we have

$$\begin{aligned} L_{\alpha_{/\omega}}\, \Xi_{\mathbf{Mo}_\alpha}\, z &= \{L_{\alpha_{/\omega}}\, (\Xi_{\mathbf{Mo}_\alpha}\, z)_L^{\mathbf{Mo}_\alpha} \mid L_{\alpha_{/\omega}}\, (\Xi_{\mathbf{Mo}_\alpha}\, z)_R^{\mathbf{Mo}_\alpha}, \Xi_{\mathbf{Mo}_\alpha}\, z\}_{\mathbf{Mo}_\alpha} \\ &= \{L_{\alpha_{/\omega}}\, \Xi_{\mathbf{Mo}_\alpha}\, z_L \mid L_{\alpha_{/\omega}}\, \Xi_{\mathbf{Mo}_\alpha}\, z_R, \Xi_{\mathbf{Mo}_\alpha}\, z\}_{\mathbf{Mo}_\alpha} \\ &= \{\Xi_{\mathbf{Mo}_\alpha}(z_L - 1) \mid \Xi_{\mathbf{Mo}_\alpha}(z_R - 1), \Xi_{\mathbf{Mo}_\alpha}\, z\}_{\mathbf{Mo}_\alpha} \qquad \text{(by the inductive hypothesis)} \\ &= \Xi_{\mathbf{Mo}_\alpha}\, \{z_L - 1 \mid z_R - 1, z\} \\ &= \Xi_{\mathbf{Mo}_\alpha}(z-1) \quad \text{by (8.2.2).} \end{aligned}$$

We conclude by induction. □

Noting that $E_{\alpha_{/\omega}} = E_\alpha\, T_1\, L_\alpha$ on $\mathbf{No}^{>,\succ}$, the previous relation further generalizes as follows.

Proposition 12.3.10. *Assume that ν is a successor ordinal and let $r \in \mathbb{R}$. Then*

$$\Xi_{\mathbf{Mo}_\alpha}\, T_r \;=\; E_\alpha\, T_r\, L_\alpha\, \Xi_{\mathbf{Mo}_\alpha} \tag{12.3.9}$$

Proof. We proceed by induction. Let $(z, r) \in \mathbf{No} \times \mathbb{R}$ be such that

$$\Xi_{\mathbf{Mo}_\alpha}\, T_s\, y \;=\; E_\alpha\, T_s\, L_\alpha\, \Xi_{\mathbf{Mo}_\alpha}\, y$$

for all strictly simpler $(y, s) \in \mathbf{No} \times \mathbb{R}$ with respect to the product order $\sqsubset \times \sqsubset$. For $s \in \mathbb{R}$, let ϕ_s be the function $b \longmapsto E_\alpha\, T_s\, L_\alpha\, b$ on $\mathbf{No}^{>,\succ}$ and let $\mathfrak{a} := \Xi_{\mathbf{Mo}_\alpha}\, z$. By (8.2.2) and Proposition 10.2.3, we have

$$\begin{aligned} \Xi_{\mathbf{Mo}_\alpha}(z+r) &= \{\mathbb{R}, \mathcal{E}_\alpha\, \Xi_{\mathbf{Mo}_\alpha}(z_L + r), \mathcal{E}_\alpha\, \Xi_{\mathbf{Mo}_\alpha}(z + r_L) \mid \mathcal{E}_\alpha\, \Xi_{\mathbf{Mo}_\alpha}(z_R + r), \mathcal{E}_\alpha\, \Xi_{\mathbf{Mo}_\alpha}(z + r_R)\} \\ &= \{\mathbb{R}, \mathcal{E}_\alpha\, \phi_r(\mathfrak{a}_L^{\mathbf{Mo}_\alpha}), \mathcal{E}_\alpha\, \phi_{r_L}(\mathfrak{a}) \mid \mathcal{E}_\alpha\, \phi_r(\mathfrak{a}_R^{\mathbf{Mo}_\alpha}), \mathcal{E}_\alpha\, \phi_{r_R}(\mathfrak{a})\}. \end{aligned}$$

By (12.3.1), Lemma 12.3.7 and (8.2.2), we have:

$$T_r\, L_\alpha\, \mathfrak{a} \;=\; \{T_r\, L_\alpha\, \mathfrak{a}_L^{\mathbf{Mo}_\alpha}, T_{r_L}\, L_\alpha\, \mathfrak{a} \mid T_{r_R}\, L_\alpha\, \mathfrak{a}, T_r\, L_\alpha\, \mathfrak{a}_R^{\mathbf{Mo}_\alpha}, L_{<\alpha}\, \mathfrak{a}\}_{\mathbf{Tr}_\alpha}.$$

We deduce that

$$\begin{aligned}\phi_r(\mathfrak{a}) &= \{E_{<\alpha} T_r L_\alpha \mathfrak{a}, \mathcal{E}_\alpha \phi_r(\mathfrak{a}_L^{\mathbf{Mo}_\alpha}), \mathcal{E}_\alpha \phi_{r_L}(\mathfrak{a}) \mid \mathcal{E}_\alpha \phi_{r_R}(\mathfrak{a}), \mathcal{E}_\alpha \phi_r(\mathfrak{a}_R^{\mathbf{Mo}_\alpha}), \mathcal{E}_\alpha E_\alpha L_{<\alpha} \mathfrak{a}\} \\ &= \{E_{<\alpha} L_\alpha \mathfrak{a}, \mathcal{E}_\alpha \phi_r(\mathfrak{a}_L^{\mathbf{Mo}_\alpha}), \mathcal{E}_\alpha \phi_{r_L}(\mathfrak{a}) \mid \mathcal{E}_\alpha \phi_{r_R}(\mathfrak{a}), \mathcal{E}_\alpha \phi_r(\mathfrak{a}_R^{\mathbf{Mo}_\alpha}), E_\alpha L_{<\alpha} \mathfrak{a}\}.\end{aligned}$$

It is enough to prove that $E_{<\alpha} L_\alpha \mathfrak{a} < \Xi_{\mathbf{Mo}_\alpha}(z+r) < E_\alpha L_{<\alpha} \mathfrak{a}$ to conclude that $\phi_r(\mathfrak{a}) = \Xi_{\mathbf{Mo}_\alpha}(z+r)$. Toward this, fix an $n \in \mathbb{N}$ with $-n \leqslant r \leqslant n$. Lemma 12.3.9 yields

$$\begin{aligned}\Xi_{\mathbf{Mo}_\alpha}(z+r) &\leqslant \Xi_{\mathbf{Mo}_\alpha}(z+n) = E_{\alpha/\omega^n} \mathfrak{a} < E_\alpha L_{<\alpha} \mathfrak{a} \\ \Xi_{\mathbf{Mo}_\alpha}(z+r) &\geqslant \Xi_{\mathbf{Mo}_\alpha}(z-n) = L_{\alpha/\omega^n} \mathfrak{a} > E_{<\alpha} L_\alpha \mathfrak{a}.\end{aligned}$$

We conclude by induction that (12.3.9) holds. □

Remark 12.3.11. For $r, s \in \mathbb{R}$, we have $\phi_{r+s} = \phi_r \circ \phi_s$, and $\phi_1 = E_{\alpha/\omega}$. Therefore we can see $(\phi_r)_{r\in\mathbb{R}}$ as a system of fractional and real iterates of the hyperexponential function $E_{\alpha/\omega}$ on $\mathbf{No}^{>,\succ}$. The previous proposition shows that the action of those iterates on $L_{<\alpha}$-atomic numbers reduces to translations, modulo the parametrization $\Xi_{\mathbf{Mo}_\alpha}$. In particular, one can compute the functional square root of exp on $\mathbf{Mo}_\omega$ in terms of sign sequences using the material from [9].

Proposition 12.3.12. *If ν is a successor ordinal, then $\mathbf{Mo}_\alpha^* = \mathbf{Mo}_\alpha \prec \mathbf{No}_\succ$.*

Proof. For $\theta \in \mathbf{No}_\succ$, we have $\theta_L + \mathbb{N} < \theta < \theta_R - \mathbb{N}$. By Lemma 12.3.9, it follows that $E_{\alpha/\omega\mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_L < \Xi_{\mathbf{Mo}_\alpha} \theta < L_{\alpha/\omega\mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_R$. This implies that $\mathcal{E}_\alpha^* \Xi_{\mathbf{Mo}_\alpha} \theta_L < \Xi_{\mathbf{Mo}_\alpha} \theta < \mathcal{E}_\alpha^* \Xi_{\mathbf{Mo}_\alpha} \theta_R$, so $\Xi_{\mathbf{Mo}_\alpha} \theta$ is $\mathcal{E}_\alpha^*$-simple.

Conversely, consider $\theta \in \mathbf{No}^{>,\succ}$ such that $\Xi_{\mathbf{Mo}_\alpha} \theta$ is $\mathcal{E}_\alpha^*$-simple. We have $\Xi_{\mathbf{Mo}_\alpha} \theta_L \subseteq (\Xi_{\mathbf{Mo}_\alpha} \theta)_L$ and $\Xi_{\mathbf{Mo}_\alpha} \theta_R \subseteq (\Xi_{\mathbf{Mo}_\alpha} \theta)_R$, whence $E_{\alpha/\omega\mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_L < \Xi_{\mathbf{Mo}_\alpha} \theta < L_{\alpha/\omega\mathbb{N}} \Xi_{\mathbf{Mo}_\alpha} \theta_R$. We obtain $\theta_L + \mathbb{N} < \theta < \theta_R - \mathbb{N}$, which proves that $\theta \in \mathbf{No}_\succ$. □

Proposition 12.3.13. *We have $E_\alpha \mathbf{Tr}_\alpha^* = \mathbf{Mo}_\alpha^*$.*

Proof. Let $\varphi \in \mathbf{Tr}_\alpha^*$. So $\varphi \in \mathbf{Tr}_\alpha$. By Proposition 9.2.24, the number $E_\alpha \varphi$ is simplest in

$$E_\alpha(\mathcal{E}_\alpha^*[\varphi] \cap \mathbf{Tr}_\alpha) = \mathcal{E}_\alpha^*[E_\alpha \varphi] \cap \mathbf{Mo}_\alpha.$$

Since $\mathbf{Mo}_\alpha^* \subseteq \mathbf{Mo}_\alpha$, we have $E_\alpha \varphi \sqsubseteq \mathcal{E}_\alpha^*[E_\alpha \varphi] \cap \mathbf{Mo}_\alpha^*$ so $E_\alpha \varphi \sqsubseteq \mathfrak{d}_\alpha^*(E_\alpha \varphi)$. We deduce that $E_\alpha \varphi = \mathfrak{d}_\alpha^*(E_\alpha \varphi)$, so $E_\alpha \varphi$ is $\mathcal{E}_\alpha^*$-simple. Conversely, let $\mathfrak{a} \in \mathbf{Mo}_\alpha^*$. By Proposition 9.2.24 the number $L_\alpha \mathfrak{a}$ is simplest in $L_\alpha(\mathcal{E}_\alpha^*[\mathfrak{a}] \cap \mathbf{Mo}_\alpha) = \mathcal{L}_\alpha^*[L_\alpha \mathfrak{a}] \cap \mathbf{No}_{\succ,\alpha}$. Since $\mathbf{Tr}_\alpha^* \subseteq \mathbf{No}_{\succ,\alpha}$, we have $L_\alpha \mathfrak{a} \sqsubseteq \mathcal{L}_\alpha^*[L_\alpha \mathfrak{a}] \cap \mathbf{Tr}_\alpha^*$ so $L_\alpha \mathfrak{a} \sqsubseteq \sharp_\alpha^*(L_\alpha \mathfrak{a})$. We deduce that $L_\alpha \mathfrak{a} \sqsubseteq \sharp_\alpha^*(L_\alpha \mathfrak{a})$ is $\mathcal{L}_\alpha^*$-simple. □

Corollary 12.3.14. *If ν is a successor ordinal, then $\mathbf{Mo}_\alpha^* = E_\alpha \mathbf{No}_\succ^{>}$.*

Chapter 13

Well-nestedness

This chapter and the two subsequent ones are contained in essence in the pre-print [13] with van der Hoeven. In this chapter, we prove Theorem 13.2.7, i.e. that each number is well-nested. Using the terminology from the introduction, the general idea of the proof is as follows:

i. We assume for contradiction that there exists a number a that is not well-nested, and we choose a simplest (i.e. $\sqsubseteq$-minimal) such number.

ii. By definition, there exists a bad path $P=(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}}$ in a. Recall that a and P give rise to a sequence $(\varphi_i,\psi_{i+1},r_i,\iota_i,\alpha_i,\beta_i,a_{i+1},\delta_i)_{i\in\mathbb{N}}$ that describes the path P within a. Now consider the lowest level i at which the branching phenomenon occurs. Then

$$a_i=\varphi_i+r_i\,\mathrm{e}^{\psi_{i+1}}\,(L_{\beta_i}\,E_{\alpha_i}^{u_{i+1}})^{\iota_i}+\delta_i \tag{13.0.1}$$

and

$$\begin{aligned}\mathfrak{m}_{i+1} &\in \operatorname{supp}\psi_{i+1}, \text{ or}\\ r_i &\notin \{-1,1\}, \text{ or}\\ \beta_i &\neq 0, \text{ or}\\ \delta_i &\neq 0.\end{aligned}$$

Here we regard a_i as a subexpression of a at level i and we may write $a=H_i(a_i)$ for a suitable function that involves hyperexponentials and hyperlogarithms .

iii. If $\mathfrak{m}_{i+1}\in\operatorname{supp}\psi_{i+1}$, then we show that the number $b_i=\varphi_i+r_i\,\mathrm{e}^{\psi_{i+1}}$ is strictly simpler than a_i. Otherwise, we show the same thing for $b_i=\varphi_i+\operatorname{sign}(r_i)\,\mathrm{e}^{\psi_{i+1}}\,(E_{\alpha_i}^{u_{i+1}})^{\iota_i}$.

iv. We next show that the substitution $H_i(b_i)$ of a_i by b_i in a is strictly simpler than a.

v. We finally show that P is a path in $H_i(b_i)$, contradicting the $\sqsubset$-minimality of a.

The second step requires a way to expand numbers as hyperseries, as in the formula (13.0.1). This gives rise to the notion of hyperserial expansions that will be studied in section 13.1.1. The third and fourth steps requires techniques to derive a relation $x\sqsubseteq y$ from similar relations for subexpressions of x and y. This will be the subject of sections 13.1.3 and 13.2. For the last step, we must know how to deduce the existence of paths in a number x from the existence of paths in certain subexpressions of x. Since hyperexponentials in particular have involved expansions around truncated series (see (5.3.1)), this step requires a careful study of paths which is carried out in sections 13.2 and 13.1.2. We prove Theorem 13.2.7 in section 13.2.2.

Before we start, we recall a few inequalities. Let $\nu\in\mathbf{On}^{>}$ and write

$$\alpha:=\omega^{\nu}.$$

The first inquality below is immediate by definition and by the fact that $\mathcal{H}<E_\alpha$. The others are Lemmas 12.2.7 and 12.2.9 and Proposition 12.2.15, in that order:

$$\begin{aligned}\mathcal{E}_{\alpha\omega} &< E_\alpha && (13.0.2)\\ E_{<\alpha} &< E_\alpha\,H_2\,L_\alpha && (13.0.3)\\ \langle E_\gamma:\gamma<\alpha\rangle &\lneqq \mathcal{E}_\alpha \quad \text{if } \nu \text{ is a limit} && (13.0.4)\\ \forall\gamma<\rho<\alpha,\forall r,s>1,\quad E_\gamma\,H_r\,L_\gamma &< E_\rho\,H_s\,L_\rho. && (13.0.5)\end{aligned}$$

From (13.0.5), we also deduce that

$$\{E_\gamma H_r L_\gamma : \gamma < \alpha, r \in \mathbb{R}\} \not\leqslant \mathcal{E}_\alpha. \tag{13.0.6}$$

13.1 Hyperserial expansions

We start by introducing our notion of hyperserial expansions of monomials, which will play a crucial role in the sequel of the thesis.

13.1.1 Hyperserial expansions

Recall that any number can be written as a well-based series. In order to represent numbers as hyperseries, it therefore suffices to devise a means to represent the infinitely large monomials $\mathfrak{m}$ in $\mathbf{Mo}^{\succ}$. We do this by taking a hyperlogarithm $L_\alpha \mathfrak{m}$ of the monomial and then recursively applying the same procedure for the monomials in this new series. This procedure stops when we encounter a monomial in $L_{\mathbf{On}} \omega$.

Technically speaking, instead of directly applying a hyperlogarithm L_α to the monomial, it turns out to be necessary to first decompose $\mathfrak{m}$ as a product $\mathfrak{m} = \mathrm{e}^\psi \mathfrak{n}$ and write $\mathfrak{n}$ as a hyperexponential (or more generally as the hyperlogarithm of a hyperexponential). This naturally leads to the introduction of *hyperserial expansions* of monomials $\mathfrak{m} \in \mathbf{Mo}^{\neq 1}$, as we will detail now.

Definition 13.1.1. *We say that a purely infinite number $\varphi \in \mathbf{No}_{\succ}$ is* **tail-atomic** *if $\varphi = \psi + \iota \mathfrak{a}$, for certain $\psi \in \mathbf{No}_{\succ}$, $\iota \in \{-1, 1\}$, and $\mathfrak{a} \in \mathbf{Mo}_\omega$.*

Definition 13.1.2. *Let $\mathfrak{m} \in \mathbf{Mo}^{\neq 1}$. Assume that there are $\psi \in \mathbf{No}_{\succ}$, $\iota \in \{-1, 1\}$, $\alpha \in \{0\} \cup \omega^{\mathbf{On}}$, $\beta \in \mathbf{On}$ and $u \in \mathbf{No}^{>,\succ}$ such that*

$$\mathfrak{m} = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota, \tag{13.1.1}$$

with $\operatorname{supp} \psi \succ L_{\beta+1} E_\alpha^u$. *Then we say that (13.1.1) is a* **hyperserial expansion of type I** *if*

- $\beta \omega < \alpha$;
- $E_\alpha^u \in \mathbf{Mo}_\alpha \setminus L_{<\alpha} \mathbf{Mo}_{\alpha\omega}$;
- $\alpha = 1 \Longrightarrow (\psi = 0$ *and u is not tail-atomic).*

We say that (13.1.1) is a **hyperserial expansion of type II** *if $\alpha = 0$ and $u = \omega$, so that $E_\alpha^u = \omega$ and*

$$\mathfrak{m} = \mathrm{e}^\psi (L_\beta \omega)^\iota. \tag{13.1.2}$$

Note that u is α-truncated in expansions of type I, since E_α^u is in particular $L_{<\alpha}$-atomic. Expansions of type II are those for which $E_\alpha^u = \omega$. Formally speaking, hyperserial expansions can be represented by tuples $(\psi, \iota, \alpha, \beta, u)$. By convention, we also consider

$$1 = \mathrm{e}^0 (L_0 E_0 0)^0,$$

to be a hyperserial expansion of the monomial $\mathfrak{m} = 1$; this expansion is represented by the tuple $(0, 0, 0, 0, 0)$.

Example 13.1.3. We will give a hyperserial expansion for the monomial

$$\mathfrak{m} = \exp(2 E_\omega \omega - \sqrt{\omega} + L_{\omega+1} \omega),$$

and show how it can be expressed as a hyperseries. Note that

$$\log \mathfrak{m} = 2 E_\omega \omega - \sqrt{\omega} + L_{\omega+1} \omega$$

is tail-atomic since $L_\omega \omega$ is log-atomic. Now $L_\omega \omega = L_\omega \omega$ is a hyperserial expansion of type II and we have $L_{\omega+1} \omega \prec E_\omega \omega, \sqrt{\omega}$. Hence $\mathfrak{m} = \mathrm{e}^{2 E_\omega \omega - \sqrt{\omega}} (L_\omega \omega)$ is a hyperserial expansion of type II.

Let $\psi := 2\, E_\omega\, \omega - \sqrt{\omega}$, so $\mathfrak{m} = \mathrm{e}^{\psi}\,(L_\omega\, \omega)$. We may further expand each monomial in $\operatorname{supp}\psi$. We clearly have $E_\omega\, \omega \in \mathbf{Mo}_{\omega^2}$. We claim that $E_\omega\, \omega \in \mathbf{Mo}_{\omega^2} \setminus L_{<\omega^2}\,\mathbf{Mo}_{\omega^3}$. Indeed, if we could write $E_\omega\, \omega = L_n\, L_{\omega m}\, \mathfrak{a}$ for some $\mathfrak{a} \in \mathbf{Mo}_{\omega^3}$ and $n, m \in \mathbb{N}^>$, then $\omega = L_\omega(L_n\, L_{\omega m}\, \mathfrak{a}) = L_{\omega(m+1)}\, \mathfrak{a} - n$ and $L_{\omega(m+1)}\, \mathfrak{a}$ would both be monomials, which cannot be. Note that $E_\omega\, \omega = E_{\omega^2}(L_{\omega^2}\, E_\omega\, \omega) = E_{\omega^2}^{L_{\omega^2}\omega + 1}$, so $E_\omega\, \omega = E_{\omega^2}^{L_{\omega^2}\omega + 1}$ is a hyperserial expansion of type I. We also have $\sqrt{\omega} = \exp\!\left(\frac{1}{2}\log \omega\right)$ where $\frac{1}{2}\log\omega$ is tail atomic. Thus $\sqrt{\omega} = E_1^{\frac{1}{2}\log\omega}$ is a hyperserial expansion of type I. Note finally that $\log \omega = L_1\, \omega$ is a hyperserial expansion of type II. We thus have the following "recursive" expansion of $\mathfrak{m}$:

$$\mathfrak{m} = \mathrm{e}^{2 E_{\omega^2}^{L_{\omega^2}\omega+1} - E_1^{\frac{1}{2} L_1 \omega}}\,(L_\omega\, \omega). \tag{13.1.3}$$

We will study this type of recursive expansion by studying paths in Section 13.1.2

Lemma 13.1.4. *Any $\mathfrak{m} \in \mathbf{Mo}$ has a hyperserial expansion.*

Proof. We first prove the result for $\mathfrak{m} \in \mathbf{Mo}_\omega$, by induction with respect to the simplicity relation $\sqsubseteq$. The $\sqsubseteq$-minimal element of $\mathbf{Mo}_\omega$ is ω, which satisfies (13.1.2) for $\psi = \beta = 0$ and $\iota = 1$. Consider $\mathfrak{m} \in \mathbf{Mo}_\omega \setminus \{\omega\}$ such that the result holds on $\mathfrak{m}_{\sqsubset}^{\mathbf{Mo}_\omega}$. By [12, Proposition 6.20], the monomial $\mathfrak{m}$ is not $L_{<\mathbf{On}}$-atomic. So there is a maximal $\lambda \in \omega^{\mathbf{On}}$ with $\mathfrak{m} \in \mathbf{Mo}_\lambda$, and we have $\lambda \geqslant \omega$ by our hypothesis.

If there is no ordinal $\gamma < \lambda$ such that $E_\gamma^{\mathfrak{m}} \in \mathbf{Mo}_{\lambda\omega}$, then we have $\mathfrak{m} \in \mathbf{Mo}_\lambda \setminus L_{<\lambda}\,\mathbf{Mo}_{\lambda\omega}$. So setting $\alpha = \lambda$, $\beta = 0$ and $u = L_\lambda\, \mathfrak{m}$, we are done. Otherwise, let $\gamma < \lambda$ be such that $\mathfrak{a} := E_\gamma^{\mathfrak{m}} \in \mathbf{Mo}_{\lambda\omega}$. We cannot have $\gamma = 0$ by definition of λ. So there is a unique ordinal η and a unique natural number $n \in \mathbb{N}^>$ such that $\gamma = \gamma' + \omega^\eta n$ and $\gamma' \gg \omega^\eta$. Note that $\lambda \geqslant \omega^{\eta+1}$. We must have $\lambda = \omega^{\eta+1}$: otherwise, $L_{\omega^{\eta+1}}\, \mathfrak{m} = L_{\gamma' + \omega^{\eta+1}}(\mathfrak{a}) + n$ where $L_{\omega^{\eta+1}}\, \mathfrak{m}$ and $L_{\gamma'+\omega^{\eta+1}}\, \mathfrak{a}$ are monomials. We deduce that $\gamma' = 0$ and $\gamma = \omega^\eta n$. Note that $L_\lambda\, \mathfrak{a} \asymp L_\lambda\, \mathfrak{m}$, $\lambda < \lambda\omega$, and $\mathfrak{a} \in \mathbf{Mo}_{\lambda\omega}$, so $\mathfrak{a} = \mathfrak{d}_{\lambda\omega}(\mathfrak{m})$. We deduce that $\mathfrak{a} \sqsubset \mathfrak{m}$. The induction hypothesis yields a hyperserial expansion $\mathfrak{a} = \mathrm{e}^{\psi}\,(L_\beta\, E_\alpha^u)^\iota$. Since $\mathfrak{a}$ is log-atomic, we must have $\psi = 0$ and $\iota = 1$. If $\mathfrak{a} = L_\beta \omega$, then $\beta \geqq \lambda_{/\omega} = \omega^\eta$, since $\mathfrak{a} \in \mathbf{Mo}_{\lambda\omega}$. Thus $\mathfrak{m} = L_\gamma\, \mathfrak{a} = L_{\beta+\gamma}\omega$ is a hyperexponential expansion of type II. If $\mathfrak{a} = L_\beta\, E_\alpha^u$, then likewise $\beta \geqq \omega^\eta$ and thus $\mathfrak{m} = L_{\beta+\gamma}\, E_\alpha^u$ is a hyperexponential expansion of type I. This completes the inductive proof.

Now let $\mathfrak{m} \in \mathbf{Mo}^{\neq} \setminus \mathbf{Mo}_\omega$ and set $\varphi := \log \mathfrak{m}$. If φ is tail-atomic, then there are $\psi \in \mathbf{No}_\succ$, $\iota \in \{-1, 1\}$ and $\mathfrak{a} \in \mathbf{Mo}_\omega$ with $\varphi = \psi +\!\!\!\!+\, \iota\, \mathfrak{a}$. Applying the previous arguments to $\mathfrak{a}$, we obtain elements $\alpha \geqslant \omega$, β, u with $\mathfrak{a} = L_\beta\, E_\alpha^u$ and $\beta\,\omega < \alpha$, or an ordinal β with $\mathfrak{a} = L_\beta\, \omega$. Then $\mathfrak{m} = \mathrm{e}^{\psi}\,(L_\beta\, E_\alpha^u)^\iota$ or $\mathfrak{m} = \mathrm{e}^{\psi}\,(L_\beta\, \omega)^\iota$ is a hyperserial expansion. If φ is not tail-atomic, then we have $\mathfrak{m} = E_1^\varphi$ is a hyperserial expansion of type I. □

Lemma 13.1.5. *Let $\mathfrak{a} \in \mathbf{Mo}^{\neq}$ and assume that $\mathfrak{a} = L_\beta\, E_\alpha^u$ is a hyperserial expansion. Let $\mu > 0$ and define $\mu_- := \mu - 1$ if μ is a successor ordinal and $\mu_- := \mu$ if μ is a limit ordinal. Let*

$$\begin{aligned}
\beta &=: \beta' + \beta'' \quad \textit{where} \\
\beta' &:= \beta_{\succcurlyeq \omega^{\mu_-}} \geqq \omega^{\mu_-} \quad \textit{and} \\
\beta'' &:= \beta_{\prec \omega^{\mu_-}} < \omega^{\mu_-}.
\end{aligned}$$

a) Then $\mathfrak{a}$ is $L_{<\omega^\mu}$-atomic if and only if $\beta'' = 0$ and either $\alpha \geqslant \omega^\mu$ or $\alpha = 0$.

b) If $\alpha \geqslant \omega^\mu$, then $\mathfrak{d}_{\omega^\mu}(\mathfrak{a}) = L_{\beta'}\, E_\alpha^u$.

Proof. We first prove *a*). Assume that $\mathfrak{a}$ is $L_{<\omega^\mu}$-atomic. Assume for contradiction that $\beta'' \neq 0$ and let $\omega^\eta m$ denote the least non-zero term in the Cantor normal form of β''. Since $\beta'' < \omega^{\mu_-}$, we have $\omega^{\eta+1} < \omega^\mu$ so $L_{\omega^{\eta+1}}\, \mathfrak{a}$ is a monomial. But $L_{\omega^{\eta+1}}\, \mathfrak{a} = L_{\beta''_{\succ\omega^\eta}}\, E_\alpha^u - m$ where $L_{\beta''_{\succ\omega^\eta}}\, E_\alpha^u$ is a monomial: a contradiction. So $\beta'' = 0$. If $\alpha = 0$ then we are done. Otherwise $E_\alpha^u \notin \mathbf{Mo}_{\alpha\omega}$, so we must have $\alpha\,\omega > \omega^\mu$, whence $\alpha \geqslant \omega^\mu$. Conversely, assume that $\alpha \geqslant \omega^\mu$ or $\alpha = 0$, and that $\beta'' = 0$. If $\alpha \neq 0$, then then for all $\gamma < \omega^\mu$, we have $L_\gamma\, \mathfrak{a} = L_{\beta'+\gamma}\, E_\alpha^u$ where $\beta' + \gamma < \alpha$, so $L_\gamma\, \mathfrak{a}$ is a monomial, whence $\mathfrak{a} \in \mathbf{Mo}_{\omega^\mu}$. If $\alpha = 0$, then for all $\gamma < \omega^\mu$, we have $L_\gamma\, \mathfrak{a} = L_{\beta'+\gamma}\omega \in \mathbf{Mo}$, whence $\mathfrak{a} \in \mathbf{Mo}_{\omega^\mu}$. This proves *a*).

Now assume that $\alpha \geqslant \omega^\mu$. So $L_{\beta'} E_\alpha^u$ is $L_{<\omega^\mu}$-atomic by *a*). If $\beta'' = 0$ then we conclude that $\mathfrak{a} = L_{\beta'} E_\alpha^u = \mathfrak{d}_{\omega^\mu}(\mathfrak{a})$. If $\beta'' \neq 0$, then let $\omega^\eta m$ denote the least non-zero term in its Cantor normal form. We have $\omega^{\eta+1} < \omega^\mu$ and $L_{\omega^{\eta+1}} \mathfrak{a} = L_{\omega^{\eta+1}} L_{\beta'} E_\alpha^u - m \asymp L_{\omega^{\eta+1}} L_{\beta'} E_\alpha^u$, so $L_{\beta'} E_\alpha^u = \mathfrak{d}_{\omega^\mu}(\mathfrak{a})$. □

Corollary 13.1.6. *Let $\mu \in \mathbf{On}^>$, $\alpha := \omega^\mu$, $\gamma < \alpha$, and $\mathfrak{b} \in \mathbf{Mo}_{\alpha\omega}$. If $L_\gamma \mathfrak{b} \in \mathbf{Mo}_\alpha \setminus \mathbf{Mo}_{\alpha\omega}$, then μ is a successor ordinal and $\gamma = \alpha_{/\omega} n$ for some $n \in \mathbb{N}^>$.*

Proof. Since $L_\gamma \mathfrak{b} \in \mathbf{Mo}_\alpha \setminus \mathbf{Mo}_{\alpha\omega}$, we must have $\gamma \neq 0$. By Lemma 13.1.4, we have a hyperserial expansion $\mathfrak{b} = \mathrm{e}^\psi (L_\beta E_\rho^u)^\iota$. Since $\mathfrak{b}$ is log-atomic, we have $\log \mathfrak{b} = \psi + \iota L_{\beta+1} E_\alpha^u \in \mathbf{Mo}$, whence $\psi = 0$ and $\iota = 1$. So $\mathfrak{b} = L_\beta E_\rho^u$. We have $\mathfrak{b} \in \mathbf{Mo}_{\alpha\omega}$ so by Lemma 13.1.5(a), we have $\beta \gtrsim \alpha$. It follows that $L_\gamma \mathfrak{b} = L_{\beta+\gamma} E_\rho^u$ is a hyperserial expansion. But then $L_{\beta+\gamma} E_\rho^u \in \mathbf{Mo}_\alpha$ and Lemma 13.1.5(*a*) imply that $\gamma \gtrsim \omega^{\mu_-}$. The condition that $\gamma < \alpha$ now gives $\mu_- < \mu$, whence μ is a successor and $\gamma = \omega^{\mu_-} n$ for a certain $n \in \mathbb{N}^>$, as claimed. □

Lemma 13.1.7. *Any $\mathfrak{m} \in \mathbf{Mo}$ has a unique hyperserial expansion (that we will call* the *hyperserial expansion, henceforth).*

Proof. Consider a monomial $\mathfrak{m} \neq 1$ with

$$\mathfrak{m} = \mathrm{e}^\psi (L_\beta \mathfrak{a})^\iota,$$

where $\psi \in \mathbf{No}_\succ$, $\iota \in \{-1, 1\}$, $\beta, \alpha \in \omega^{\mathbf{On}}$, $\mathfrak{a} \in \mathbf{Mo}_\alpha$, $\beta\omega < \alpha$, and $\operatorname{supp} \psi \succ L_{\beta+1} \mathfrak{a}$. Assume for contradiction that we can write $\mathfrak{m} = \mathrm{e}^{\psi'} (L_{\beta'} E_{\alpha'} u')^{\iota'}$ as a hyperserial expansion of type I with $\alpha' < \alpha$. Note in particular that $\alpha > 1$, so $L_{\beta+1} \mathfrak{a}$ is log-atomic. We have

$$\log \mathfrak{m} = \psi + \iota L_{\beta+1} \mathfrak{a} = \psi' + \iota' L_{\beta'+1} E_{\alpha'}^{u'}.$$

If $\alpha' = 1$, then $\beta' = 0$, $\psi' = 0$, $\iota' = 1$, and u' is not tail-atomic. But $\psi + \iota L_{\beta+1} \mathfrak{a} = u'$, where $L_{\beta+1} \mathfrak{a} \in \mathbf{Mo}_\omega$, so u' is tail-atomic: a contradiction. Hence $\alpha' > 1$. Note that $\iota L_{\beta+1} \mathfrak{a}$ and $\iota' L_{\beta'+1} E_{\alpha'}^{u'}$ are both the least term of $\log \mathfrak{m}$. It follows that $\psi = \psi'$, $\iota = \iota'$, and

$$L_\beta \mathfrak{a} = L_{\beta'} E_{\alpha'}^{u'}. \tag{13.1.4}$$

Since $\beta' \omega < \alpha'$, we have

$$E_{\alpha'}^{u'} = \mathfrak{d}_{\alpha'}(L_{\beta'} E_{\alpha'}^{u'}) = \mathfrak{d}_{\alpha'}(L_\beta \mathfrak{a}).$$

Now $E_{\alpha'}^{u'} \notin \mathbf{Mo}_{\alpha'\omega}$, so $\mathfrak{d}_{\alpha'}(L_\beta \mathfrak{a}) \neq \mathfrak{a}$ and thus $\beta\omega \geqslant \alpha'$. In particular $\beta > \beta'$. Taking hyperexponentials on both sides of (13.1.4), we may assume without loss of generality that $\beta' = 0$ or that the least exponents η and η' in the Cantor normal forms of β resp. β' differ. If $\beta' = 0$, then we decompose $b = \gamma \dotplus \omega^\eta n$ where $n \in \mathbb{N}^>$ and $\gamma \gg \omega^\eta$. Since $L_\beta \mathfrak{a} = E_{\alpha'}^{u'} \in \mathbf{Mo}_{\alpha'} \setminus \mathbf{Mo}_{\alpha'\omega}$, applying Lemma 13.1.5(*a*) twice (for $\omega^\mu = \alpha'$ and $\omega^\mu = \alpha'\omega$) gives $\omega^{\eta+1} \geqslant \alpha'$ and $\omega^{\eta+1} \ngeqslant \alpha'\omega$, whence $\alpha' = \omega^{\eta+1}$. But then $E_{\alpha'}^{u'} = L_{\omega^\eta n} \mathfrak{b}$, where $\mathfrak{b} := L_\gamma \mathfrak{a} \in \mathbf{Mo}_{\alpha'} \omega$ by Lemma 13.1.5(*a*). So $E_{\alpha'}^{u'} \in L_{<\alpha'} \mathbf{Mo}_{\alpha'\omega}$: a contradiction. Assume now that $\beta' \neq 0$. Lemma 13.1.5(*a*) yields both $L_\beta \mathfrak{a} \in \mathbf{Mo}_{\omega^{\eta+1}} \setminus \mathbf{Mo}_{\omega^{\eta+2}}$ and $L_{\beta'} E_{\alpha'}^{u'} \in \mathbf{Mo}_{\omega^{\eta'+1}} \setminus \mathbf{Mo}_{\omega^{\eta'+2}}$, which contradicts (13.1.4).

Taking $\mathfrak{a} = \omega \in \mathbf{No}$ and $\alpha := \max(\alpha'\omega, \beta\omega^2)$, this proves that no two hyperserial expansions of distinct types I and II can be equal. Taking $\mathfrak{a} = E_\alpha^u$ with $\alpha > \alpha'$, this proves that no two hyperserial expansions $\mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota, \mathrm{e}^{\psi'} (L_{\beta'} E_{\alpha'}^{u'})^{\iota'}$ of type I with $\alpha \neq \alpha'$ can be equal.

The two remaining cases are hyperserial expansions of type II and hyperserial expansions $\mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota$ and $\mathrm{e}^{\psi'} (L_{\beta'} E_{\alpha'}^{u'})^{\iota'}$ of type I with $\alpha = \alpha'$. Consider a monomial $\mathfrak{m} \in \mathbf{Mo}^{\neq}$ with the hyperserial expansions $\mathfrak{m} = \mathrm{e}^\psi (L_\gamma \omega)^\iota = \mathrm{e}^{\psi'} (L_{\gamma'} \omega)^{\iota'}$ of type II. As above we have $\psi = \psi'$, $\iota = \iota'$, and $L_\gamma \omega = L_{\gamma'} \omega$. We deduce that $\gamma = \gamma'$, so the expansions coincide.

Finally, consider a monomial $\mathfrak{m} \neq 1$ with two hyperserial expansions of type I

$$\mathfrak{m} = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota = \mathrm{e}^{\psi'} (L_{\beta'} E_\alpha^{u'})^{\iota'}. \tag{13.1.5}$$

If $\alpha = 1$, then we have $\psi = \psi' = 0$ and $\beta = \beta' = 0$ and $\iota = \iota' = 1$, whence $u = u'$, so we are done.

Assume now that $\alpha > 1$. Taking logarithms in (13.1.5), we see that $\psi = \psi'$, $\iota = \iota'$, and

$$L_\beta E_\alpha^u = L_{\beta'} E_\alpha^{u'}. \tag{13.1.6}$$

We may assume without loss of generality that $\beta \geqslant \beta'$. Assume for contradiction that $\beta > \beta'$. Taking hyperexponentials on both sides of (13.1.6), we may assume without loss of generality that $\beta' = 0$ or that the least exponents η and η' in the Cantor normal forms of β resp. β' differ. On the one hand, Lemma 13.1.5(a) yields $L_\beta E_\alpha^u \in \mathbf{Mo}_{\omega^{\eta+1}} \setminus \mathbf{Mo}_{\omega^{\eta+2}}$. Note in particular that $L_\beta E_\alpha^u \notin \mathbf{Mo}_\alpha$, since $\beta\omega < \alpha$. On the other hand, if $\beta \neq 0$, then Lemma 13.1.5(a) yields $L_{\beta'} E_\alpha^{u'} \in \mathbf{Mo}_{\omega^{\eta'}} \setminus \mathbf{Mo}_{\omega^{\eta'+1}}$; if $\beta' = 0$, then $L_{\beta'} E_\alpha^{u'} \in \mathbf{Mo}_\alpha$. Thus (13.1.6) is absurd: a contradiction. We conclude that $\beta = \beta'$. Finally $E_\alpha^u = E_\alpha^{u'}$ yields $u = u'$, so the expansions are identical. □

Lemma 13.1.8. *If $\mathfrak{m} = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota$ is a hyperserial expansion of type I, then we have*

$$\operatorname{supp} \psi \cap \operatorname{supp} u = \varnothing.$$

Proof. Assume for contradiction that $\mathfrak{n} \in \operatorname{supp} \psi \cap \operatorname{supp} u$. Since $\mathfrak{n} \in \operatorname{supp} \psi$, we have $\mathfrak{n} \succ L_{\beta+1} E_\alpha^u$. Since $u > 0$, there is $r \in \mathbb{R}^>$ with $u \geqslant r\,\mathfrak{n}$, so $L_{\beta+1} E_\alpha^u \succcurlyeq \mathfrak{n}$: a contradiction. □

13.1.2 Paths and subpaths

Let λ be an ordinal with $0 < \lambda \leqslant \omega$ and note that $i < 1 \dotplus \lambda \Longleftrightarrow (i \leqslant \lambda < \omega \vee i < \omega = \lambda)$ for all $i \in \mathbb{N}$. Consider a sequence

$$P = (P(i))_{i<\lambda} = (\tau_{P,i})_{i<\lambda} = (r_{P,i}\,\mathfrak{m}_{P,i})_{i<\lambda} \quad \text{in} \quad (\mathbb{R}^{\neq}\,\mathbf{Mo})^\lambda.$$

We say that P is a *path* if there exist sequences $(u_{P,i})_{i<1\dotplus\lambda}$, $(\psi_{P,i})_{i<1\dotplus\lambda}$, $(\iota_{P,i})_{i<\lambda}$, $(\alpha_{P,i})_{i<\lambda}$, and $(\beta_{P,i})_{i<1\dotplus\lambda}$ with

- $u_{P,0} = \tau_{P,0}$ and $\psi_{P,0} = 0$;
- $\tau_{P,i} \in \operatorname{term} \psi_{P,i}$ or $\tau_{P,i} \in \operatorname{term} u_{P,i}$, for all $i < \lambda$;
- $\tau_{P,i} \in \mathbb{R}^{\neq} \cup \{\omega\} \Longrightarrow \lambda = i+1$, for all $i < \lambda$;
- For $i < \lambda$, the hyperserial expansion (of type I or II) of $\mathfrak{m}_{P,i}$ is

$$\mathfrak{m}_{P,i} \;=\; \mathrm{e}^{\psi_{P,i+1}} (L_{\beta_{P,i}} E_{\alpha_{P,i}}^{u_{P,i+1}})^{\iota_{P,i}}.$$

We call λ the *length* of P and we write $|P| := \lambda$. We say that P is *infinite* if $|P| = \omega$ and *finite* otherwise. For $a \in \mathbf{No}$, we say that P is a path in a if $P(0) \in \operatorname{term} a$. We then set $a_{P,0} := a$. For $0 < i < |P|$, we define

$$(s_{P,i}, a_{P,i}) \;:=\; \begin{cases} (-1, \psi_{P,i}) & \text{if } \mathfrak{m}_{P,i} \in \operatorname{supp} \psi_{P,i} \\ (1, u_{P,i}) & \text{if } \mathfrak{m}_{P,i} \in \operatorname{supp} u_{P,i}. \end{cases}$$

By Lemma 13.1.8, those cases are mutually exclusive so $(s_{P,i}, a_{P,i})$ is well-defined.

For $k \leqslant |P|$, we let $P_{\nearrow k}$ denote the path of length $|P| - k$ in $a_{P,k}$ with

$$\forall i < |P| - k, \tau_{P_{\nearrow k},i} := \tau_{P,k+i}.$$

So $P_{\nearrow k}$ is the path obtained by removing the first k elements of P and reindexing.

Example 13.1.9. Let us find all the paths in the monomial $\mathfrak{m}$ of Example 13.1.3. We have a representation (13.1.3) of $\mathfrak{m}$ as a hyperseries

$$\mathfrak{m} = \mathrm{e}^{2 E_{\omega^2}^{L_{\omega^2 \omega + 1}} - E_1^{\frac{1}{2} L_1 \omega}} (L_\omega\, \omega)$$

which by Lemma 13.1.7 is unique. There are nine paths in $\mathfrak{m}$, namely

- one path $(\mathfrak{m})$ of length 1;
- three paths $(\mathfrak{m}, 2 E_{\omega^2}^{L_{\omega^2 \omega+1}})$, $\left(\mathfrak{m}, -E_1^{\frac{1}{2} L_1 \omega}\right)$, and $(\mathfrak{m}, \omega)$ of length 2;
- three paths $(\mathfrak{m}, 2 E_{\omega^2}^{L_{\omega^2 \omega+1}}, L_{\omega^2}\omega)$, $(\mathfrak{m}, 2 E_{\omega^2}^{L_{\omega^2 \omega+1}}, 1)$ and $\left(\mathfrak{m}, -E_1^{\frac{1}{2} L_1 \omega}, \frac{1}{2} L_1 \omega\right)$ of length 3;
- two paths $(\mathfrak{m}, 2 E_{\omega^2}^{L_{\omega^2 \omega+1}}, L_{\omega^2}\omega, \omega)$ and $\left(\mathfrak{m}, -E_1^{\frac{1}{2} L_1 \omega}, \frac{1}{2} L_1 \omega, \omega\right)$ of length 4.

Note that the paths which cannot be extended into strictly longer paths are those whose last value is a real number or ω.

Infinite paths occur in so-called nested numbers that will be studied in more detail in Section 7.3.2.

Definition 13.1.10. *Let $a \in \mathbf{No}$ and let P be a path in a. We say that an index $i < |P|$ is* **bad** *for (P, a) if one of the following conditions is satisfied*

1. $\mathfrak{m}_{P,i}$ *is not the $\preccurlyeq$-minimum of* $\operatorname{supp} u_{P,i}$*;*
2. $\mathfrak{m}_{P,i} = \min \operatorname{supp} u_{P,i}$ *and* $\beta_{P,i} \neq 0$*;*
3. $\mathfrak{m}_{P,i} = \min \operatorname{supp} u_{P,i}$ *and* $\beta_{P,i} = 0$ *and* $r_{P,i} \notin \{-1, 1\}$*;*
4. $\mathfrak{m}_{P,i} = \min \operatorname{supp} u_{P,i}$ *and* $\beta_{P,i} = 0$ *and* $r_{P,i} \in \{-1, 1\}$ *and* $\mathfrak{m}_{P,i} \in \operatorname{supp} \psi_{P,i}$.

The index i is **good** *for (P, a) if it is not bad for (P, a).*

If P is infinite, then we say that it is **good** *if $(P, \tau_{P,0})$ is good for all but a finite number of indexes. In the opposite case, we say that P is a* **bad** *path. An element $a \in \mathbf{No}$ is said to be* **well-nested** *every path in a is good.*

Remark 13.1.11. The above definition extends the former definitions of paths in [60, 92, 18]. More precisely, a path P with $\alpha_{P,i} = 1$ (whence $\psi_{P,i} = 0$) for all $i < |P|$, corresponds to a path for these former definitions. The validity of the axiom **T4** for **No** means that those paths are good. With Theorem 13.2.7, we will extend this result to all paths.

Lemma 13.1.12. *For $\mathfrak{m} \in (L_{\mathbf{On}}\, \omega)^{\pm 1}$ and for any path P in $\mathfrak{m}$, we have $|P| \leqslant 2$. For $a \in \mathbb{L} \circ \omega$ and for any path P in a, we have $|P| \leqslant 3$.*

Proof. Let $\mathfrak{l} \in \mathfrak{L} \setminus \{-1\}$ and let P be a path in $\mathfrak{l} \circ \omega$. If there is an ordinal γ with $\mathfrak{l} = \ell_\gamma$, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $L_\gamma \omega$, so $|P| = 1$ if $\gamma = 0$ and $|P| = 2$ otherwise. If there is an ordinal γ with $\mathfrak{l} = \ell_\gamma^{-1}$, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $(L_\gamma \omega)^{-1}$ and $|P| = 2$.

Assume now that $\mathfrak{l} \notin \ell_{\mathbf{On}}^{\pm 1}$. If $\log \mathfrak{l} \circ \omega$ is not tail-atomic, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $\mathfrak{l} \circ \omega = \mathrm{e}^{\log \mathfrak{l} \circ \omega}$. If $\log \mathfrak{l} \circ \omega$ is tail-atomic, then the hyperserial expansion of $\mathfrak{l} \circ \omega$ is $\mathfrak{l} \circ \omega = \mathrm{e}^{\psi \circ \omega} (\mathfrak{a} \circ \omega)^\iota$ for a certain log-atomic $\mathfrak{a} \in \mathbb{L}$. Since $\psi \in \log \mathbb{L}$, we have $\operatorname{supp} \psi \subseteq \{\ell_\rho : \rho \in \mathbf{On}\}$. We also have $\mathfrak{a} \in \mathfrak{L}_\omega = \{\ell_\rho : \rho \in \mathbf{On}\}$ by Section 4.2.4. So in both cases, $P_{\nearrow 1}$ is a path in some monomial in $L_{\mathbf{On}}\, \omega$, whence $|P_{\nearrow 1}| \leqslant 2$ and $|P| \leqslant 3$, by the previous argument. □

Definition 13.1.13. *Let P, Q be paths. We say that Q is a* **subpath of** *P, or equivalently that P* **extends** *Q, if there exists a $k < |P|$ with $Q = P_{\nearrow k}$. For $a \in \mathbf{No}$, we say that Q is a* **subpath in** *a if there is a path P in a such that Q is a subpath of P. We say that P* **shares a subpath with** *a if there is a subpath of P which is a subpath in a.*

So our subpaths are always initial subsequences. Paths can sometimes be concatenated. Indeed, let P be a finite path and let Q be a path with $Q(0) \in \operatorname{supp} u_{P,|P|} \cup \operatorname{supp} \psi_{P,|P|}$. Then we define $P * Q$ to be the path $(P(0), \ldots, P(|P|), Q(0), \ldots)$ of length $|P| + |Q|$.

Lemma 13.1.14. *Let $\lambda \in \omega^{\mathbf{On}}$ and $\mathfrak{m} \in \mathbf{Mo}_\lambda$. Let P be a path in $\mathfrak{m}$ with $|P| > 2$. Then $P_{\nearrow 1}$ is a subpath in $L_\lambda \mathfrak{m}$.*

Proof. By Lemma 13.1.12, we have $\mathfrak{m} \notin (L_{\mathbf{On}}\, \omega)^{\pm 1}$. If $\mathfrak{m}$ has a hyperserial expansion of the form $\mathfrak{m} = \mathrm{e}^\psi (L_\gamma \omega)^\iota$, then $P_{\nearrow 1}$ must be a path in ψ. So ψ is non-zero and thus $\lambda = 1$. It follows that $P_{\nearrow 1}$ is a path in $\log \mathfrak{m} = \psi + \iota\, (L_\gamma \omega)^\iota$. Otherwise, let $\mathfrak{m} = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota$ be the hyperserial expansion of $\mathfrak{m}$. If $P_{\nearrow 1}$ is a path in ψ, then it is a path in $\log \mathfrak{m}$ as above. Otherwise, it is a path in u. Assume that $\lambda = 1$. If $\alpha = 1$, then we have $\psi = 0$ and $\log \mathfrak{m} = \iota\, u$ so $P_{\nearrow 1}$ is a path in $\log \mathfrak{m}$. If $\alpha > 1$, then $\log \mathfrak{m} = \psi + \iota\, L_{\beta+1} E_\alpha^u$ where $L_{\beta+1} E_\alpha^u$ is a hyperserial expansion, so $P_{\nearrow 1}$ is a path in $\log \mathfrak{m}$. Assume now that $\lambda > 1$, so $\psi = 0$, $\iota = 1$, and $\alpha \geqslant \omega$. We must have $\beta \gtrsim \lambda_{/\omega}$ so there are $\beta' \in \mathbf{On}$ and $n \in \mathbb{N}$ with $\beta' \gg \lambda_{/\omega}$ and $\beta = \beta' + \lambda_{/\omega}\, n$. We have $L_\lambda \mathfrak{m} = L_{\beta'+\lambda} E_\alpha^u - n$ where $L_{\beta'+\lambda} E_\alpha^u$ is a hyperserial expansion, so $P_{\nearrow 1}$ is a path in $L_\lambda \mathfrak{m}$. □

Lemma 13.1.15. *Let $a \in \mathbf{No}^{>, \succ}$, $\alpha \in \omega^{\mathbf{On}}$ and $k \in \mathbb{N}^>$. If P is a path in $\sharp_\alpha(a)$ with $|P| > 2$, then $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\alpha k} a}$.*

Proof. We prove this by induction on αk, for any number $a \in \mathbf{No}^{>,\succ}$. We consider $a \in \mathbf{No}^{>,\succ}$, and a fixed path P in $\sharp_\alpha(a)$ with $|P| > 2$.

Assume that $\alpha = k = 1$. We have $\sharp_1(a) = a_\succ$ and $\mathfrak{d}_{\exp a} = \mathrm{e}^{a_\succ}$. Assume that $a_\succ = \psi \mathbin{+\!\!\!+} \iota\,\mathfrak{a}$ for certain $\psi \in \mathbf{No}_\succ$, $\iota \in \{-1, 1\}$, and $\mathfrak{a} \in \mathbf{Mo}_\omega$. Let $\mathfrak{a} = L_\gamma E_\lambda^u$ be the hyperserial expansion of $\mathfrak{a}$. If $\lambda = \omega$, then $\gamma = 0$ and the hyperserial expansion of $\mathrm{e}^{\mathfrak{a}}$ is $\mathrm{e}^{\mathfrak{a}} = E_\omega^{u+1}$. Therefore $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a} = \mathrm{e}^\psi\,(E_\omega^{u+1})^\iota$. If $\lambda > \omega$, then the hyperserial expansion of $\mathrm{e}^{\mathfrak{a}}$ is $\mathrm{e}^{\mathfrak{a}} = L_{\gamma+1} E_\lambda^u$. Therefore $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a} = \mathrm{e}^\psi\,(L_{\gamma+1} E_\lambda^u)^\iota$. Finally, if $\mathrm{e}^{a_\succ}$ is not tail-atomic, then $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a} = (E_1^{\epsilon a_\succ})^\epsilon$, where $\epsilon \in \{-1, 1\}$ is the sign of $a_\succ$.

Now assume that $\alpha = 1$, $k > 1$, and that the result holds strictly below k. We have $E_k\,a = E_{k-1}(\exp a)$ where $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{\exp a}$ by the previous argument. We have $r\,\mathfrak{d}_{\exp a} \trianglelefteq \sharp_1(\exp a)$ for a certain $r \in \mathbb{R}^{\neq}$, so $Q := (r\,\mathfrak{d}_{\exp a}) * P_{\nearrow 1}$ is a path in $\sharp_1(\exp a)$. The induction hypothesis on $k-1$ implies that $Q_{\nearrow 1} = P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_k a}$.

Assume now that $\alpha \geqslant \omega$ and that the result holds strictly below α. Write $v := \sharp_\alpha(a)$. Recall by (5.3.1) that there exist an $\eta \in \mathbf{On}$, an $n < \omega$, and a $\delta \in \mathbf{No}$ with $\beta := \omega^\eta < \alpha$ and

$$E_\alpha\,a = E_{\beta n}(L_{\beta n} E_\alpha^v \mathbin{+\!\!\!+} \delta).$$

Assume for contradiction that there is a $\gamma \in \mathbf{On}$ with $E_\alpha^v = L_\gamma \omega$. We must have $\gamma \geqq \alpha_{/\omega}$, so there are a number $n \in \mathbb{N}$ and an ordinal $\gamma' \geqq \alpha$ with $\gamma = \gamma' + \alpha_{/\omega}\,n$. We have $v = L_{\gamma'+\alpha}\,\omega - n$. By Lemma 13.1.12, this contradicts the fact that $|P| > 2$. So by Lemma 13.1.4, there exist $\beta \in \omega^{\mathbf{On}}$ and $\gamma \in \mathbf{On}$ with $\beta \geqslant \alpha$, $\gamma\,\omega < \beta$, $E_\alpha^v = L_\gamma E_\beta^u$, and $E_\beta^u \in \mathbf{Mo}_\beta \setminus L_{<\beta} \mathbf{Mo}_{\beta\omega}$. Since $E_\alpha^v \in \mathbf{Mo}_\alpha$, we must have $\gamma \geqq \alpha_{/\omega}$ so there are a number $n \in \mathbb{N}$ and an ordinal $\gamma' \geqq \alpha$ with $\gamma = \gamma' + \alpha_{/\omega}\,n$ (note that $n = 0$ whenever $\alpha_{/\omega} = \alpha$). Thus $v + n = L_\alpha\,L_{\gamma'+\alpha_{/\omega} n} E_\beta^u + n = L_{\gamma'+\alpha} E_\beta^u$ is a monomial with hyperserial expansion $v + n = L_{\gamma'+\alpha} E_\beta^u$. There is no path in n of length >1, so P must be a path in $L_{\gamma'+\alpha} E_\beta^u$. We deduce that $P_{\nearrow 1}$ is a path in u. Consequently, $Q = (L_\gamma E_\beta^u) * P_{\nearrow 1}$ is a path in E_α^v with $|Q| = |P| > 2$. Applying n times Lemma 13.1.14, we deduce that $Q_{\nearrow 1} = P_{\nearrow 1}$ is a subpath in $L_{\beta n} E_\alpha^v$, hence in $\sharp_\beta(L_{\beta n} E_\alpha\,a)$. Consider a path R in $\sharp_\beta(L_{\beta n} E_\alpha\,a)$ with $P_{\nearrow 1} = R_{\nearrow i}$ for a certain $i > 0$. Applying the induction hypothesis for $L_{\beta n} E_\alpha\,a$ and βn in the roles of a and αk, the path $R_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\beta n}(L_{\beta n} E_\alpha a)} = \mathfrak{d}_{E_\alpha a}$. Therefore $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_\alpha a}$. We deduce as in the case $\alpha = 1$ that $P_{\nearrow 1}$ is a subpath in $\mathfrak{d}_{E_{\alpha k} a}$. □

Lemma 13.1.16. *Let $\psi \in \mathbf{No}_\succ$, and $\mathfrak{m} \in \mathbf{Mo}^{\neq}$ with $\operatorname{supp} \psi \succ \log \mathfrak{m}$. Let P be a path in $\mathfrak{m}$ with $|P| > 1$. Then $P_{\nearrow 1}$ is a subpath in $\mathrm{e}^\psi\,\mathfrak{m}$.*

Proof. Let $\mathfrak{m} = \mathrm{e}^\varphi\,(L_\beta E_\alpha^u)^\iota$ be a hyperserial expansion. The condition $\operatorname{supp} \psi \succ \log \mathfrak{m}$ implies $\varphi + \psi = \varphi \mathbin{+\!\!\!+} \psi$, whence $\mathrm{e}^\psi\,\mathfrak{m} = \mathrm{e}^{\psi \mathbin{+\!\!\!+} \varphi}\,(L_\beta E_\alpha^u)^\iota$ is also a hyperserial expansion. In particular $P_{\nearrow 1}$ is a subpath in $\mathrm{e}^\psi\,\mathfrak{m}$. □

Corollary 13.1.17. *Let $\alpha = \omega^\nu \in \mathbf{On}$, $\beta \in \mathbf{On}$ with $\beta < \alpha$, and $\varphi \in \mathbf{No}_{\succ,\alpha}$. If P is an infinite path, then P shares a subpath with φ if and only if it shares a subpath with $L_\beta E_\alpha^\varphi$.*

Proof. Write $\beta = \omega^{\eta_1} m_1 + \cdots + \omega^{\eta_k} m_k$ in Cantor normal form, with $\eta_1 > \cdots > \eta_k$ and $m_1, \ldots, m_k \in \mathbb{N}^>$ and let

$$\mathfrak{a}_i := L_{\omega^{\eta_1} m_1 + \cdots + \omega^{\eta_{i-1}} m_{i-1}} E_\alpha^\varphi$$

for all $i = 1, \ldots, k$.

Assume that P shares a subpath with φ. In other words, there is a path R in φ which has a common subpath with P. The path R must be infinite, so by Lemma 13.1.15, it shares a subpath with $E_\alpha^\varphi = \mathfrak{a}_1$. Let us prove by induction on $i = 1, \ldots, k$ that R shares a subpath with $E_\alpha^\varphi = \mathfrak{a}_i$. Assuming that this holds for $i < k$, we note that $\mathfrak{a}_i$ is $L_{<\omega^{\eta_{i-1}}\omega}$-atomic, hence $L_{<\omega^{\eta_i}}$-atomic. So P shares a subpath with $\mathfrak{a}_{i+1}$ by Lemma 13.1.14 and the induction hypothesis. We conclude by induction that P shares a subpath with $\mathfrak{a}_k = L_\beta E_\alpha^\varphi$.

Suppose conversely that P shares a subpath with $L_\beta E_\alpha^\varphi = \mathfrak{a}_k$. By induction on $i = k-1, \ldots, 1$, it follows from Lemma 13.1.15 that P shares a subpath with $\mathfrak{a}_i$. Applying Lemma 13.1.14 to $\mathfrak{a}_1 = E_\alpha^\varphi$, we conclude that P shares a subpath with φ. □

13.1.3 Deconstruction lemmas

In this subsection, we list several results on the interaction between the simplicity relation $\sqsubseteq$ and various operations in $(\mathbf{No}, +, \times, (L_\alpha)_{\alpha \in \mathbf{On}})$.

Lemma 13.1.18. [55, Theorem 3.3] *For $a,b\in\mathbf{No}$, we have*

$$a\sqsubseteq b\Longleftrightarrow -a\sqsubseteq -b.$$

Lemma 13.1.19. [55, Theorem 5.12(a)] *For $\mathfrak{m}\in\mathbf{Mo}$ and $r\in\mathbb{R}^{\neq}$, we have*

$$\operatorname{sign}(r)\,\mathfrak{m}\sqsubseteq r\,\mathfrak{m}.$$

Lemma 13.1.20. [11, Proposition 4.20] *Let $\varphi\in\mathbf{No}$. For δ,ε with $\delta,\varepsilon\prec\operatorname{supp}\varphi$, we have*

$$\varphi+\delta\sqsubseteq\varphi+\varepsilon \iff \delta\sqsubseteq\varepsilon.$$

Lemma 13.1.21. [18, Corollary 4.21] *For $\mathfrak{m},\mathfrak{n}\in\mathbf{Mo}$, we have*

$$\mathfrak{m}\sqsubseteq\mathfrak{n} \iff \mathfrak{m}^{-1}\sqsubseteq\mathfrak{n}^{-1}.$$

Lemma 13.1.22. [18, Proposition 4.23] *Given φ,a,b in $\mathbf{No}_{\succ}$ with $a,b\prec\operatorname{supp}\varphi$, we have*

$$\mathrm{e}^a\sqsubseteq\mathrm{e}^b \implies \mathrm{e}^{\varphi+a}\sqsubseteq\mathrm{e}^{\varphi+b}.$$

Lemma 13.1.23. [18, Proposition 4.24] *Given $\mathfrak{m},\mathfrak{n}\in\mathbf{Mo}^{\succ}$ with $\log\mathfrak{m}\prec\mathfrak{n}$, we have*

$$\mathfrak{m}\sqsubseteq\mathfrak{n} \implies \mathrm{e}^{\mathfrak{m}}\sqsubseteq\mathrm{e}^{\mathfrak{n}}.$$

Lemma 13.1.24. *Let $\varphi\in\mathbf{No}_{\succ}$ and $r\in\mathbb{R}^{\neq}$, let $\mathfrak{m},\mathfrak{n}\in\mathbf{Mo}^{\succ}\cap\mathbf{No}^{\prec\operatorname{supp}\varphi}$ with $\mathfrak{m}\in\mathcal{E}_\omega[\mathfrak{n}]$, and let $\delta\in\mathbf{No}_{\succ}$ with $\delta\prec\operatorname{supp}\mathfrak{n}$. Then*

$$\mathfrak{m}\sqsubseteq\mathfrak{n}\Longrightarrow\mathrm{e}^{\varphi+\operatorname{sign}(r)\mathfrak{m}}\sqsubseteq\mathrm{e}^{\varphi+r\mathfrak{n}+\delta}.$$

Proof. The condition $\mathfrak{m}\in\mathcal{E}_\omega[\mathfrak{n}]$ yields $\log\mathfrak{m}\prec\mathfrak{n}$. We have $\mathrm{e}^{\mathfrak{m}}\sqsubseteq\mathrm{e}^{\mathfrak{n}}$ by Lemma 13.1.23. The identity $\mathrm{e}^{\mathbf{Mo}^{\succ}}=\mathbf{Smp}_{\mathcal{P}}$ implies that $\mathrm{e}^{\mathfrak{m}}\sqsubseteq\mathrm{e}^{|r|\mathfrak{n}}$, whence $\mathrm{e}^{\operatorname{sign}(r)\mathfrak{m}}\sqsubseteq\mathrm{e}^{r\mathfrak{n}}$ by Lemma 13.1.21. Consequently, $\mathrm{e}^{\varphi+\operatorname{sign}(r)\mathfrak{m}}\sqsubseteq\mathrm{e}^{\varphi+r\mathfrak{n}}$, by Lemma 13.1.22. Since $\mathrm{e}^0=1\sqsubseteq\mathrm{e}^{\delta}\in\mathbf{No}^{>}$, we may apply Lemma 13.1.22 to $\varphi+r\,\mathfrak{n}$ and $\varphi+r\,\mathfrak{n}+\delta$ to obtain $\mathrm{e}^{\varphi+r\mathfrak{n}}\sqsubseteq\mathrm{e}^{\varphi+r\mathfrak{n}+\delta}$. We conclude using the transitivity of $\sqsubseteq$. □

Lemma 13.1.25. *Let $\alpha\in\omega^{\mathbf{On}}$ with $\alpha>1$. For $\varphi,\psi\in\mathbf{No}_{\succ,\alpha}$ with $L_\alpha\,E_{<\alpha}\,\varphi<\psi$, we have*

$$\varphi\sqsubseteq\psi\Longrightarrow E_\alpha^\varphi\sqsubseteq E_\alpha^\psi.$$

Proof. By (12.3.3), we have

$$E_\alpha\,\varphi=\left\{E_{<\alpha}\,\varphi,\mathcal{E}_\alpha\,E_\alpha^{\varphi_L^{\mathbf{No}_{\succ,\alpha}}}\;\middle|\;\mathcal{E}_\alpha\,E_\alpha^{\varphi_R^{\mathbf{No}_{\succ,\alpha}}}\right\}.$$

Since $\varphi\sqsubseteq\psi$, we have $\varphi_L^{\mathbf{No}_{\succ,\alpha}}\subseteq\psi_L^{\mathbf{No}_{\succ,\alpha}}$ and $\varphi_R^{\mathbf{No}_{\succ,\alpha}}\subseteq\psi_R^{\mathbf{No}_{\succ,\alpha}}$, whence

$$\mathcal{E}_\alpha\,E_\alpha^{\varphi_L^{\mathbf{No}_{\succ,\alpha}}}<E_\alpha\,\psi<\mathcal{E}_\alpha\,E_\alpha^{\varphi_R^{\mathbf{No}_{\succ,\alpha}}}.$$

Furthermore, we have $L_\alpha\,E_{<\alpha}\,\varphi<\psi$, so $E_{<\alpha}\,\varphi<E_\alpha^\psi$. We conclude that $E_\alpha^\varphi\sqsubseteq E_\alpha^\psi$. □

13.2 Nested truncation and well-nestedness

In [18, Section 8], the authors prove the well-nestedness axiom **T4** for **No** by relying on a well-founded partial order $\blacktriangleleft_{\mathrm{BM}}$ that is defined by induction. This relation has the additional property that

$$\forall a,b\in\mathbf{No}^{\neq},\quad a\blacktriangleleft_{\mathrm{BM}}b\Longrightarrow a\sqsubseteq b.$$

In this subsection, we define a similar relation $\lesssim$ on **No** that will be instrumental in deriving results on the structure of $(\mathbf{No},(L_\gamma)_{\gamma\in\omega^{\mathbf{On}}})$. However, this relation does *not* satisfy $a\lesssim b\Longrightarrow a\sqsubseteq b$ for all $a,b\in\mathbf{No}$.

13.2.1 Nested truncation

Given $a,b\in\mathbf{No}$, we define

$$a\lesssim b \;\overset{\text{def}}{\Longleftrightarrow}\; \exists n\in\mathbb{N},\ a\lesssim_n b,$$

where $(\lesssim_n)_{n\in\mathbb{N}}$ is a sequence of relations that are defined by induction on n, as follows. For $n=0$, we set $a\lesssim_0 b$, if $a\trianglelefteq b$ or if there exist decompositions

$$\begin{aligned} a &= \varphi+\operatorname{sign}(r)\,\mathfrak{m}\\ b &= \varphi+r\,\mathfrak{m}+\delta, \end{aligned}$$

with $r\in\mathbb{R}^{\neq}$ and $\mathfrak{m}\in\mathbf{Mo}$. Assuming that $\lesssim_n$ has been defined, we set $a\lesssim_{n+1} b$ if we are in one of the two following configurations:

Configuration I. We may decompose a and b as

$$a = \varphi+\operatorname{sign}(r)\,\mathrm{e}^{\psi}\,(E_\alpha^u)^\iota \tag{13.2.1}$$
$$b = \varphi+r\,\mathrm{e}^{\psi}\,(L_\beta E_\alpha^v)^\iota+\delta, \tag{13.2.2}$$

where $r\in\mathbb{R}^{\neq}$, $\psi\in\mathbf{No}_\succ$, $\alpha\in\omega^{\mathbf{On}}$, $\beta\omega<\alpha$, $\iota\in\{-1,1\}$, $u,v\in\mathbf{No}_{\succ,\alpha}$,

$$\operatorname{supp}\psi\succ\log E_\alpha^u,\ L_{\beta+1}E_\alpha^v,$$

and $u\lesssim_n v$. If $\alpha=1$, then we also require that $\psi=0$.

Configuration II. We may decompose a and b as

$$a = \varphi+\operatorname{sign}(r)\,\mathrm{e}^{\psi} \tag{13.2.3}$$
$$b = \varphi+r\,\mathrm{e}^{\psi'}\,\mathfrak{a}^\iota+\delta, \tag{13.2.4}$$

where $r\in\mathbb{R}^{\neq}$, $\psi,\psi'\in\mathbf{No}_\succ$, $\iota\in\{-1,1\}$, $\mathfrak{a}\in\mathbf{Mo}_\omega$, $\delta\in\mathbf{No}$, $\operatorname{supp}\psi'\succ\log\mathfrak{a}$, and $\psi\lesssim_n\psi'$.

Warning 13.2.1. Taking $\alpha=1$ in the first configuration, we see that $\lesssim$ extends $\blacktriangleleft_{\mathrm{BM}}$. However, the relation $\lesssim$ is neither transitive nor anti-symmetric. Furthermore, as we already noted above, we do *not* have $\forall a,b\in\mathbf{No},\ a\lesssim b\Longrightarrow a\sqsubseteq b$.

Lemma 13.2.2. *Let $\alpha\in\omega^{\mathbf{On}}$. Let $a,b\in\mathbf{No}^{>,\succ}$ be numbers of the form*

$$\begin{aligned} a &= \varphi+r\,\mathfrak{m}\\ b &= \varphi+s\,\mathfrak{n}+\delta \end{aligned}$$

where $\varphi,\delta\in\mathbf{No}$, $r,s\in\mathbb{R}^{\neq}$ with $\operatorname{sign}(r)=\operatorname{sign}(s)$, and $\mathfrak{m},\mathfrak{n}\in\mathbf{Mo}^{\prec}$. If $\mathfrak{m}^{-1}<E_\rho\,\mathfrak{n}^{-1}$ for sufficiently large $\rho<\alpha$, then

$$b\in\mathbf{No}_{\succ,\alpha}\Longrightarrow a\in\mathbf{No}_{\succ,\alpha}.$$

Proof. Let $\nu\in\mathbf{On}$ and $\alpha:=\omega^\nu$. Assume for contradiction that $b\in\mathbf{No}_{\succ,\alpha}$ and $a\notin\mathbf{No}_{\succ,\alpha}$. Assume first that $a\triangleleft b$, so $b=a+\delta$. Then $\operatorname{supp}b\succ\frac{1}{L_{<\alpha}E_\alpha b}$. Let $k\in\mathbb{N}^>$ be such that $a+k\,\mathfrak{d}_\delta\geqslant b$. Since $\operatorname{supp}(a+k\,\mathfrak{d}_\delta)\subseteq\operatorname{supp}b$, we deduce that $\operatorname{supp}(a+k\,\mathfrak{d}_\delta)\succ\frac{1}{L_{<\alpha}E_\alpha(a+k\,\mathfrak{d}_\delta)}$, whence $a+k\,\mathfrak{d}_\delta\in\mathbf{No}_{\succ,\alpha}$. Modulo replacing b by $a+k\,\mathfrak{d}_\delta$, it follow that we may assume without loss of generality that $\delta=k\,\mathfrak{p}$ for some $k\in\mathbb{N}^>$ and some monomial $\mathfrak{p}$.

On the one hand, a is not α-truncated, so there are $\mathfrak{q}\in(\operatorname{supp}\varphi)_\prec$ and γ with $0<\gamma<\alpha$ and $a<L_\alpha^{\uparrow\gamma}(\mathfrak{q}^{-1})$. We may choose $\gamma=\omega^\eta n$ for certain $\eta<\nu$ and $n\in\mathbb{N}^>$, so $a<L_\alpha^{\uparrow\omega^\eta n}(\mathfrak{p}^{-1})$. On the other hand, $a+k\,\mathfrak{p}$ is α-truncated, so we have

$$a+k\,\mathfrak{p}>L_\alpha^{\uparrow\omega^\eta(n+\mathbb{N}^>)}(\mathfrak{p}^{-1})>L_\alpha^{\uparrow\omega^\eta n}(\mathfrak{p}^{-1})>a.$$

We deduce that $k\,\mathfrak{p}>L_\alpha^{\uparrow\omega^\eta(n+\mathbb{N}^>)}(\mathfrak{p}^{-1})-L_\alpha^{\uparrow\omega^\eta n}(\mathfrak{p}^{-1})$. If ν is a successor, then choosing $\eta=\nu_-$, we obtain $k\,\mathfrak{p}>L_\alpha^{\uparrow\omega^\eta n}(\mathfrak{p}^{-1})+\mathbb{N}^>-L_\alpha^{\uparrow\omega^\eta n}(\mathfrak{p}^{-1})$, so $k\,\mathfrak{p}\succ1$: a contradiction. Otherwise, $k\,\mathfrak{p}>\ell_{[\omega^{\eta+1},\alpha)}^{-1}\circ\mathfrak{p}^{-1}$ by [14, (2.4)], where $\ell_{[\omega^{\eta+1},\alpha)}:=\prod_{\omega^{\eta+1}\leqslant\gamma<\alpha}\ell_\gamma$. Thus $k^{-1}\,\mathfrak{p}^{-1}<\ell_{[\omega^{\eta+1},\alpha)}\circ\mathfrak{p}^{-1}$, whence $k^{-1}\,\ell_0<\ell_{[\omega^{\eta+1},\alpha)}$: a contradiction.

We now treat the general case. By a similar argument as above, we may assume without loss of generality that $b = \varphi + s\,\mathfrak{n}$. Assume that $b \leqslant a$. Since a is not α-truncated, there exists a $\gamma < \alpha$ with $\mathfrak{m} \prec (L_\gamma E_\alpha a)^{-1} \leqslant (L_\gamma E_\alpha b)^{-1}$, whence $\mathfrak{m}^{-1} \succ L_\gamma E_\alpha b$. But b is α-truncated, so $\mathfrak{n}^{-1} \prec L_{<\alpha} E_\alpha b$. In particular $\mathfrak{n}^{-1} \prec L_\gamma E_\alpha b$, so our hypothesis $\mathfrak{m}^{-1} < L_\rho \mathfrak{n}^{-1}$ implies that $\mathfrak{m}^{-1} \prec L_\rho L_\gamma E_\alpha b \preccurlyeq L_\gamma E_\alpha b$: a contradiction.

Assume now that $b > a$. As in the first part of the proof, there are $\eta < \nu$ and $n < n' < \omega$ with $\varphi + s\,\mathfrak{n} > L_\alpha^{\uparrow \omega^\eta n'}(\mathfrak{n}^{-1})$ and $L_\alpha^{\uparrow \omega^\eta n}(\mathfrak{m}^{-1}) > \varphi + r\,\mathfrak{m}$. Recall that $\mathfrak{m}^{-1} < E_\rho \mathfrak{n}^{-1}$ for sufficiently large $\rho < \alpha$. Take $\eta < \nu$ and $n' < \omega$ such that

$$\begin{array}{rcll} L_\alpha^{\uparrow \omega^\eta n'}(\mathfrak{n}^{-1}) & > & L_\alpha^{\uparrow \omega^\eta (n+1)}(\mathfrak{m}^{-1}) & \\ L_{\omega^\eta} \mathfrak{m}^{-1} & \prec & \mathfrak{n}^{-1} & \text{if } \nu \text{ is a limit.} \end{array} \tag{13.2.5}$$

Then $b - a > L_\alpha^{\uparrow \omega^\eta (n+1)}(\mathfrak{m}^{-1}) - L_\alpha^{\uparrow \omega^\eta n}(\mathfrak{m}^{-1})$. If ν is a successor, then choosing $\eta = \nu_-$ yields $b - a > 1$, which contradicts the fact that $\mathfrak{m}$ and $\mathfrak{n}$ are infinitesimal. So ν is a limit. Writing $\mathfrak{q} := \max(\mathfrak{m}, \mathfrak{n})$, we have $b - a \asymp \mathfrak{q}$. As in the first part of the proof, we obtain $\mathfrak{q} \succcurlyeq \ell_{[\omega^{\eta+1},\alpha)}^{-1} \circ \mathfrak{m}^{-1}$, so $\mathfrak{q}^{-1} \preccurlyeq \ell_{[\omega^{\eta+1},\alpha)} \circ \mathfrak{m}^{-1} \prec \mathfrak{m}^{-1}$. In view of (13.2.5), we also obtain $\mathfrak{q}^{-1} \prec \mathfrak{n}^{-1}$, so $\mathfrak{q}^{-1} \prec \max(\mathfrak{m}, \mathfrak{n})^{-1}$: a contradiction. □

Lemma 13.2.3. *Let $\alpha, \alpha' \in \omega^{\mathbf{On}}$ with $\alpha' \geqslant \alpha$. For $u, v \in \mathbf{No}^{>,\succ}$, we have*

$$L_\alpha u < \mathcal{E}_\alpha v \implies L_{\alpha'} E_\alpha u < \mathcal{E}_{\alpha'} E_\alpha v.$$

Proof. Assume that $L_\alpha u < \mathcal{E}_\alpha v$. Let $h \in \mathcal{E}_{\alpha'}$ and let h^{inv} be its functional inverse in $\mathcal{E}_{\alpha'}$. We have $h^{\mathrm{inv}} < E_{\alpha'} H_2 L_{\alpha'}$ by (13.0.5, 13.0.6), whence $h > E_{\alpha'} H_{1/2} L_{\alpha'}$. Furthermore, $u < E_\alpha \mathcal{E}_\alpha v$, so

$$E_\alpha u < E_\alpha E_\alpha \mathcal{E}_\alpha v. \tag{13.2.6}$$

We want to prove that $E_\alpha u < (E_{\alpha'} h E_\alpha) v$. By (13.2.6), it is enough to prove that there is a $g \in \mathcal{E}_\alpha$ such that the inequality $E_\alpha E_\alpha g \leqslant E_{\alpha'} h E_\alpha$ holds on $\mathbf{No}^{>,\succ}$.

Assume that $\alpha = \alpha'$. Setting $g := H_{1/2} \in \mathcal{E}_\alpha$, we have $L_\alpha h E_\alpha > g$, whence $E_\alpha g \leqslant h E_\alpha$, and $E_\alpha E_\alpha g \leqslant E_\alpha h E_\alpha$.

Assume that $\alpha' > \alpha$. We have $E_{\alpha'} H_{1/2} > H_2$ so $E_{\alpha' 2} H_{1/2} > E_{\alpha'} H_2 > E_\alpha E_{\alpha'}$ by (13.0.3). Thus $E_{\alpha'} h > E_{\alpha' 2} H_{1/2} L_{\alpha'} > E_\alpha$. Consequently, $E_{\alpha'} h E_\alpha > E_\alpha E_\alpha$, as claimed. □

If a, b are numbers, then we write $[a \leftrightarrow b]$ for the interval $[\min(a,b), \max(a,b)]$.

Proposition 13.2.4. *For $a, b, c \in \mathbf{No}$ with $a \lesssim c$ and $b \in [a \leftrightarrow c]$, any infinite path in a shares a subpath with b.*

Proof. We prove this by induction on n with $a \lesssim_n c$. Let P be an infinite path in a. Assume that $a \lesssim_0 c$. If $a \trianglelefteq c$, then we have $a \trianglelefteq b$ so P is a path in b. Otherwise, there are $\varphi, \delta \in \mathbf{No}$, $r \in \mathbb{R}^{\neq}$ and $\mathfrak{m} \in \mathbf{Mo}$ with $a = \varphi + \operatorname{sign}(r)\,\mathfrak{m}$ and $c = \varphi + r\,\mathfrak{m} + \delta$. Then $b = \varphi + s\,\mathfrak{n} + t$ for certain $t \in \mathbf{No}$, $s \in \mathbb{R}^{\neq}$ and $\mathfrak{n} \in \mathbf{Mo}$ with $s\,\mathfrak{n} \in [\operatorname{sign}(r)\,\mathfrak{m} \leftrightarrow r\,\mathfrak{m}]$. We must have $\mathfrak{n} = \mathfrak{m}$. If P is a path in φ, then it is a path in b. Otherwise, it is a path in $\operatorname{sign}(r)\,\mathfrak{m}$, so $P_{\nearrow 1}$ is a subpath in $s\,\mathfrak{m}$, hence in b.

We now assume that $a \lesssim_n c$ where $n > 0$ and that the result holds for all $a', b', c' \in \mathbf{No}$ and $k < n$ with $a' \lesssim_k c'$ and $b' \in [a' \leftrightarrow c']$. Assume first that (a, c) is in Configuration I, and write

$$\begin{array}{rcl} a & = & \varphi + \operatorname{sign}(r)\,\mathrm{e}^\psi\,(E_\alpha^u)^\iota \\ c & = & \varphi + r\,\mathrm{e}^\psi\,(L_\beta E_\alpha^v)^\iota + \delta \end{array} \quad \text{with} \quad u \lesssim_{n-1} v.$$

Then we can write $b = \varphi + s\,\mathfrak{m} + t$ like in the case when $n = 0$. If P is a path in φ, then it is a path in b. So we may assume that P is a path in $\operatorname{sign}(r)\,\mathrm{e}^\psi\,(E_\alpha^u)^\iota$. Note that we have $\mathfrak{m} \in [\mathrm{e}^\psi\,(E_\alpha^u)^\iota \leftrightarrow \mathrm{e}^\psi\,(L_\beta E_\alpha^v)^\iota]$. Setting $\mathfrak{n} := (\mathfrak{m}\,\mathrm{e}^{-\psi})^\iota \in [E_\alpha^u \leftrightarrow L_\beta E_\alpha^v]$, we observe that $\operatorname{supp}\log\mathfrak{n} \prec \operatorname{supp}\psi$, whence $\mathrm{e}^\psi\,\mathfrak{n}^\iota$ is the hyperserial expansion of $\mathfrak{m}$. If $P_{\nearrow 1}$ is a path in ψ, then it is a path in $\mathfrak{m}$.

Suppose that $P_{\nearrow 1}$ is not a path in ψ. Assume first that $\alpha = 1$, so $\psi = 0$, $\beta = 0$, and P is a path in $(E_1^u)^\iota$. Then Lemma 13.1.14 implies that $P_{\nearrow 1}$ is a subpath in $\iota\,u$, so $P_{\nearrow 2}$ is a subpath in u. Otherwise, consider the hyperserial expansion $E_\alpha^u = L_{\beta'} E_{\alpha'}^w$, $E_{\alpha'}^w \in \mathbf{Mo}_{\alpha'} \setminus L_{<\alpha'} \mathbf{Mo}_{\alpha' \omega}$ of E_α^u. Since $P_{\nearrow 1}$ is not a path in ψ, it must be a path in w. The number $L_{\beta'} E_{\alpha'}^w$ is $L_{<\alpha}$-atomic, so we must have $\alpha' \geqslant \alpha$ and $\beta' \gtrsim \alpha_{/\omega}$. There are $n \in \mathbb{N}$ and $\beta'' \gg \alpha_{/\omega}$ such that $\beta' = \beta'' + \alpha_{/\omega}\,n$. Therefore $u = L_{\beta'' + \alpha} E_{\alpha'}^w - n$. It follows by Corollary 13.1.17 that $P_{\nearrow 1}$ shares a subpath with u, whence so does P.

Let $z := \sharp_\alpha(L_\alpha \mathfrak{n})$. Recall that $\mathfrak{n} \in [E_\alpha^u \leftrightarrow L_\beta E_\alpha^v]$, so $L_\alpha \mathfrak{n} \in [u \leftrightarrow L_\alpha L_\beta E_\alpha^v]$. Now (13.0.3) implies that $L_\beta E_\alpha^v \in \mathcal{E}_\alpha[E_\alpha^v]$, so $L_\alpha L_\beta E_\alpha^v \in L_\alpha \mathcal{E}_\alpha[E_\alpha^v] = \mathcal{L}_\alpha[v]$. The function $\sharp_\alpha = \pi_{\mathbf{Smp}_{\mathcal{L}_\alpha}}$ is non-decreasing, so $z = \sharp_\alpha(L_\alpha \mathfrak{n}) \in [u \leftrightarrow \sharp_\alpha(L_\alpha L_\beta E_\alpha^v)] = [u \leftrightarrow v]$. But $u \lesssim_{n-1} v$, so the induction hypothesis yields that $P_{\nearrow 2}$, and thus P, shares a subpath with z. We deduce with Lemma 13.1.15 that P shares a subpath with $\mathfrak{n}$, hence with b.

Assume now that (a, c) is in Configuration II, and write

$$\begin{array}{rcl} a & = & \varphi + \operatorname{sign}(r)\,\mathrm{e}^{\psi} \\ c & = & \varphi + r\,\mathrm{e}^{\psi'}\,\mathfrak{a}^{\iota} + \delta \end{array} \quad \text{with} \quad \psi \lesssim_{n-1} \psi'.$$

Note that we also have $\psi \lesssim_{n-1} \psi' + \iota \log \mathfrak{a}$. We may again assume that $P_{\nearrow 1}$ is a path in ψ. Write $b = \varphi + s'\,\mathfrak{q} + t'$, where $s' \in \mathbb{R}^{\neq}$, $t' \in \mathbf{No}$, and $\mathfrak{q} \in [\mathrm{e}^{\psi} \leftrightarrow \mathrm{e}^{\psi'}\,\mathfrak{a}^{\iota}] \cap \mathbf{Mo}$. Then $\log \mathfrak{q} \in [\psi \leftrightarrow \psi' + \iota \log \mathfrak{a}]$ where $\psi \lesssim_{n-1} \psi' + \iota \log \mathfrak{a}$. We deduce by induction that P shares a subpath with $\log \mathfrak{q}$. By Lemma 13.1.15, it follows that P shares a subpath with $\mathfrak{q}$, hence with b. This concludes the proof. □

Lemma 13.2.5. *Let $\lambda, \alpha \in \omega^{\mathbf{On}}$ and $\beta \in \mathbf{On}$ with $\beta\omega < \alpha$. Let $a \in \mathbf{No}_{\succ,\lambda}$ be of the form*

$$a = \varphi + r\,\mathrm{e}^{\psi}\,(L_\beta E_\alpha^b)^{\iota} + \delta,$$

with $\varphi \in \mathbf{No}$, $r \in \mathbb{R}^{\neq}$, $\psi \in \mathbf{No}_{\succ}$, $b \in \mathbf{No}_{\succ,\alpha}$, $\iota \in \{-1, 1\}$, $\delta \in \mathbf{No}$ and $\log L_\beta E_\alpha^b \prec \operatorname{supp} \psi$. Consider an infinite path P in $c \in \mathbf{No}_{\succ,\alpha}$ with $c \lesssim b$.

i. If $\log E_\alpha^c \not\prec \operatorname{supp} \psi$, then P shares a subpath with ψ.

ii. If $\log E_\alpha^c \prec \operatorname{supp} \psi$ and $\mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota} \not\prec \operatorname{supp} \varphi$, then P shares a subpath with φ.

iii. If $\log E_\alpha^c \prec \operatorname{supp} \psi$ and $\mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota} \prec \operatorname{supp} \varphi$ and $a' := \varphi + \operatorname{sign}(r)\,\mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota} \notin \mathbf{No}_{\succ,\lambda}$, then P shares a subpath with φ.

Proof. i. If $\log E_\alpha^c \not\prec \operatorname{supp} \psi$, then we have $\psi \neq 0$, so $\alpha > 1$. Let $\mathfrak{m} \in \operatorname{supp} \psi$ with $\log E_\alpha^c \succcurlyeq \mathfrak{m}$. Since $\log E_\alpha^c$ and $\mathfrak{m}$ are monomials, we have $\mathfrak{m} \leqslant \log E_\alpha^c$, whence $\mathrm{e}^{\mathfrak{m}} \leqslant E_\alpha^c$. Our assumption that $\mathfrak{m} \in \operatorname{supp} \psi \succ \log L_\beta E_\alpha^b$ also implies $\mathrm{e}^{\mathfrak{m}} \leqslant L_\beta E_\alpha^b$. Hence $\mathrm{e}^{\mathfrak{m}} \in [E_\alpha^c \leftrightarrow L_\beta E_\alpha^b]$. Now P shares a subpath with E_α^c, by Lemma 13.1.15. Since $E_\alpha^c \lesssim L_\beta E_\alpha^b$, Proposition 13.2.4 next implies that P shares a subpath with $\mathrm{e}^{\mathfrak{m}}$. Using Lemma 13.1.14, we conclude that P shares a subpath with $\mathfrak{m}$, and hence with ψ.

ii. Let $\mathfrak{m} \in \operatorname{supp} \varphi$ with $\mathfrak{m} \preccurlyeq \mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota}$. It is enough to prove that P shares a subpath with $\mathfrak{m}$. Since $\mathfrak{m}$, $\mathrm{e}^{\psi}\,(L_\beta E_\alpha^b)^{\iota}$, and $\mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota}$ are monomials, we have $\mathrm{e}^{\psi}\,(L_\beta E_\alpha^b)^{\iota} \leqslant \mathfrak{m} \leqslant \mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota}$. Let $\mathfrak{n} := (\mathrm{e}^{-\psi}\,\mathfrak{m})^{\iota}$, so that $\mathfrak{n} \in [L_\beta E_\alpha^b \leftrightarrow E_\alpha^c]$. In particular, we have $\operatorname{supp} \psi \succ \log \mathfrak{n} \succ 1$. Moreover $E_\alpha^c \lesssim L_\beta E_\alpha^b$, so using Lemma 13.1.15 and Proposition 13.2.4, we deduce in the same way as above that P shares a subpath with $\mathfrak{n}$. If $\mathfrak{n} \notin \mathbf{Mo}_\omega$, then $\mathfrak{m} = \mathrm{e}^{\psi + \iota \log \mathfrak{n}}$ is the hyperserial expansion of $\mathfrak{m}$, so P shares a subpath with $\mathfrak{m}$. If $\mathfrak{n} \in \mathbf{Mo}_\omega$, then the hyperserial expansion of $\mathfrak{n}$ must be of the form $\mathfrak{n} = E_{\beta'} E_{\alpha'}^u$, since otherwise $\log \mathfrak{n}$ would have at least two elements in its support. We deduce that P shares a subpath with u and that the hyperserial expansion of $\mathfrak{m}$ is $\mathrm{e}^{\psi}\,(E_{\beta'} E_{\alpha'}^u)^{\iota}$. Therefore P shares a subpath with $\mathfrak{m}$.

iii. We assume that a' is not λ-truncated whereas $\log E_\alpha^c \prec \operatorname{supp} \psi$ and $\mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota} \prec \operatorname{supp} \varphi$. If $\lambda = 1$, then we must have $\mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota} \preccurlyeq 1$, which means that $\psi < 0$ or that $\psi = 0$ and $\iota = -1$. But then $\mathrm{e}^{\psi}\,(L_\beta E_\alpha^b)^{\iota} \preccurlyeq 1$: a contradiction.

Assume that $\lambda > 1$. By Lemma 13.2.2, we may assume without loss of generality that $\delta = 0$. The assumption on a' and the fact that $a \in \mathbf{No}^{>,\succ}$ imply that φ is non-zero. Write

$$\begin{array}{rcl} \mathfrak{p} & := & \mathrm{e}^{\psi}\,(E_\alpha^c)^{\iota} \quad \text{and} \\ \mathfrak{q} & := & \mathrm{e}^{\psi}\,(L_\beta E_\alpha^b)^{\iota}. \end{array}$$

So $a = \varphi + r\,\mathfrak{q}$ and $a' = \varphi + \operatorname{sign}(r)\,\mathfrak{p}$. Note that $\mathfrak{p}$ must be infinitesimal since a' is not λ-truncated. Thus $\mathfrak{q}$ is also infinitesimal. By Lemma 13.2.2, we deduce that $E_{<\lambda}\,\mathfrak{q}^{-1} \prec \mathfrak{p}^{-1}$. We have $\sharp_\lambda(a') \trianglelefteq a'$, so $\sharp_\lambda(a') = \varphi$, since a and $\varphi \trianglelefteq a$ are both λ-truncated. Since a' is not λ-truncated, there is an ordinal $\gamma < \lambda$ with $\mathfrak{p} \prec (L_\gamma E_\lambda^{\varphi})^{-1}$. If $\varphi \geqslant a$, then $\mathfrak{q} \succ (L_{<\lambda} E_\lambda^{a})^{-1}$, because a is λ-truncated. Thus $\mathfrak{q} \succ (L_{<\lambda} E_\lambda^{\varphi})^{-1}$. If $\varphi < a$, then $\varphi + (L_{<\lambda} E_\lambda^{\varphi})^{-1} \in \mathcal{L}_\lambda[\varphi] < \mathcal{L}_\lambda[a] \ni a = \varphi + r\,\mathfrak{q}$, because φ and a are λ-truncated. Now $r > 0$, since $\varphi < a$. We again deduce that $\mathfrak{q} \succ (L_{<\lambda} E_\lambda^{\varphi})^{-1}$.

In both cases, we have $L_\gamma E_\lambda^{\varphi} \in [\mathfrak{p}^{-1} \leftrightarrow \mathfrak{q}^{-1}]$ where $\mathfrak{p}^{-1} \lesssim \mathfrak{q}^{-1}$, so P shares a subpath with $L_\gamma E_\lambda^{\varphi}$, by Proposition 13.2.4. It follows by Corollary 13.1.17 that P shares a subpath with φ. □

13.2.2 Well-nestedness

We now prove that every number is well-nested. Throughout this subsection, P will be an infinite path inside a number $a\in\mathbf{No}$. At the beginning of Section 13.1.2 we have shown how to attach sequences $(r_{P,i})_{i<\omega}$, $(\mathfrak{m}_{P,i})_{i<\omega}$, etc. to this path. In order to alleviate notations, we will abbreviate $r_i:=r_{P,i}$, $\mathfrak{m}_i:=\mathfrak{m}_{P,i}$, $u_i:=u_{P,i}$, $\psi_i:=\psi_{P,i}$, $\iota_i:=\iota_{P,i}$, $\alpha_i:=\alpha_{P,i}$, and $\beta_i:=\beta_{P,i}$ for all $i\in\mathbb{N}$.

We start with a technical lemma that will be used to show that the existence of a bad path P in a implies the existence of a bad path in a strictly simpler number than a.

Lemma 13.2.6. *Let $a\in\mathbf{No}$, let P be an infinite path in a and let $i\in\mathbb{N}$ such that every index $k\leqslant i$ is good for (P,a). For $k\leqslant i$, let $\varphi_k:=(u_k)_{\succ\mathfrak{m}_k}$, $\varepsilon_k:=r_k$, and $\rho_k:=(u_k)_{\prec\mathfrak{m}_k}$, so that $\varepsilon_0,\ldots,\varepsilon_{i-1}\in\{-1,1\}$ and*

$$\begin{aligned} u_k &= \varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}\,(E_{\alpha_k}^{u_{k+1}})^{\iota_k} && (k<i)\\ u_i &= \varphi_i+r_i\,\mathrm{e}^{\psi_{i+1}}\,(L_{\beta_i}E_{\alpha_i}^{u_{i+1}})^{\iota_i}+\rho_i. \end{aligned}$$

Let $\chi\in\{0,1\}$ and let $c_i\in\mathbf{No}_{\succ,\alpha_{i-1}}$ be a number with $c_i\lesssim u_i$ and

$$c_i=\varphi_i+\chi\,\mathrm{sign}(r_i)\,\mathrm{e}^{\psi_{i+1}}\,\mathfrak{p}^{\iota_i}, \tag{13.2.7}$$

for a certain $\mathfrak{p}\in\mathbf{Mo}^{\succ}$ with $\log\mathfrak{p}\prec\operatorname{supp}\psi_{i+1}$, $\mathfrak{p}\sqsubseteq E_{\alpha_i}^{u_{i+1}}$ and $\mathfrak{p}\in\mathcal{E}_\omega[E_{\alpha_i}^{u_{i+1}}]$ whenever $\psi_{i+1}=0$. For $k=i-1,\ldots,0$, we define

$$c_k:=\varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}\,(E_{\alpha_k}^{c_{k+1}})^{\iota_k} \tag{13.2.8}$$

Assume that P shares a subpath with c_i. If P shares no subpath with any of the numbers $\varphi_0,\psi_1,\ldots,\varphi_{i-1},\psi_i$, then we have $c_0\sqsubseteq a$, and P shares a subpath with c_0.

Proof. Using backward induction on k, let us prove for $k=i-1,\ldots,0$ that

$$\begin{aligned} L_{\alpha_k}c_{k+1} &< \mathcal{E}_{\alpha_k}u_{k+1} && (13.2.9)_k\\ \log E_{\alpha_k}^{c_{k+1}} &\prec \operatorname{supp}\psi_{k+1} && (13.2.10)_k\\ \mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k} &\prec \operatorname{supp}\varphi_k && (13.2.11)_k\\ c_k &\lesssim u_k && (13.2.12)_k\\ P\text{ shares a subpath with } c_{k+1} & && (13.2.13)_k\\ c_{k+1} &\in \mathbf{No}_{\succ,\alpha_k} && (13.2.14)_k\\ c_{k+1} &\sqsubseteq u_{k+1} && (13.2.15)_k \end{aligned}$$

and that $(13.2.13)_k$ and $(13.2.15)_k$ also hold for $k=-1$.

We first treat the case when $k=i-1$. Note that $c_i\neq0$ since it contains a subpath, so $\varphi_i\in\mathbf{No}^{>,\succ}$ or $\chi=1$. From our assumption that $c_i=\varphi_i+\chi\,\mathrm{sign}(r_i)\,\mathrm{e}^{\psi_{i+1}}\,\mathfrak{p}^{\iota_i}$ and the fact that $\mathfrak{p}\in\mathcal{E}_\omega[E_{\alpha_i}^{u_{i+1}}]$ if $\psi_{i+1}=0$, we deduce that $c_i\in\mathcal{E}_\omega[u_i]$. Hence $L_{\alpha_{i-1}}c_i<\mathcal{E}_{\alpha_{i-1}}u_i$ and $(13.2.9)_{i-1}$. Note that $(13.2.13)_{i-1}$ and $(13.2.14)_{i-1}$ follow immediately from the other assumptions on c_i. If $\chi=0$ then $c_i=\varphi_i\trianglelefteq u_i$. If $\chi=1$, then $\mathfrak{p}\sqsubseteq L_{\beta_i}E_{\alpha_i}^{u_{i+1}}$, since $L_{\beta_i}E_{\alpha_i}^{u_{i+1}}\in\mathcal{E}_{\alpha_i}[E_{\alpha_i}^{u_{i+1}}]$ and $\mathfrak{p}\sqsubseteq E_{\alpha_i}^{u_{i+1}}\sqsubseteq\mathcal{E}_{\alpha_i}[E_{\alpha_i}^{u_{i+1}}]$. Hence $\mathfrak{p}^{\iota_i}\sqsubseteq(L_{\beta_i}E_{\alpha_i}^{u_{i+1}})^{\iota_i}$ by Lemma 13.1.21 and $\mathrm{sign}(r_i)\,\mathrm{e}^{\psi_{i+1}}\,\mathfrak{p}^{\iota_i}\sqsubseteq r_i\,\mathrm{e}^{\psi_{i+1}}\,(L_{\beta_i}E_{\alpha_i}^{u_{i+1}})^{\iota_i}$ by Lemmas 13.1.19 and 13.1.22. Finally, $c_i\sqsubseteq u_i$ by Lemma 13.1.20, so $(13.2.15)_{i-1}$ holds in general. Recall that P is a subpath in c_i, but that it shares no subpath with ψ_i or φ_{i-1}. In view of $(13.2.14)_{i-1}$, we deduce $(13.2.10)_{i-1}$ from Lemma 13.2.5(i) and $(13.2.11)_{i-1}$ from Lemma 13.2.5(ii). Combining $(13.2.10)_{i-1}$, $(13.2.11)_{i-1}$ and $(13.2.14)_{i-1}$ with the relation $c_i\lesssim u_i$, we finally obtain $(13.2.12)_{i-1}$.

Let $k\in\{0,\ldots,i-1\}$ and assume that $(13.2.9\text{–}13.2.15)_\ell$ hold for all $\ell\geqslant k$. We shall prove $(13.2.9\text{–}13.2.15)_{k-1}$ if $k>0$, as well as $(13.2.13)_{-1}$ and $(13.2.15)_{-1}$. Recall that

$$c_k=\varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}\,(E_{\alpha_k}^{c_{k+1}})^{\iota_k}.$$

$(13.2.9)_{k-1}$. Recall that $k>0$. If $\varphi_k\neq0$ or $\psi_{k+1}\neq0$, then $c_k\in\mathcal{P}[u_k]$ and $(13.2.10\text{–}13.2.11)_k$ imply $(13.2.9)_{k-1}$. Assume now that $\varphi_k=\psi_{k+1}=0$. It follows since $k>0$ that $\iota_k=1$, so $c_{k-1}=E_{\alpha_{k-1}}^{c_k}$ and $u_{k-1}=E_{\alpha_{k-1}}u_k$. Since $E_{\alpha_{k-1}}^{u_k}$ is a hyperserial expansion, we must have $u_k\notin\mathbf{Mo}_{\alpha_{k-1}\omega}$, so $\alpha_{k-1}\geqslant\alpha_k$. The result now follows from $(13.2.9)_k$ and Lemma 13.2.3.

(13.2.13)$_{k-1}$. We know by (13.2.13)$_k$ that P shares a subpath with c_{k+1}. Since $c_{k+1}\in\mathbf{No}_{\succ,\alpha_k}$, we deduce with Corollary 13.1.17 that P also shares a subpath with $E_{\alpha_k}^{c_{k+1}}$, hence with $(E_{\alpha_k}^{c_{k+1}})^{\iota_k}$. In view of (13.2.10)$_k$ and Lemma 13.1.16, we see that P shares a subpath with $\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}$. Hence (13.2.11)$_k$ gives that P shares a subpath with c_k.

(13.2.10)$_{k-1}$. By (13.2.12)$_k$, we have $c_k\lesssim u_k$. Now P shares a subpath with c_k by (13.2.13)$_k$, but it shares no subpath with ψ_k. Lemma 13.2.5(i) therefore yields the desired result $\log E_{\alpha_{k-1}}^{c_k}\prec\operatorname{supp}\psi_k$.

(13.2.11)$_{k-1}$. As above, P shares a subpath with c_k, but no subpath with φ_{k-1}. We also have $c_k\lesssim u_k$ and $\log E_{\alpha_{k-1}}^{c_k}\prec\operatorname{supp}\psi_k$, so (13.2.11)$_{k-1}$ follows from Lemma 13.2.5(ii).

(13.2.12)$_{k-1}$. We obtain (13.2.12)$_{k-1}$ by combining (13.2.9–13.2.12)$_k$ and (13.2.14)$_k$.

(13.2.14)$_{k-1}$. The path P shares a subpath with c_k, but no subpath with φ_k. By what precedes, we also have $\log E_{\alpha_k}^{c_{k+1}}\prec\operatorname{supp}\psi_k$ and $\mathrm{e}^{\psi_k}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\prec\operatorname{supp}\varphi_k$. Note finally that $u_k\in\mathbf{No}_{\succ,\alpha_{k-1}}$. Hence $c_k\in\mathbf{No}_{\succ,\alpha_{k-1}}$, by applying Lemma 13.2.5(iii) with α_k, α_{k-1}, u_k, u_{k+1}, and c_{k+1} in the roles of α, λ, a, b, and c.

(13.2.15)$_{k-1}$. It suffices to prove that $E_{\alpha_k}^{c_{k+1}}\sqsubseteq E_{\alpha_k}^{u_{k+1}}$, since

$$\begin{array}{lll} & E_{\alpha_k}^{c_{k+1}}\sqsubseteq E_{\alpha_k}^{u_{k+1}} & \\ \Longrightarrow & (E_{\alpha_k}^{c_{k+1}})^{\iota_k}\sqsubseteq(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \text{(by Lemma 13.1.21)}\\ \Longrightarrow & \mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\sqsubseteq\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \text{(by Lemma 13.1.22)}\\ \Longrightarrow & \varepsilon_k\,\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\sqsubseteq\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \\ \Longrightarrow & \varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\sqsubseteq\varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{u_{k+1}})^{\iota_k} & \text{(by Lemma 13.1.20)}\\ \Longrightarrow & c_k\sqsubseteq u_k. & \end{array}$$

Assume that $\alpha_k>1$ and recall that

$$\begin{aligned} c_k &= \varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_{k+1}}(E_{\alpha_k}^{c_{k+1}})^{\iota_k}\\ c_{k+1} &= \varphi_{k+1}+\varepsilon_{k+1}\,\mathrm{e}^{\psi_{k+2}}(E_{\alpha_{k+1}}^{c_{k+2}})^{\iota_{k+1}}. \end{aligned}$$

By Lemma 13.1.25, it suffices to prove that $c_{k+1}\sqsubseteq u_{k+1}$ and that $E_\gamma\,c_{k+1}<E_{\alpha_k}^{u_{k+1}}$ for all $\gamma<\alpha_k$. The first relation holds by (13.2.15)$_k$. By (13.2.9)$_k$, we have $L_{\alpha_k}c_{k+1}<\mathcal{E}_{\alpha_k}u_{k+1}$. Therefore $c_{k+1}<E_{\alpha_k}\frac{1}{2}u_{k+1}<L_{<\alpha_k}E_{\alpha_k}u_{k+1}$ by Lemma 13.1.25. This yields the result.

Assume now that $\alpha_k=1$. For $d=0,\ldots,i$, let

$$\begin{aligned} \mathfrak{c}_d &:= \mathfrak{d}_{c_d-\varphi_d}\\ \mathfrak{u}_d &:= \mathfrak{d}_{u_d-\varphi_d}. \end{aligned}$$

We will prove, by a second descending induction on $d=i,\ldots,k-1$, that the monomials $\mathfrak{c}_d$ and $\mathfrak{u}_d$ satisfy the premises of Lemma 13.1.24, i.e. $\mathfrak{c}_d,\mathfrak{u}_d\succ1$, $\mathfrak{c}_d\in\mathcal{E}_\omega[\mathfrak{u}_d]$, and $\mathfrak{c}_d\sqsubseteq\mathfrak{u}_d$. It will then follow by Lemma 13.1.24 that $\mathrm{e}^{c_k}\sqsubseteq\mathrm{e}^{u_k}$, thus concluding the proof.

If $d=i$, then $\operatorname{supp}c_i,\operatorname{supp}u_i\succ1$, because $\alpha_{i-1}=1$. In particular $\mathfrak{c}_i,\mathfrak{u}_i\succ1$. Moreover, $\mathfrak{c}_i\sqsubseteq\mathfrak{u}_i$ follows from our assumption that $\mathfrak{p}\sqsubseteq E_{\alpha_i}^{u_{i+1}}$, the fact that $E_{\alpha_i}^{u_{i+1}}\sqsubseteq\mathcal{E}_{\alpha_i}[E_{\alpha_i}^{u_{i+1}}]\ni L_{\beta_i}E_{\alpha_i}^{u_{i+1}}$, and Lemmas 13.1.22 and 13.1.21. If $\psi_{i+1}\neq0$, then we have $\mathfrak{c}_i\in\mathcal{E}_\omega[\mathfrak{u}_i]$ because $\operatorname{supp}\psi_{i+1}\succ\log\mathfrak{p},\log E_{\alpha_i}^{u_{i+1}}$. Otherwise, we have $\mathfrak{c}_i=\mathfrak{p}\in\mathcal{E}_\omega[E_{\alpha_i}^{u_{i+1}}]=\mathcal{E}_\omega[\mathfrak{u}_i]$.

Now assume that $d<i$, that the result holds for $d+1$, and that $\alpha_d=1$. Again $\alpha_d=1$ implies that $\mathfrak{c}_{d+1},\mathfrak{u}_{d+1}\succ1$. The relation $c_{d+1}\sqsubseteq u_{d+1}$ and Lemmas 13.1.18, 13.1.19, and 13.1.20 imply that $\mathfrak{c}_{d+1}\sqsubseteq\mathfrak{u}_{d+1}$. If $\psi_{d+2}\neq0$, then $\mathfrak{c}_{d+1}\in\mathcal{E}_\omega[\mathfrak{u}_{d+1}]$ by (13.2.10)$_{d+1}$. Otherwise, we have $\iota_{d+1}=1$, because $c_d\in\mathbf{No}_{\succ,1}$. Since $\alpha_d=1$, the number $u_{d+1}=\varphi_{d+1}+\varepsilon_{d+1}E_{\alpha_{d+1}}^{u_{d+2}}$ is not tail-atomic, so we must have $\alpha_{d+1}=1$. This entails that $\mathfrak{c}_{d+1}=\mathrm{e}^{c_{d+2}}$ and $\mathfrak{u}_{d+1}=\mathrm{e}^{u_{d+2}}$. By the induction hypothesis at $d+1$, we have $\mathfrak{c}_{d+2}\in\mathcal{E}_\omega[\mathfrak{u}_{d+2}]$. We deduce that $c_{d+2}\in\mathcal{E}_\omega[u_{d+2}]$, so

$$\mathfrak{c}_{d+1}\in\exp\mathcal{E}_\omega[u_{d+2}]=\mathcal{E}_\omega[\mathrm{e}^{u_{d+2}}]=\mathcal{E}_\omega[\mathfrak{u}_{d+1}].$$

It follows by induction that (13.2.15)$_{k-1}$ is valid.

This concludes our inductive proof. The lemma follows from (13.2.15)$_{-1}$ and (13.2.13)$_{-1}$. □

We are now in a position to prove Theorem **D**.

Theorem 13.2.7. *Every surreal number is well-nested.*

Proof. Assume for contradiction that the theorem is false. Let a be a $\sqsubseteq$-minimal ill-nested number and let P be a bad path in a. Let $i \in \mathbb{N}$ be the smallest bad index in (P, a). As in Lemma 13.2.6, we define $\varphi_k := (u_k)_{\prec \mathfrak{m}_k}$, $\rho_k := (u_k)_{\succ \mathfrak{m}_k}$, and $\varepsilon_k := r_k$ for all $k \leqslant i$. We may assume that $i > 0$, otherwise the number $c_0 := \varphi_0 + \operatorname{sign}(r_0)\, \mathrm{e}^{\psi_1}\, (E_{\alpha_0}^{u_1})^{\iota_0}$ is ill-nested and satisfies $c_0 \sqsubset a$: a contradiction.

Assume for contradiction that there is a $j < i$ such that φ_j or ψ_{j+1} is ill-nested. Set $\chi := 0$ if φ_j is ill-nested and $\chi := 1$ otherwise. If $\chi = 1$, then P cannot share a subpath with φ_j, so $\operatorname{supp} \varphi_j \succ \mathrm{e}^{\psi_{j+1}}$ by Lemma 13.2.5, and $\varphi_j + \varepsilon_j\, \mathrm{e}^{\psi_{j+1}}$ is ill-nested. In general, it follows that $c_j := \varphi_j + \chi\, \varepsilon_j\, \mathrm{e}^{\psi_{j+1}}$ is ill-nested. Let Q be a bad path in c_j and set $P' := (P(0), \ldots, P(j-1)) * Q$. Then we may apply Lemma 13.2.6 to j, c_j, and P' in the roles of i, c_i, and P. Since $c_j \neq u_j$, this yields an ill-nested number $c_0 \sqsubset a$: a contradiction.

Therefore the numbers $\varphi_0, \psi_1, \ldots, \varphi_{i-1}, \psi_i$ are well-nested. Since i is bad for (P, a), one of the four cases listed in Definition 13.1.10 must occur. We set

$$d_i := \begin{cases} \varphi_i + \operatorname{sign}(r_i)\, \mathrm{e}^{\psi_{i+1}} & \text{if Definition 13.1.10(4) occurs} \\ \varphi_i + \operatorname{sign}(r_i)\, \mathrm{e}^{\psi_{i+1}}\, (E_{\alpha_i}^{u_{i+1}})^{\iota_i} & \text{otherwise.} \end{cases}$$

By construction, we have $d_i \lesssim u_i$. Furthermore P shares a subpath with d_i, so there exists a bad path Q in d_i. We have $d_i \in \mathbf{No}_{\succ, \alpha_{j-1}}$ by Lemma 13.2.2. If Definition 13.1.10(4) occurs, then we must have $\psi_{i+1} \neq 0$ so d_i is written as in (13.2.7) with d_i in the role of c_i and $\mathfrak{p} = \chi = 1$. Otherwise, d_i is as in (13.2.7) for $\mathfrak{p} = E_{\alpha_i}^{u_{i+1}}$. Setting $P' := (P(0), \ldots, P(i-1)) * Q$, it follows that we may apply Lemma 13.2.6 to d_i and P' in the roles of c_i and P. We conclude that there exists an ill-nested number $d_0 \sqsubset a$: a contradiction. □

Chapter 14
Nested numbers

In the previous chapter, we have examined the nature of infinite paths in surreal number and shown that they are ultimately "well-behaved". In this section, we work in the opposite direction and show how to construct surreal numbers that contain infinite paths of a specified kind. We follow the same method as in [11, Section 8].

Let us briefly outline the main ideas. Our aim is to construct "nested numbers" that correspond to nested expressions like

$$a = \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\sqrt{\log \log \log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{14.0.1}$$

Nested expressions of this kind will be presented through so-called *coding sequences* Σ. Once we have fixed such a coding sequence Σ, numbers a of the form (14.0.1) need to satisfy a sequence of natural inequalities: for any $c \in \mathbb{R}$ with $c > 1$, we require that

$$\begin{array}{rcccl} c^{-1}\sqrt{\omega} & < & a & < & c\sqrt{\omega} \\ \sqrt{\omega} + \mathrm{e}^{c^{-1}\sqrt{\log \omega}} & < & a & < & \sqrt{\omega} + \mathrm{e}^{c\sqrt{\log \omega}} \\ \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{c\sqrt{\log \log \omega}}} & < & a & < & \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{c^{-1}\sqrt{\log \log \omega}}} \\ \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log \log \omega} + c^{-1}\mathrm{e}^{\sqrt{\log \log \log \omega}}}} & < & a & < & \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log \log \omega} + c\,\mathrm{e}^{\sqrt{\log \log \log \omega}}}} \\ & & \vdots & & \end{array}$$

Numbers that satisfy these constraints are said to be *admissible*. Under suitable conditions, the class $\mathbf{Ad}$ of admissible numbers forms a convex surreal substructure. This will be detailed in Section 14.1, where we will also introduce suitable coordinates

$$\begin{array}{rcll} a_{;0} & = & \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\sqrt{\log \log \log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} & = \; a \\ a_{;1} & = & \sqrt{\log \omega} - \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\sqrt{\log \log \log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}} & = \; \log (a_{;0} - \sqrt{\omega}) \\ a_{;2} & = & \sqrt{\log \log \omega} + \mathrm{e}^{\sqrt{\log \log \log \omega} - \mathrm{e}^{\cdot^{\cdot^{\cdot}}}} & = \; \log \left(\sqrt{\log \omega} - a_{;1} \right) \\ & & \vdots & \end{array}$$

for working with numbers in $\mathbf{Ad}$.

The notation (14.0.1) also suggests that each of the numbers $a_{;0} - \sqrt{\omega}$, $\sqrt{\log \omega} - a_{;1}$, ... should be a monomial. An admissible number $a \in \mathbf{Ad}$ is said to be *nested* if this is indeed the case. The main result of this section is Theorem **E**, i.e. that the class $\mathbf{Ne}$ of nested numbers forms a surreal substructure. In other words, the notation (14.0.1) is ambiguous, but can be disambiguated using a single surreal parameter.

14.1 Coding sequences for nested numbers

Let us first define and study the basic properties of sequences of numbers that can occur in nested expansions.

14.1.1 Coding sequences

Definition 14.1.1. *Let* $\Sigma := (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}} \in (\mathbf{No} \times \{-1, 1\} \times \mathbf{No} \times \{-1, 1\} \times \omega^{\mathbf{On}})^{\mathbb{N}}$. *We say that* Σ *is a* **coding sequence** *if for all* $i \in \mathbb{N}$, *we have*

a) $\psi_i \in \mathbf{No}_{\succ}$;

b) $\varphi_{i+1} \in \mathbf{No}_{\succ, \alpha_i} \cup \{0\}$;

c) $(\alpha_i = 1) \Longrightarrow (\psi_i = 0 \wedge (\psi_{i+1} = 0 \Longrightarrow \alpha_{i+1} = 1))$;

d) $(\varphi_{i+1} = \psi_{i+1} = 0) \Longrightarrow (\alpha_i \geqslant \alpha_{i+1} \wedge \varepsilon_{i+1} = \iota_{i+1} = 1)$;

e) $\exists j > i, (\varphi_j \neq 0 \vee \psi_j \neq 0)$.

Taking $\alpha_i = 1$ for all $i \in \mathbb{N}$, we obtain a reformulation of the notion of coding sequences in [11, Section 8.1]. If $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$ is a coding sequence and $k \in \mathbb{N}$, then we write

$$\Sigma_{\nearrow k} := (\varphi_{k+i}, \varepsilon_{k+i}, \psi_{k+i}, \iota_{k+i}, \alpha_{k+i})_{i \in \mathbb{N}},$$

which is also a coding sequence.

Lemma 14.1.2. *Let* P *be an infinite path in a number* $a \in \mathbf{No}$ *without any bad index for* a. *Let* $\varphi_0 := a_{\succ \mathfrak{m}_{P,0}}$ *and* $\varphi_i := (a_{P,i})_{\succ \mathfrak{m}_{P,i}}$ *for all* $i \in \mathbb{N}^{>}$. *Then* $\Sigma_P := (\varphi_i, r_{P,i}, \psi_{P,i+1}, \iota_{P,i}, \alpha_{P,i})_{i \in \mathbb{N}}$ *is a coding sequence.*

Proof. Let $i \in \mathbb{N}$. We have $r_{P,i} \in \{-1, 1\}$ because i is a good index for (P, a). We have $\psi_{P,i+1} \in \mathbf{No}_{\succ}$ and $a_{P,i+1} \in \mathbf{No}_{\succ, \alpha_i}$ by the definition of hyperserial expansions. If $i > 0$ and $\varphi_i \neq 0$, then we have $\varphi_i \in \mathbf{No}^{>, \succ}$ because $a_{P,i} \in \mathbf{No}^{>, \succ}$ by the definition of paths. Lemma 13.2.2 also yields $\varphi_i \in \mathbf{No}_{\succ, \alpha_i}$. This proves the conditions *a)* and *b)* for coding sequences. Assume that $\alpha_i = 1$. Then by the definition of hyperserial expansions, we have $\psi_{P,i+1} = 0$ and $u_{P,i+1} = a_{P,i+1}$ is not tail-atomic. Assume that $\psi_{P,i+2} = 0$. Then $\operatorname{supp} u_{P,i+1} \succ 1$ so $\iota_{P,i+2} = 1$. We have $u_{P,i+1} = \varphi_{i+1} \mathbin{+\!\!\!+} r_{P,i+1}\, \mathfrak{a}$ where $\mathfrak{a} := E_{\alpha_{P,i+1}}^{u_{P,i+2}}$ and $u_{P,i+1}$ is not tail-atomic. This implies that $\mathfrak{a}$ is not log-atomic, so $\alpha_{P,i+1} = 1$. Thus *c)* is valid.

Assume that $\varphi_{i+1} = \psi_{P,i+2} = 0$. Recall that $a_{P,i+1} = r_{P,i+1}\, (E_{\alpha_{P,i+1}}^{u_{P,i+2}})^{\iota_{P,i+1}} = u_{P,i+1} \in \mathbf{No}^{>, \succ}$, so $r_{P,i+1} = \iota_{P,i+1} = 1$. Since $E_{\alpha_{P,i}}^{u_{P,i+1}} \notin \mathbf{Mo}_{\alpha_{P,i}\omega}$, we have $u_{P,i+1} \notin \mathbf{Mo}_{\alpha_{P,i}\omega}$, whence $\alpha_{P,i+1} \leqslant \alpha_{P,i}$. This proves *d)*.

Assume now for contradiction that there is an $i_0 \in \mathbb{N}$ with $\varphi_{P,j} = \psi_{P,j+1} = 0$ for all $j > i_0$. By *d)*, we have $r_{P,j} = \iota_{P,j} = 1$ for all $j > i_0$, and the sequence $(\alpha_{P,j})_{j > i_0}$ is non-increasing, hence eventually constant. Let $i_1 > i_0$ with $\alpha_{P,i_1} = \alpha_{P,j}$ for all $j > i_1$. For $k \in \mathbb{N}$, we have $a_{P,i_1} = E_{\alpha_{P,i_1} k}\, a_{P,i_1+k}$ so $a_{P,i_1} \in \bigcap_{k \in \mathbb{N}} E_{\alpha_{P,i_1} k} \mathbf{Mo}_{\alpha_{P,i_1}} = \mathbf{Mo}_{\alpha_{P,i_1}\omega}$. Therefore $E_{\alpha_{P,i_1}}^{a_{P,i_1+1}}$ is $L_{<\alpha_{P,i_1+1}\omega}$-atomic: a contradiction. We deduce that *e)* holds as well. □

We next fix some notations. For all $i, j \in \mathbb{N}$ with $i \leqslant j$, we define partial functions Φ_i, $\Phi_{i;}$ and $\Phi_{j;i}$ on $\mathbf{No}$ by

$$\begin{aligned} \Phi_i(a) &:= \varphi_i + \varepsilon_i\, \mathrm{e}^{\psi_{i-1}} (E_{\alpha_{i-1}} a)^{\iota_{i-1}}, \\ \Phi_{j;i}(a) &:= (\Phi_i \circ \cdots \circ \Phi_{j-1})(a), \\ \Phi_{i;} &:= \Phi_{i;0}. \end{aligned}$$

The domains of these functions are assumed to be largest for which these expressions make sense. We also write

$$\begin{aligned} \sigma_{i;} \;=\; \sigma_{;i} &:= \prod_{k<i} \varepsilon_k\, \iota_k \\ \sigma_{j;i} \;=\; \sigma_{i;j} &:= \prod_{i \leqslant k < j} \varepsilon_k\, \iota_k \end{aligned}$$

We note that on their respective domains, the functions Φ_i, $\Phi_{i;}$, and $\Phi_{j;i}$ are strictly increasing if $\varepsilon_i\, \iota_i = 1$, $\sigma_{i;} = 1$, and $\sigma_{j;i} = 1$, respectively, and strictly decreasing in the contrary cases. We will write $\Phi_{;i}$ and $\Phi_{i;j}$ for the partial inverses of $\Phi_{i;}$ and $\Phi_{j;i}$. We will also use the abbreviations

$$\begin{aligned} a_{i;} &:= \Phi_{i;}(a) \qquad & a_{j;i} &:= \Phi_{j;i}(a) \\ a_{;i} &:= \Phi_{;i}(a) \qquad & a_{i;j} &:= \Phi_{i;j}(a) \end{aligned}$$

For all $i \in \mathbb{N}$, we set

$$\begin{array}{rcl} L_i^{[1]} & := & (\varphi_i - \sigma_{;i}\,\mathbb{R}^{>} \operatorname{supp} \varphi_i)_{i;} \\ L_i^{[2]} & := & (\varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i - \varepsilon_i \sigma_{;i} \mathbb{R}^{>} \operatorname{supp} \psi_i})_{i;} \\ L_i^{[3]} & := & \begin{cases} \varnothing & \text{if } \varphi_{i+1} = 0 \\ & \text{or } \sigma_{;i+1}\,\varepsilon_{i+1} = -1 \\ (\mathcal{L}_{\alpha_i} \varphi_{i+1})_{i+1;} & \text{otherwise} \end{cases} \\ L_i & := & L_i^{[1]} \cup L_i^{[2]} \cup L_i^{[3]} \\ L & := & \bigcup_{i \in \mathbb{N}} L_i. \end{array} \qquad \begin{array}{rcl} R_i^{[1]} & := & (\varphi_i + \sigma_{;i}\,\mathbb{R}^{>} \operatorname{supp} \varphi_i)_{i;} \\ R_i^{[2]} & := & (\varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i + \varepsilon_i \sigma_{;i} \mathbb{R}^{>} \operatorname{supp} \psi_i})_{i;} \\ R_i^{[3]} & := & \begin{cases} \varnothing & \text{if } \varphi_{i+1} = 0 \\ & \text{or } \sigma_{;i+1}\,\varepsilon_{i+1} = 1 \\ (\mathcal{L}_{\alpha_i} \varphi_{i+1})_{i+1;} & \text{otherwise} \end{cases} \\ R_i & := & R_i^{[1]} \cup R_i^{[2]} \cup R_i^{[3]} \\ R & := & \bigcup_{i \in \mathbb{N}} R_i. \end{array}$$

Note that

$$\begin{array}{rcl} \varphi_i = 0 & \Longleftrightarrow & L_i^{[1]} = R_i^{[1]} = \varnothing \quad \text{and} \\ \psi_i = 0 & \Longleftrightarrow & L_i^{[2]} = R_i^{[2]} = \varnothing. \end{array}$$

The following lemma generalizes [11, Lemma 8.1].

Lemma 14.1.3. *If $a \in (L \,|\, R)$, then $a_{;i}$ is well defined for all $i \in \mathbb{N}$.*

Proof. Let us prove the lemma by induction on i. The result clearly holds for $i = 0$. Assuming that $a_{;i}$ is well defined, let $j > i$ be minimal such that $\varphi_j \neq 0$ or $\psi_j \neq 0$. Note that we have $\alpha_i \geqslant \alpha_{i+1} \geqslant \cdots \geqslant \alpha_j$, so $E_{\alpha_i} \circ E_{\alpha_{i+1}} \circ \cdots \circ E_{\alpha_j} = E_\gamma$ where $\gamma = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. Applying $\Phi_{;i}$ to the inequality

$$L_j < a < R_j\,,$$

we obtain

$$\sigma_{;i}\,(L_j)_{;i} < \sigma_{;i}\,a_{;i} < \sigma_{;i}\,(R_j)_{;i}.$$

Now if $\varphi_j \neq 0$, then

$$\begin{array}{rcl} (L_j)_{;i} & \supseteq & \varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_\gamma(\varphi_j - \sigma_{;j}\,\mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i} \\ (R_j)_{;i} & \supseteq & \varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_\gamma(\varphi_j + \sigma_{;j}\,\mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i}, \end{array}$$

whence

$$\sigma_{;i}\,\mathrm{e}^{\psi_i}\,(E_\gamma(\varphi_j - \sigma_{;j}\,\mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i} < \sigma_{;i}\,\frac{a_{;i} - \varphi_i}{\varepsilon_i} < \sigma_{;i}\,\mathrm{e}^{\psi_i}\,(E_\gamma(\varphi_j + \sigma_{;j}\,\mathbb{R}^{>} \operatorname{supp} \varphi_j))^{\iota_i}.$$

Both in the cases when $\sigma_{;i} = 1$ and when $\sigma_{;i} = -1$, it follows that $((a_{;i} - \varphi_i) / \varepsilon_i \,\mathrm{e}^{\psi_i})^{\iota_i}$ is bounded from below by the hyperexponential E_γ of a number. Thus $a_{;j} = L_\gamma(((a_{;i} - \varphi_i) / (\varepsilon_i \,\mathrm{e}^{\psi_i}))^{\iota_i})$ is well defined and so is each $a_{;k}$ for $i \leqslant k < j$. If $\varphi_j = 0$, then we have $\psi_j \neq 0$ and

$$\begin{array}{rcl} (L_j)_{;i} & \supseteq & \varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_\gamma(\mathrm{e}^{\psi_j - \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}))^{\iota_i}, \\ (R_j)_{;i} & \supseteq & \varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_\gamma(\mathrm{e}^{\psi_j + \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}))^{\iota_i}. \end{array}$$

Hence

$$\varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_\gamma(\mathrm{e}^{\psi_j - \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}))^{\iota_i} < a_{;i} - \varphi_i < \varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_\gamma(\mathrm{e}^{\psi_j + \varepsilon_j \sigma_{;j} \mathbb{R}^{>} \operatorname{supp} \psi_j}))^{\iota_i}$$

Both in the cases when $\varepsilon_i = 1$ and when $\varepsilon_i = -1$, it follows that $((a_{;i} - \varphi_i) / \varepsilon_i \,\mathrm{e}^{\psi_i})^{\iota_i}$ is bounded from below by the hyperexponential E_γ of a number, so $a_{;j}$ is well defined and so is each $a_{;k}$ for $i \leqslant k < j$. □

14.1.2 Admissible sequences

Definition 14.1.4. *Let $\Sigma := (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$ be a coding sequence and let $a \in \mathbf{No}$. We say that a is* Σ**-admissible** *if $a_{;i}$ is well defined for all $i \in \mathbb{N}$ and*

$$\begin{array}{rcl} a_{;i} & = & \varphi_i + \varepsilon_i \,\mathrm{e}^{\psi_i}\,(E_{\alpha_i}\,a_{;i+1})^{\iota_i}, \\ \operatorname{supp} \psi_i & \succ & \log E_{\alpha_i}\,a_{;i+1}, \ \text{ and} \\ \varphi_{i+1} & \lhd & \sharp_{\alpha_i}(a_{;i+1}) \quad \text{if } \varphi_{i+1} \neq 0. \end{array}$$

We say that Σ *is* **admissible** *if there exists a* Σ*-admissible number.*

Note that we do not ask that $\mathrm{e}^{\psi_i}(E_{\alpha_i} a_{;i+1})^{\iota_i}$ be a hyperserial expansion, nor even that $E_{\alpha_i} a_{;i+1}$ be a monomial. For the rest of the section, we fix a coding sequence $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$. We write **Ad** for the class of Σ-admissible numbers. If $a \in \mathbf{Ad}$, then the definition of **Ad** implicitly assumes that $a_{;i}$ is well defined for all $i \in \mathbb{N}$. Note that if Σ is admissible, then so is $\Sigma_{\nearrow k}$ for $k \in \mathbb{N}$. We denote by $\mathbf{Ad}_{\nearrow k}$ the corresponding class of $\Sigma_{\nearrow k}$-admissible numbers.

The main result of this subsection is the following generalization of [11, Proposition 8.2]:

Proposition 14.1.5. *We have* $\mathbf{Ad} = (L \mid R)$.

Proof. Let $a \in \mathbf{Ad} \cup (L \mid R)$ and let $i \in \mathbb{N}$. We have $a_{;i} \in \mathbf{No}^{>,\succ}$. If $\sigma_{;i} = 1$, then $\Phi_{i;}$ is strictly increasing so we have

$$\begin{aligned} L_i^{[1]} < a < R_i^{[1]} &\iff \left(L_i^{[1]}\right)_{;i} < a_{;i} < \left(R_i^{[1]}\right)_{;i} \\ &\iff \varphi_i - \mathbb{R}^{>} \operatorname{supp} \varphi_i < a_{;i} < \varphi_i + \mathbb{R}^{>} \operatorname{supp} \varphi_i \\ &\iff a_{;i} - \varphi_i \prec \operatorname{supp} \varphi_i \\ &\iff \varphi_i \trianglelefteq a_{;i}. \end{aligned}$$

If $\sigma_{;i} = -1$, then $\Phi_{i;}$ is strictly decreasing and likewise we obtain $L_{i;} < a < R_{i;} \iff \varphi_i \trianglelefteq a_{;i}$.

We have $\iota_i \log E_{\alpha_i} a_{;i+1} = \iota_i \left(\log \frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}}\right)$. If $\sigma_{;i} = 1$, then $\Phi_{i;}$ is strictly increasing so we have

$$\begin{aligned} L_i^{[2]} < a < R_i^{[2]} &\iff \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i - \varepsilon_i \mathbb{R}^{>} \operatorname{supp} \psi_i} < a_{;i} < \varphi_i + \varepsilon_i \mathrm{e}^{\psi_i + \varepsilon_i \mathbb{R}^{>} \operatorname{supp} \psi_i} \\ &\iff -\mathbb{R}^{>} \operatorname{supp} \psi_i < \log \frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} < \mathbb{R}^{>} \operatorname{supp} \psi_i \\ &\iff \operatorname{supp} \psi_i \succ \log \frac{a_{;i} - \varphi_i}{\varepsilon_i \mathrm{e}^{\psi_i}} \\ &\iff \log E_{\alpha_i} a_{;i+1} \prec \operatorname{supp} \psi_i. \end{aligned}$$

Likewise, we have $L_i^{[2]} < a < R_i^{[2]} \iff \log E_{\alpha_i} a_{;i+1} \prec \operatorname{supp} \psi_i$ if $\sigma_{;i} = -1$.

Assume that $\varphi_{i+1} \neq 0$ and $\sigma_{;i+1} = 1$. If $\varepsilon_{i+1} = 1$, then we have $a_{;i+1} > \varphi_{i+1}$. Hence

$$\begin{aligned} L_i^{[3]} \cup L_{i+1}^{[1]} < a < R_i^{[3]} \cup R_{i+1}^{[1]} &\iff \mathcal{L}_{\alpha_i} \varphi_{i+1} < a_{;i+1} \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\iff \varphi_{i+1} < \sharp_{\alpha_i}(a_{;i+1}) \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\iff \varphi_{i+1} \triangleleft \sharp_{\alpha_i}(a_{;i+1}). \end{aligned}$$

If $\varepsilon_{i+1} = -1$, then we have $a_{;i+1} > \varphi_{i+1}$, whence

$$\begin{aligned} L_i^{[3]} \cup L_{i+1}^{[1]} < a < R_i^{[3]} \cup R_{i+1}^{[1]} &\iff a_{;i+1} < \mathcal{L}_{\alpha_i} \varphi_{i+1} \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\iff \sharp_{\alpha_i}(a_{;i+1}) < \varphi_{i+1} \wedge \varphi_{i+1} \trianglelefteq a_{;i+1} \\ &\iff \varphi_{i+1} \triangleleft \sharp_{\alpha_i}(a_{;i+1}). \end{aligned}$$

Symmetric arguments apply when $\varphi_{i+1} \neq 0$ and $\sigma_{;i+1} = -1$.

We deduce by definition of **Ad** that $\mathbf{Ad} = \bigcap_{i \in \mathbb{N}} (L_i \mid R_i) = (L \mid R)$. □

As a consequence of this last proposition and [11, Proposition 4.29(a)], the class **Ad** is a surreal substructure if and only if Σ is admissible.

Example 14.1.6. Consider the coding sequence $\Sigma_0 = (\varphi_i, \varepsilon_i, \iota_i, \psi_i, \alpha_i)_{i \in \mathbb{N}}$ where for all $i \in \mathbb{N}$, we have

$$\begin{aligned} \varphi_i &= L_{\omega^{2i}} \omega + L_{\omega^{2i} 2} \omega + L_{\omega^{2i} 3} \omega + \cdots, \\ \varepsilon_i &= 1, \\ \psi_i &= L_{\omega^{2i+1}} \omega + L_{\omega^{2i+1} 2} \omega + L_{\omega^{2i+1} 3} \omega + \cdots, \\ \iota_i &= -1 \quad \text{and} \\ \alpha_i &= \omega^{2i+1}. \end{aligned}$$

We use the notations from Section 14.1. We claim that Σ_0 is admissible. Indeed for $i \in \mathbb{N}$, set

$$a_i := \varphi_0 + \mathrm{e}^{\psi_0} \left(E_{\omega}^{\varphi_1 + \mathrm{e}^{\psi_1} \left(E_{\omega^3}^{\cdot^{\cdot^{\cdot \varphi_i}}} \right)^{-1}} \right)^{-1}.$$

Given $j \in \mathbb{N}$ and $i > j$, we have $L_j < a_i$ and $a_i < R_j$. We deduce that $L < R$, whence Σ_0 is admissible.

Lemma 14.1.7. *Let $a \in \mathbf{Ad}$ and $b \in \mathbf{No}$ be such that $a - \varphi_0$ and $b - \varphi_0$ have the same sign and the same dominant monomial. Then $b \in \mathbf{Ad}$.*

Proof. For $x, y \in \mathbf{No}^{\neq}$, we write $x \equiv y$ if $x \asymp y$ and x and y have the same sign. Let us prove by induction on $i \in \mathbb{N}$ that $b_{;i}$ is defined and that $a_{;i} - \varphi_i \equiv b_{;i} - \varphi_i$. Since this implies that $\varphi_i \trianglelefteq b_{;i}$, that $\log \frac{b_{;i} - \varphi_i}{\varepsilon_i \, \mathrm{e}^{\psi_i}} \prec \operatorname{supp} \psi_i$, and that $\varphi_i \lhd \sharp_{\alpha_{i-1}}(b_{;i})$ if $i > 0$, this will yield $b \in \mathbf{Ad}$.

The result follows from our hypothesis if $i = 0$. Assume now that $a_{;i} - \varphi_i \equiv b_{;i} - \varphi_i$ and let us prove that $a_{;i+1} - \varphi_{i+1} \equiv b_{;i+1} - \varphi_{i+1}$. Let

$$c_i := \left(\frac{b_{;i} - \varphi_i}{\varepsilon_i \, \mathrm{e}^{\psi_i}} \right).$$

We have $c_i \equiv \left(\frac{a_{;i} - \varphi_i}{\varepsilon_i \, \mathrm{e}^{\psi_i}} \right)^{\iota_i} = E_{\alpha_i} a_{;i+1} \in \mathbf{No}^{>, \succ}$, so $b_{;i+1} = L_{\alpha_i}(c_i)$ is defined. Moreover $c_i \in \mathcal{E}_{\alpha_i}[E_{\alpha_i} a_{;i+1}]$ so $b_{;i+1} \in \mathcal{L}_{\alpha_i}[a_{;i+1}]$. Since $\varphi_{i+1} \lhd \sharp_{\alpha_i}(a_{;i+1}) = \sharp_{\alpha_i}(b_{;i+1})$, we deduce that $b_{;i+1} - \varphi_{i+1} \sim a_{;i+1} - \varphi_{i+1}$, whence in particular $b_{;i+1} - \varphi_{i+1} \equiv a_{;i+1} - \varphi_{i+1}$. This concludes the proof. □

Corollary 14.1.8. *We have $\mathbf{Ad}_{\nearrow 1} = \mathcal{L}_{\alpha_0}[\mathbf{Ad}_{\nearrow 1}]$.*

Proof. For $b \in \mathbf{Ad}_{\nearrow 1}$, and $c \in \mathcal{L}_{\alpha_0}[b]$, we have $\varphi_1 \lhd \sharp_{\alpha_0}(b) = \sharp_{\alpha_0}(c)$ so $c - \varphi_1 \sim b - \varphi_1$. We conclude with the previous lemma. □

Lemma 14.1.9. *For $a, b \in \mathbf{Ad}$ and $i \in \mathbb{N}^{>}$, we have $L_{\alpha_{i-1}} a_{;i} < \mathcal{E}_{\alpha_{i-1}} b_{;i}$.*

Proof. Let $j > i$ be minimal with $\varphi_j \neq 0$ or $\psi_j \neq 0$. We thus have $a_{;j}, b_{;j} \in \mathcal{P}[\varphi_j \mathbin{+\!\!\!+} \varepsilon_j \, \mathrm{e}^{\psi_j}]$ so $\log a_{;j} \prec b_{;j}$. We have $a_{;i} = E_{\alpha_i + \cdots + \alpha_{j-1}} a_{;j}$ and $b_{;i} = E_{\alpha_i + \cdots + \alpha_{j-1}} b_{;j}$ where $\alpha_i \geqslant \cdots \geqslant \alpha_j \geqslant 1$. We deduce by induction using Lemma 13.2.3 that $L_{\alpha_{i-1}} a_{;i} < \mathcal{E}_{\alpha_{i-1}} b_{;i}$. □

14.1.3 Nested sequences

In this subsection, we assume that Σ is admissible. For $k \in \mathbb{N}$ we say that a $\Sigma_{\nearrow k}$-admissible number a is $\Sigma_{\nearrow k}$*-nested* if we have $E_{\alpha_{k+i}} a_{k;i+1} \in \mathbf{Mo}_{\alpha_{k+i}} \setminus L_{<\alpha_{k+i}} \mathbf{Mo}_{\alpha_{k+i} \omega}$ for all $i \in \mathbb{N}$. We write $\mathbf{Ne}_{\nearrow k}$ for the class of $\Sigma_{\nearrow k}$-nested numbers. For $k = 0$ we simply say that a is Σ-nested and we write $\mathbf{Ne} := \mathbf{Ne}_{\nearrow 0}$.

Definition 14.1.10. *We say that Σ is* **nested** *if for all $k \in \mathbb{N}$, we have*

$$\mathbf{Ad}_{\nearrow k} = \varphi_k + \varepsilon_k \, \mathrm{e}^{\psi_k} \, (E_{\alpha_k} \mathbf{Ad}_{\nearrow k+1})^{\iota_k}.$$

Note that the inclusion $\mathbf{Ad}_{\nearrow k} \subseteq \varphi_k + \varepsilon_k \, \mathrm{e}^{\psi_k} \, (E_{\alpha_k} \mathbf{Ad}_{\nearrow k+1})^{\iota_k}$ always holds. In [11, Section 8.4], we gave examples of nested and admissible non-nested sequences in the case of transseries, i.e. with $\alpha_i = 1$ for all $i \in \mathbb{N}$. We next give an example in the hyperserial case.

Example 14.1.11. We claim that the sequence Σ_0 from Example 14.1.6 is nested. Indeed, let $k \in \mathbb{N}$ and $a \in \mathbf{Ad}_{\nearrow k+1}$. We have $a = \varphi_{k+1} \mathbin{+\!\!\!+} \mathrm{e}^{\psi_{k+1}} \, (E_{\omega^{2k+3}} b)^{-1}$ for a certain $b \in \mathbf{No}^{>, \succ}$ with $b \asymp L_{\omega^{2k+4}} \omega$. Let us check that the conditions of Definition 14.1.4 are satisfied for $c := \varphi_k + \mathrm{e}^{\psi_k} \, (E_{\omega^{2k+1}} a)^{-1}$.

First let $\mathfrak{m} \in \operatorname{supp} \psi_k$. We want to prove that $\mathfrak{m} \succ \log E_{\omega^{2k+1}} a$. We have $\mathfrak{m} = L_{\omega^{2k+1} n} \, \omega$ for a certain $n \in \mathbb{N}^{>}$. Now $a < 2 \, L_{\omega^{2k+2}} \omega$, so $\log E_{\omega^{2k+1}} a \prec E_{\omega^{2k+1} 2}^{L_{\omega^{2k+2}} \omega} = L_{\omega^{2k+2}}(\omega + 2) \prec \mathfrak{m}$.

Secondly, let $\mathfrak{n} \in \operatorname{supp} \varphi_k$. We want to prove that $\mathfrak{n} \succ \mathrm{e}^{\psi_k} \, (E_{\omega^{2k+1}} a)^{-1}$. We have $\mathfrak{n} = L_{\omega^{2k} n} \, \omega$ for a certain $n \in \mathbb{N}^{>}$. Then $\mathrm{e}^{\psi_k} \, (E_{\omega^{2k+1}} a)^{-1} \prec \mathrm{e}^{2 \psi_k}$ by the previous paragraph. Now $2 \, \psi_k + \mathbb{N} < 3 \, L_{\omega^{2k+1}} \omega$ so $\mathrm{e}^{2 \psi_k} \prec \mathrm{e}^{3 L_{\omega^{2k+1}} \omega} \prec \mathfrak{n}$.

Finally, we claim that $\varphi_{k+1} \lhd \sharp_{\omega^{2k+1}}(a)$. This is immediate since the dominant term τ of $\mathrm{e}^{\psi_{k+1}} (E_{\omega^{2k+3}} b)^{-1}$ is positive infinite, so $\varphi_{k+1} \lhd \varphi_{k+1} + \tau \trianglelefteq \sharp_{\omega^{2k+1}}(a)$. Therefore Σ_0 is nested.

A crucial feature of nested sequences is that they are sufficient to describe nested expansions. This is the content of Theorem 14.1.15 below.

Lemma 14.1.12. *Let $b \in \mathbf{Ad}_{\nearrow 1}$. If $\alpha_0 > 1$, or $\alpha_0 = 1$ and $b_\succ$ is not tail-atomic, then the hyperserial expansion of $E_{\alpha_0} \sharp_{\alpha_0}(b)$ is*

$$E_{\alpha_0} \sharp_{\alpha_0}(b) = E_{\alpha_0}^{\sharp_{\alpha_0}(b)}$$

If $\alpha_0 = 1$, $b_\succ = \psi + \iota\, \mathfrak{b}$ is tail-atomic, and $\mathrm{e}^{\mathfrak{b}} = L_\beta E_\alpha^u$ is a hyperserial expansion, then $\psi \in \mathbf{Ad}_{\nearrow 1}$ and the hyperserial expansion of $\exp b_\succ$ is

$$\exp b_\succ = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota.$$

Proof. Recall that $\sharp_1(b) = b_\succ$. By Corollary 14.1.8, we have $\sharp_{\alpha_0}(b) \in \mathbf{Ad}_{\nearrow 1},$. So we may assume without loss of generality that $b = \sharp_{\alpha_0}(b)$.

We claim that $E_{\alpha_0}^b \in \mathbf{Mo}_{\alpha_0} \setminus L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$. Assume for contradiction that $E_{\alpha_0}^b \in L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$ and write $E_{\alpha_0}^b = L_\gamma \mathfrak{a}$ accordingly. Then Corollary 13.1.6 implies that $\gamma = 0$, in which case we define $n := 0$, or $\alpha_0 = \omega^{\mu+1}$ for some ordinal μ and $\gamma = (\alpha_0)_{/\omega} n$ for some $n \in \mathbb{N}^>$. Therefore $E_{\alpha_0}^{b+n} \in \mathbf{Mo}_{\alpha_0 \omega}$, so $b + n \in \mathbf{Mo}_{\alpha_0 \omega}$. This implies that

$$b = (b + n) + (-n).$$

Recall that $\varphi_1 \lhd b$. Assume that $n = 0$, so $\varphi_1 = 0$. Since b is log-atomic, we also have $\psi_1 = 0$. Let $j > 1$ be minimal with $\varphi_j \neq 0$ or $\psi_j \neq 0$. We have $\alpha_1 \geqslant \cdots \geqslant \alpha_{j-1}$ and $b_{1;j} = L_{\alpha_1 + \cdots + \alpha_{j-1}} b \in \mathbf{Mo}_{\alpha_{j-1} \omega}$. In particular, the number $b_{1;j}$ is log-atomic. If $\varphi_j \neq 0$, this contradicts the fact that $\varphi_j \lhd b_{1;j}$. If $\psi_j \neq 0$, then $\operatorname{supp} \psi_j \succ \log ((b_{1;j} \mathrm{e}^{-\psi_j})^{\iota_j})$ implies

$$\log b_{1;j} = \psi_j + \log ((b_{1;j} \mathrm{e}^{-\psi_j})^{\iota_j}).$$

But then $\log b_{1;j}$ is not a monomial: a contradiction. Assume now that $n > 0$. So $\varphi_1 = b + n$ and $b = \varphi_1 + (-n)$. But then $b_{1;2}$ is not defined: a contradiction. We conclude that $E_{\alpha_0}^b \notin L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$.

If $\alpha_0 > 1$, or if $\alpha_0 = 1$ and b is not tail-atomic, then our claim yields the result. Assume now that $\alpha_0 = 1$ and that $b = \psi + \iota\, \mathfrak{b}$ is tail-atomic where $\iota \in \{-1, 1\}$, $\psi \in \mathbf{No}_\succ$, and $\mathrm{e}^{\mathfrak{b}} = L_\beta E_\alpha^u \in \mathbf{Mo}_\omega$ is a hyperserial expansion. Then the hyperserial expansion of $\exp b$ is $\exp b = \mathrm{e}^\psi (L_\beta E_\alpha^u)^\iota$.

We next show that $\psi \in \mathbf{Ad}_{\nearrow 1}$. If $\mathfrak{b} \notin \mathrm{e}^{\psi_1} (E_{\alpha_1} \mathbf{Ad}_{\nearrow 2})^{\iota_1}$, then $\varphi_1 \lhd \psi$, and we conclude with Lemma 14.1.7 that $\psi \in \mathbf{Ad}_{\nearrow 1}$. Assume for contradiction that $\mathfrak{b} \in \mathrm{e}^{\psi_1} (E_{\alpha_1} \mathbf{Ad}_{\nearrow 2})^{\iota_1}$. Since $\mathfrak{b}$ is log-atomic, we must have $\psi_1 = 0$. By the definition of coding sequences, this implies that $\iota_1 = 1$ and $\alpha_1 = 1$. So $b = \varphi_1 + \varepsilon_1 \exp(b_{1;2})$, whence $\psi = \varphi_1$, $\iota = \varepsilon_1$, and $\mathfrak{b} = \exp(b_{1;2})$. In particular the number $b_{1;2}$ is log-atomic, hence tail-atomic. Since $b_{1;2} \in \mathbf{Ad}_{\nearrow 2}$, the claim in the second paragraph of the proof, applied to $\Sigma_{\nearrow 1}$, gives $E_1^{b_{1;2}} \notin \mathbf{Mo}_\omega$. But then also $\mathfrak{b} \notin \mathbf{Mo}_\omega$: a contradiction. □

We pursue with two auxiliary results that will be used order to construct a infinite path required in the proof of Theorem 14.1.15 below.

Lemma 14.1.13. *For $a \in \mathbf{Ad}$, there is a finite path P in a with $u_{P,|P|} \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$ or $\psi_{P,|P|} \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$.*

Proof. By Lemma 13.1.16, it is enough to find such a path in $E_{\alpha_0} a_{;1}$. Write $\alpha_0 =: \omega^\mu$. Assume first that $\mu = 0$, so $\alpha_0 = 1$ and $\psi_0 = 0$. If $(a_{;1})_\succ$ is not tail-atomic, then the hyperserial expansion of $\exp (a_{;1})_\succ$ is $\exp (a_{;1})_\succ = E_1^{(a_{;1})_\succ}$ and $r\, E_1^{(a_{;1})_\succ}$ is the dominant term of $\exp a_{;1}$ for some $r \in \mathbb{R}^{\neq}$. Then the path P with $|P| = 1$ and $\tau_{P,0} := r\, E_1^{(a_{;1})_\succ}$ satisfies $u_{P,|P|} = (a_{;1})_\succ \in \mathbf{Ad}_{\nearrow 1}$. If $(a_{;1})_\succ$ is tail-atomic, then there exist $\psi \in \mathbf{Ad}_{\nearrow 1}$, $\iota \in \{-1, 1\}$ and $\mathfrak{a} \in \mathbf{Mo}_\omega$ such that the hyperserial expansion of $\exp (a_{;1})_\succ$ is $\exp (a_{;1})_\succ = \mathrm{e}^\psi \mathfrak{a}^\iota$. Let $r\, \mathrm{e}^\psi \mathfrak{a}^\iota$ be a term in $\exp a_{;1}$ with $r \in \mathbb{R}^{\neq}$. Then the path P with $|P| = 1$ and $P(0) := r\, \mathrm{e}^\psi \mathfrak{a}^\iota$ satisfies $\psi_{P,|P|} = \psi \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$.

Assume now that $\mu > 0$. We recall that there are an ordinal $\lambda < \alpha_0$ and a number δ with

$$E_{\alpha_0} a_{;1} = E_\lambda (L_\lambda E_{\alpha_0}^{\sharp_{\alpha_0}(a_{;1})} + \delta).$$

If μ is a limit ordinal, then by Lemma 14.1.12, we have a hyperserial expansion $\mathfrak{m} := L_\lambda E_{\alpha_0}^{\sharp_{\alpha_0}(a_{;1})}$. Let $\tau \in \operatorname{term} \sharp_{\alpha_0}(a_{;1})$ and set $Q(0) = \mathfrak{m}$ and $Q(1) := \tau$, so that Q is a path in $\mathfrak{m}$. By Lemma 13.1.15, there is a subpath in $E_{\alpha_0} a_{;1}$, hence also a path P in $E_{\alpha_0} a_{;1}$, with $\tau_{P,|P|-1} = \mathfrak{m}$. So $u_{P,|P|} = \sharp_{\alpha_0}(a_{;1}) \in \mathbf{Ad}_{\nearrow 1}$. If μ is a successor ordinal, then we may choose $\lambda = \omega^{\mu_-} n$ for a certain $n \in \mathbb{N}$. By Lemma 14.1.12, we have a hyperserial expansion $\mathfrak{m} := E_{\alpha_0}^{\sharp_{\alpha_0}(a_{;1}) - n}$. As in the previous case, there is a path P in $E_{\alpha_0} a_{;1}$ with $\tau_{P,|P|} = \mathfrak{m}$, whence $u_{P,|P|} = \sharp_{\alpha_0}(a_{;1}) - n \in \mathbf{Ad}_{\nearrow 1} - \mathbb{N}$. □

Corollary 14.1.14. *For $a \in \mathbf{Ad}$ and $k \in \mathbb{N}$, there is a finite path P in a with $|P| \geqslant k$ and $u_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$ or $\psi_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$.*

Proof. This is immediate if $k = 0$. Assume that the result holds at k and pick a corresponding path P with $u_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$ (resp. $\psi_{P,|P|} \in \mathbf{Ad}_{\nearrow k} - \mathbb{N}$). Note that the dominant term τ of $u_{P,|P|} - \varphi_k$ (resp. $\psi_{P,|P|} - \varphi_k$) lies in $\varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_k} \mathbf{Ad}_{\nearrow k+1})^{\iota_k}$ by Lemma 14.1.7. Moreover τ is a term of $u_{P,|P|}$ (resp. $\psi_{P,|P|}$). By the previous lemma, there is a path Q in τ with $u_{Q,|Q|} \in \mathbf{Ad}_{\nearrow k+1} - \mathbb{N}$ or $\psi_{Q,|Q|} \in \mathbf{Ad}_{\nearrow k+1} - \mathbb{N}$, so $(P(0), \ldots, P(|P|-1), Q(0)) * Q$ satisfies the conditions. □

Theorem 14.1.15. *There is a $k \in \mathbb{N}$ such that $\Sigma_{\nearrow k}$ is nested.*

Proof. Assume for contradiction that this is not the case. This means that the set Δ of indexes $d \in \mathbb{N}$ such that we do not have $\mathbf{Ad}_{\nearrow d} = \varphi_d + \varepsilon_d \mathrm{e}^{\psi_d} (E_{\alpha_d} \mathbf{Ad}_{\nearrow d+1})^{\iota_d}$ is infinite. We write $\Delta = \{d_i : i \in \mathbb{N}\}$ where $d_0 < d_1 < \cdots$. Fix $a \in \mathbf{Ad}$ and let $d := d_i \in \Delta$. Let $u \in \mathbf{Ad}_{\nearrow d+1}$ such that

$$\varphi_d + \varepsilon_d \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d} \notin \mathbf{Ad}_{\nearrow d}, \tag{14.1.1}$$

let $n \in \mathbb{N}$ and let P be any finite path with

$$u_{P,|P|} = \varphi_d + \varepsilon_d \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d} - n.$$

We claim that we can extend P to a path Q with $|Q| > |P|$, $u_{Q,|Q|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$ and such that $|P|$ is a bad index in Q. Indeed, in view of Definition 14.1.4 for $\mathbf{Ad}_{\nearrow d}$, the relation (14.1.1) translates into the following three possibilities:

- There is an $\mathfrak{n} \in \operatorname{supp} \psi_d$ with $\mathfrak{n} \preccurlyeq \log E_{\alpha_d} u$. We then have $\log E_{\alpha_d} a_{;d+1} \prec \mathfrak{n} \preccurlyeq \log E_{\alpha_d} u$. By Lemma 14.1.7 and the convexity of $\mathbf{Ad}_{\nearrow d+1}$, we deduce that $\iota_d (\psi_d)_{\mathfrak{n}} \mathfrak{n}$ lies in the class $\iota_d \log E_{\alpha_d} \mathbf{Ad}_{\nearrow d+1}$, so $\mathrm{e}^{(\psi_d)_{\mathfrak{n}} \mathfrak{n}} \in (E_{\alpha_d} \mathbf{Ad}_{\nearrow d+1})^{\iota_d}$. By Corollary 14.1.14 for the admissible sequence starting with $(0, 1, 0, \iota_d, \alpha_d)$ and followed by $\Sigma_{\nearrow d+1}$, there is a finite path R_0 in $\mathrm{e}^{(\psi_d)_{\mathfrak{n}} \mathfrak{n}}$ with $|R_0| \geqslant d_{i+3} - d > 2$ and $u_{R_0,|R_0|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$. Taking the logarithm and using Lemma 13.1.14, we obtain a finite path R_1 in $(\psi_d)_{\mathfrak{n}} \mathfrak{n}$, hence in ψ_d, with $|R_1| \geqslant 2$ and $u_{R_1,|R_1|} = u_{R_0,|R_0|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$. Write $(E_{\alpha_d} a_{;d+1})^{\iota_d} = r\,\mathfrak{m} + \rho$ where $r \in \mathbb{R}^{\neq}$ and $\mathfrak{m} \in \mathbf{Mo}^{\neq}$. Then $\log \mathfrak{m} \asymp E_{\alpha_d} a_{;d+1} \prec \operatorname{supp} \psi_d$, so the hyperserial expansion of $\mathrm{e}^{\psi_d} \mathfrak{m}$ has one of the following forms

$$\begin{aligned} \mathrm{e}^{\psi_d} \mathfrak{m} &= \mathrm{e}^{\psi_d + \delta} (L_\beta E_\alpha^u)^\iota \quad \text{or} \\ \mathrm{e}^{\psi_d} \mathfrak{m} &= (E_1^{\psi_d + \delta})^\iota \end{aligned}$$

  where $(L_\beta E_\alpha^u)^\iota$ is a hyperserial expansion and δ is purely large. In both cases, the path $R = (\varepsilon_d r \mathrm{e}^{\psi_d} \mathfrak{m}) * R_1$ is a finite path R in $\varepsilon_d \mathrm{e}^{\psi_d} (E_{\alpha_d} a_{;d+1})^{\iota_d}$ with $u_{R,|R|} = u_{R_1,|R_1|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$. Since $R(0)$ is a term in $u_{P,|P|}$, we may consider the path $Q := P * R$. Moreover, since $\tau_{Q,|P|}$ is a term in $\psi_d = \psi_{Q,|P|}$, the index $|P|$ is bad for Q.

- We have $\log E_{\alpha_d} u \prec \operatorname{supp} \psi_d$, but there is an $\mathfrak{m} \in \operatorname{supp} \varphi_d$ with $\mathfrak{m} \preccurlyeq \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d}$. We then have $\mathrm{e}^{\psi_d} (E_{\alpha_d} a_{;d+1})^{\iota_d} \prec \varphi_{\mathfrak{m}} \mathfrak{m} \preccurlyeq \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d}$. By Lemma 14.1.7 and the convexity of $\mathbf{Ad}_{\nearrow d+1}$, we deduce that $(\varphi_d)_{\mathfrak{m}} \mathfrak{m}$ lies in $\mathrm{e}^{\psi_d} (E_{\alpha_d} \mathbf{Ad}_{\nearrow d+1})^{\iota_d}$. So $L_{\alpha_d}((\mathrm{e}^{-\psi_d} (\varphi_d)_{\mathfrak{m}} \mathfrak{m})^{\iota_d})$ lies in $\mathbf{Ad}_{\nearrow d+1}$. But then also $v := \sharp_{\alpha_d}(L_{\alpha_d}((\mathrm{e}^{-\psi_d} (\varphi_d)_{\mathfrak{m}} \mathfrak{m})^{\iota_d}))$ lies in $\mathbf{Ad}_{\nearrow d+1}$ by Corollary 14.1.8. By Corollary 14.1.14, there is a finite path R_0 in v with $|R_0| > 2$ and $u_{R_0,|R_0|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$. Applying Lemma 13.1.15 to this path R_0 in v, we obtain is a finite path R_1 in $(\mathrm{e}^{-\psi_d} (\varphi_d)_{\mathfrak{m}} \mathfrak{m})^{\iota_d}$ with $u_{R_1,|R_1|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$. Since $(\varphi_d)_{\mathfrak{m}} \mathfrak{m} \in \mathrm{e}^{\psi_d} (E_{\alpha_d} \mathbf{Ad}_{\nearrow d+1})^{\iota_d}$, we have $\operatorname{supp} \psi_d \succ \mathrm{e}^{-\psi_d} (\varphi_d)_{\mathfrak{m}} \mathfrak{m}$. So Lemma 13.1.16 implies that there is a finite path R in $(\varphi_d)_{\mathfrak{m}} \mathfrak{m}$, hence in φ_d, with $u_{R,|R|} \in \mathbf{Ad}_{\nearrow d_{i+3}} - \mathbb{N}$. We have $R(0) \in \operatorname{term} \varphi_d \setminus \mathbb{R} \subseteq \operatorname{term} u_{P,|P|}$, so $Q := P * R$ is a path. Write τ for the dominant term of $\varepsilon_d \mathrm{e}^{\psi_d} (E_{\alpha_d} u')^{\iota_d}$. The index $|P|$ is a bad in Q because $\tau_{Q,|P|}$ and τ both lie in $\operatorname{term} a_{Q,|P|}$, and $\tau_{Q,|P|} \succ \tau$.

- We have $\log E_{\alpha_d} u \prec \operatorname{supp} \psi_d$ and $\operatorname{supp} \varphi_d \succ \mathrm{e}^{\psi_d} (E_{\alpha_d} u)^{\iota_d}$, but $\varphi_{d+1} = \sharp_{\alpha_d}(\varphi_{d+1} \mathbin{+\!\!\!+} \varepsilon_{d+1} \mathrm{e}^{\psi_{d+1}} (E_{\alpha_{d+1}} u)^{\iota_{d+1}})$. By the definition of α_d-truncated numbers, there is a $\beta < \alpha_d$ with

$$\mathrm{e}^{\psi_{d+1}} (E_{\alpha_{d+1}} u)^{\iota_{d+1}} \prec \frac{1}{L_\beta E_{\alpha_d}^{\varphi_{d+1}}} \prec \mathrm{e}^{\psi_{d+1}} (E_{\alpha_{d+1}} a_{;d+2})^{\iota_{d+1}}.$$

  Using the convexity of $\mathbf{Ad}_{\nearrow d+2}$, it follows that $L_\beta E_{\alpha_d}^{\varphi_{d+1}} \in \mathrm{e}^{-\psi_{d+1}} (E_{\alpha_{d+1}} \mathbf{Ad}_{\nearrow d+2})^{-\iota_{d+1}}$. By similar arguments as above (using Corollary 14.1.14 and Lemmas 13.1.15 and 13.1.14), we deduce that there is a finite path R in φ_{d+1} with $u_{R,|R|} \in \mathbf{Ad}_{\nearrow d_{i+2}} - \mathbb{N}$. As in the previous case $Q := P * R$ is a path and $|P|$ is a bad index in Q.

Consider a $b \in \mathbf{Ad}_{\nearrow d_1 - 1}$ and the path $P_0 := (\tau_{a - \varphi_{d_0}})$ in b. So P is a finite path with $u_{P_0,|P_0|} \in \mathbf{Ad}_{\nearrow d_1}$. Thus there exists a path P_1 which extends P_0 with $u_{P_1,|P_1|} \in \mathbf{Ad}_{\nearrow d_3}$, where $|P_0|$ is a bad index in P. Repeating this process iteratively for $i = 2, 3, \ldots$, we construct a path P_i that extends P_{i-1} and such that $u_{P_i,|P_i|} \in \mathbf{Ad}_{\nearrow d_{2i+1}}$ and such that $|P_{i-1}|$ is a bad index in P_i. At the limit, this yields an infinite path Q in a that extends each of the paths P_i. This path Q has a cofinal set of bad indexes, which contradicts Theorem 13.2.7. We conclude that there is a $k \in \mathbb{N}$ such that $\Sigma_{\nearrow k}$ is nested. □

14.2 Existence of nested numbers

We finally show that nested sequences enjoy proper classes of corresponding nested numbers.

14.2.1 Preparation lemmas

Lemma 14.2.1. *Assume that Σ is nested. Then we have $\mathbf{Ad} = \varphi_0 + \varepsilon_0 \mathrm{e}^{\psi_0} (\mathcal{E}_{\alpha_0}[E_{\alpha_0} \mathbf{Ad}_{\nearrow 1}])^{\iota_0}$.*

Proof. Note that $\mathcal{E}_{\alpha_0}[E_{\alpha_0} \mathbf{Ad}_{\nearrow 1}] = E_{\alpha_0} \mathcal{L}_{\alpha_0}[\mathbf{Ad}_{\nearrow 1}]$. The result thus follows from Corollary 14.1.8 and the assumption that Σ is nested. □

Lemma 14.2.2. *Assume that Σ is nested. Let $k \in \mathbb{N}$, $a \in \mathbf{Ad}$ and $c_k \in \mathbf{No}$ with*

$$c_k = \varphi_k \mathbin{+\!\!\!+} \varepsilon_k \mathrm{e}^{\psi_k} \mathfrak{p}^{\iota_k} \tag{14.2.1}$$

for a certain $\mathfrak{p} \in \mathbf{Mo}^{\succ}$ with $\mathfrak{p} \sqsubseteq E_{\alpha_j} a_{;k+1}$ and $\mathfrak{p} \in \mathcal{E}_\omega[E_{\alpha_k} a_{;k+1}]$ whenever $\psi_k = 0$. If $c_k \in \mathbf{Ad}_{\nearrow k}$, then we have

$$(c_k)_{k;} \sqsubseteq a.$$

Proof. The proof is similar to the proof of Lemma 13.2.6. We have $a_{;k} = \varphi_k \mathbin{+\!\!\!+} \varepsilon_k \mathrm{e}^{\psi_k} (E_{\alpha_k} a_{;k+1})^{\iota_k}$ and we must have $\operatorname{supp} \psi_k \succ \log \mathfrak{p}$ since $c_k = \varphi_k \mathbin{+\!\!\!+} \varepsilon_k \mathrm{e}^{\psi_k} \mathfrak{p}^{\iota_k} \in \mathbf{Ad}_{\nearrow k}$. If follows from the deconstruction lemmas in Section 13.1.3 that $c_k \sqsubseteq a_{;k}$. This proves the result in the case when $k = 0$.

Now assume that $k > 0$. Setting $c_{k-p} := \Phi_{k-p;k}(c_k)$, let us prove by induction on $p \leqslant k$ that

$$\begin{aligned} c_{k-p} &\in \mathbf{Ad}_{\nearrow k-p} \\ c_{k-p} &\in \mathbf{No}_{\succ,\alpha_{k-p-1}} \\ c_{k-p} &\sqsubseteq a_{;k-p}. \end{aligned}$$

For $p = k$, the last relation yields the desired result.

If $p = 0$, then we have $c_k \in \mathbf{Ad}_{\nearrow k}$ by assumption and we have shown above that $c_k \sqsubseteq a_{;k}$. We have $\varphi_k \lhd \sharp_{\alpha_{k-1}}(c_k)$ and $\mathrm{e}^{\psi_k} \mathfrak{p}^{\iota_j}$ is a monomial, so (14.2.1) yields $c_k = \sharp_{\alpha_{k-1}}(c_k) \in \mathbf{No}_{\succ,\alpha_{k-1}}$. This deals with the case $p = 0$. In addition, we have $c_k > 0$ because $k > 0$ and $c_k \in \mathbf{Ad}_{\nearrow k}$. Let us show that

$$\log c_k \prec a_{;k}. \tag{14.2.2}$$

If $\varphi_k \neq 0$, then this follows from the facts that $\varphi_k \lhd a_{;k}$ and $\varphi_k \lhd c_k$. If $\varphi_k = 0$ and $\psi_k \neq 0$, then $\log (c_k / \varepsilon_k) \sim \psi_k \sim \log (a_{;k} / \varepsilon_k) \prec a_{;k}$. If $\varphi_k = \psi_k = 0$, then $a_{;k} = E_{\alpha_k} a_{;k+1}$ and $c_k = \mathfrak{p} \in \mathcal{E}_\omega[a_{;k}]$, so $\log c_k \prec a_{;k}$.

Assume now that $0<p\leqslant k$ and that the induction hypothesis holds for all smaller p. We have

$$c_{k-p}=\Phi_{k-p}(c_{k-p+1})=\varphi_k+\varepsilon_k\,\mathrm{e}^{\psi_k}\,(E_{\alpha_{k-p}}^{c_{k-p+1}})^{\iota_k} \tag{14.2.3}$$

Since Σ is nested, we immediately obtain $\varphi_{k-p}\lhd\sharp_{\alpha_{k-p-1}}(c_{k-p})$, whence $c_{k-p}\in\mathbf{No}_{\succ,\alpha_{k-p-1}}$ as above. Since $c_{k-p-1}\in\mathbf{Ad}_{\nearrow(k-p-1)}$ and Σ is nested, we have $c_{k-p}\in\mathbf{Ad}_{\nearrow(k-p)}$. Using (14.2.3), (14.2.2), and the decomposition lemmas, we observe that the relation $c_{k-p}\sqsubseteq a_{;k-p}$ is equivalent to

$$E_{\alpha_{k-p}}^{c_{k-p+1}}\sqsubseteq E_{\alpha_{k-p}}\,a_{;k-p+1}. \tag{14.2.4}$$

We have $c_{k-p+1}\sqsubseteq a_{;k-p+1}$, so $c_{k-p+1}\sqsubseteq\sharp_{\alpha_{k-p}}(a_{;k-p+1})$. Note that

$$E_{\alpha_{k-p}}^{\sharp_{\alpha_{k-p}}(a_{;k-p+1})}=\mathfrak{d}_{\alpha_{k-p}}(E_{\alpha_{k-p}}\,a_{;k-p+1})\sqsubseteq E_{\alpha_{k-p}}\,a_{;k-p+1}.$$

So it is enough, in order to derive (14.2.4), to prove that $E_{\alpha_{k-p}}^{c_{k-p+1}}\sqsubseteq E_{\alpha_{k-p}}^{\sharp_{\alpha_{k-p}}(a_{;k-p+1})}$. Now

$$L_{\alpha_{k-p}}\,c_{k-p+1}<\mathcal{E}_{\alpha_{k-p}}\,\sharp_{\alpha_{k-p}}(a_{;k-p+1})$$

by Lemma 14.1.9, whence $E_{\alpha_{k-p}}^{c_{k-p+1}}\sqsubseteq E_{\alpha_{k-p}}^{\sharp_{\alpha_{k-p}}(a_{;k-p+1})}$ by Lemma 13.1.25. □

14.2.2 Surreal substructures of nested numbers

For $i\in\mathbb{N}$, $g\in\mathcal{E}_{\alpha_i}$ and $a\in\mathbf{Ad}$, we have $\varphi_i+\varepsilon_i\,\mathrm{e}^{\psi_i}\,g(E_{\alpha_i}\,a_{;i+1})^{\iota_i}\in\mathbf{Ad}_{\nearrow i}$ by Lemma 14.2.1. We may thus consider the strictly increasing bijection

$$\Psi_{i,g}:=\mathbf{Ad}\longrightarrow\mathbf{Ad};a\longmapsto(\varphi_i+\varepsilon_i\,\mathrm{e}^{\psi_i}\,g(E_{\alpha_i}\,a_{;i+1})^{\iota_i})_{i;}.$$

We will prove Theorem 4.4 by proving that the function group $\mathcal{G}:=\{\Psi_{i,g}:i\in\mathbb{N},g\in\mathcal{E}_{\alpha_i}\}$ on $\mathbf{Ad}$ generates the class $\mathbf{Ne}$, i.e. that we have $\mathbf{Ne}=\mathbf{Smp}_{\mathcal{G}}$. We first need the following inequality:

Lemma 14.2.3. *Assume that Σ is nested. Let $i,j\in\mathbb{N}$ with $i<j$ and let $g\in\mathcal{E}_{\alpha_i}$. On $\mathbf{Ad}$, we have $\Psi_{i,g}<\Psi_{j,H_2}$ if $\sigma_{j+1;i+1}=1$ and $\Psi_{i,g}<\Psi_{j,H_{1/2}}$ if $\sigma_{j+1;i+1}=-1$.*

Proof. It is enough to prove the result for $j=i+1$. Assume that $\sigma_{i+2;i+1}=1$. Let $a\in\mathbf{Ad}$ and set $a':=(\Psi_{i+1,H_2}(a))_{i+1;}$, so that

$$\begin{aligned}a_{;i+1}&=\varphi_{i+1}+\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,(E_{\alpha_{i+1}}\,a_{;i+2})^{\iota_{i+1}}\\ a'&=\varphi_{i+1}+\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,(2\,E_{\alpha_{i+1}}\,a_{;i+2})^{\iota_{i+1}}.\end{aligned}$$

Note that

$$(\Psi_{i,g}(a))_{i+1;}\;\in\;\mathcal{T}_{\alpha_i}[a_{;i+1}].$$

If $\sigma_{;i+1}=1$, then $\varepsilon_{i+1}\,\iota_{i+1}=\sigma_{;i+2}/\sigma_{;i+1}=1$ and $\Psi_{;i+1}$ is strictly increasing. So we only need to prove that $\mathcal{T}_{\alpha_i}[a_{;i+1}]<a'$, which reduces to proving that $\sharp_{\alpha_i}(a_{;i+1})<\sharp_{\alpha_i}(a')$. Let τ be the dominant term of $E_{\alpha_{i+1}}\,a_{;i+2}$. Our assumption that Σ is nested gives $\varphi_i+\varepsilon_i\,\mathrm{e}^{\psi_i}\,(E_{\alpha_i}\,a')^{\iota_i}\in\mathbf{Ad}_{\nearrow i}$, whence $\varphi_{i+1}\lhd\sharp_{\alpha_i}(a')$. We deduce that $\varphi_{i+1}+\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,(2\,\tau)^{\iota_{i+1}}\trianglelefteq\sharp_{\alpha_i}(a')$. Lemma 13.2.2 implies that $\varphi_{i+1}+\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,(2\,\tau)^{\iota_{i+1}}$ is α_i-truncated.

$$\begin{aligned}\sharp_{\alpha_i}(a_{;i+1})-\varphi_{i+1}&\sim\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,\tau^{\iota_{i+1}},\\ \sharp_{\alpha_i}(a')-\varphi_{i+1}&\sim\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,(2\,\tau)^{\iota_{i+1}}\end{aligned}$$

and $\varepsilon_{i+1}\,\iota_{i+1}=1$ implies that

$$\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,(2\,\tau)^{\iota_{i+1}}-\varepsilon_{i+1}\,\mathrm{e}^{\psi_{i+1}}\,\tau^{\iota_{i+1}}$$

is a strictly positive term. We deduce that $\sharp_{\alpha_i}(a_{;i+1})-\varphi_{i+1}<\sharp_{\alpha_i}(a')-\varphi_{i+1}$, whence $\sharp_{\alpha_i}(a_{;i+1})<\sharp_{\alpha_i}(a')$. The other cases when $\sigma_{;i+1}=-1$ or when $\sigma_{i+2;i+1}=-1$ are proved similarly, using symmetric arguments. □

We are now in a position to prove the following refinement of Theorem 4.4.

Theorem 14.2.4. *If Σ is nested, then $\mathbf{Ne}$ is a surreal substructure with $\mathbf{Ne}=\mathbf{Smp}_{\mathcal{G}}$.*

Proof. By Theorem 10.2.3, the class $\mathbf{Smp}_{\mathcal{G}}$ is a surreal substructure, so it is enough to prove the equality. We first prove that $\mathbf{Smp}_{\mathcal{G}} \subseteq \mathbf{Ne}$.

Assume for contradiction that there are an $a \in \mathbf{Smp}_{\mathcal{G}}$ and a $k \in \mathbb{N}$, which we choose minimal, such that $a_{;k}$ cannot be written as $a_{;k} = \varphi_k \mathbin{+\!\!\!\!+} \varepsilon_k \mathfrak{m}_k$ where $\mathfrak{m}_k = \mathrm{e}^{\psi_k} (E_{\alpha_k}^{a_{;k+1}})^{\iota_k}$ is a hyperserial expansion. Set $\mathfrak{m} := \mathfrak{d}_{a_{;k} - \varphi_k}$, $r := (a_{;k})_{\mathfrak{m}}$ and $\delta := (a_{;k})_{\succ \mathfrak{m}}$.

Our goal is to prove that there is a number $m \in \{k, k+1\}$ and $\mathfrak{p} \in \mathbf{Mo}^{\succ}$ with

$$\begin{aligned} \mathfrak{p} &\in \mathcal{E}_{\alpha_m}[E_{\alpha_m} a_{;m+1}] \\ \mathfrak{p} &\sqsubseteq E_{\alpha_m} a_{;m+1} \\ \mathfrak{p} &\sqsubset E_{\alpha_m} a_{;m+1}, \quad \text{whenever } \delta = 0 \text{ and } r \in \{-1, 1\}. \end{aligned} \tag{14.2.5}$$

Assume that this is proved and set $c_m := \varphi_m + \varepsilon_m \mathrm{e}^{\psi_m} \mathfrak{p}^{\iota_m}$. The first condition and Lemma 14.2.1 yield $c_m \in \mathbf{Ad}_{\nearrow m}$ and the relations $\log \mathfrak{p} \prec \operatorname{supp} \psi_m$ and $\mathrm{e}^{\psi_m} \mathfrak{p}^{\iota_m} \prec \operatorname{supp} \varphi_m$. The second and third condition, together with Lemma 14.2.2, imply $c := (c_m)_{m;} \sqsubset a$. The first condition also implies that $c \in \mathcal{G}[a]$: a contradiction. Proving the existence of m and $\mathfrak{p}$ is therefore sufficient.

If $\mathfrak{m} \neq \min \operatorname{supp} a_{;k}$ or $\mathfrak{m} = \min \operatorname{supp} a_{;k}$ and $r \notin \{-1, 1\}$, then $m := k$ and $\mathfrak{p} := \mathfrak{d}_{E_{\alpha_k} a_{;k+1}}$ satisfy (14.2.5). Assume now that $\mathfrak{m} = \min \operatorname{supp} a_{;k}$ and that $r \in \{-1, 1\}$, whence $r = \varepsilon_k$. If $a_{;k+1} \notin \mathbf{No}_{\succ, \alpha_k}$ then $m := k$ and $\mathfrak{p} := E_{\alpha_k}^{\sharp_{\alpha_k}(a_{;k+1})}$ satisfy (14.2.5). Assume therefore that $a_{;k+1} \in \mathbf{No}_{\succ, \alpha_k}$. This implies that there exist $\gamma < \alpha_k$ and $\mathfrak{a} \in \mathbf{Mo}_{\alpha_k \omega}$ with $E_{\alpha_k}^{a_{;k+1}} = L_\gamma \mathfrak{a}$. By the definition of coding sequences, there is a least index $j > k$ with $\varphi_j \neq 0$ or $\psi_j \neq 0$, so

$$E_{\alpha_k}^{a_{;k+1}} = E_{\alpha_k + \cdots + \alpha_{j-1}} (\varphi_j \mathbin{+\!\!\!\!+} \varepsilon_j \mathrm{e}^{\psi_j} (E_{\alpha_j}^{a_{;j+1}})^{\iota_j}) \notin \mathbf{Mo}_{\alpha_k \omega}.$$

We have $\mathfrak{a} \in \mathbf{Mo}_{\alpha_k \omega}$ and $L_\gamma \mathfrak{a} \in \mathbf{Mo}_{\alpha_k} \setminus \mathbf{Mo}_{\alpha_k \omega}$. So by Corollary 13.1.6, we must have $\alpha_k = \omega^{\mu+1}$ for a certain $\mu \in \mathbf{On}$ and $\gamma = (\alpha_k)_{/\omega} n$ for a certain $n \in \mathbb{N}^{>}$. Note that $a_{;k+1} = L_{\alpha_k} \mathfrak{a} - n$. Recall that $\varphi_{k+1} \trianglelefteq a_{;k+1}$ and $L_{\alpha_k} \mathfrak{a} \in \mathbf{Mo}^{\succ}$, so $\varphi_{k+1} \in \{L_{\alpha_k} \mathfrak{a}, 0\}$. The case $\varphi_{k+1} = L_{\alpha_k} \mathfrak{a}$ cannot occur for otherwise

$$a_{;k+2} = \left(\frac{a_{;k+1} - \varphi_{k+1}}{\varepsilon_{k+1} \mathrm{e}^{\psi_{k+1}}} \right)^{\iota_{k+1}} = \frac{n^{\iota_{k+1}}}{\varepsilon_{k+1} \mathrm{e}^{\psi_{k+1}}}$$

would not lie in $\mathbf{No}^{>,\succ}$. So $\varphi_{k+1} = 0$. Let $m := k+1$ and

$$\mathfrak{p} := \left(\frac{L_{\alpha_k} \mathfrak{a}}{\mathrm{e}^{\psi_{k+1}}} \right)^{\iota_{k+1}} = \left(\frac{\mathfrak{d}_{a_{;k+1}}}{\mathrm{e}^{\psi_{k+1}}} \right)^{\iota_{k+1}} = \mathfrak{d}_{E_{\alpha_{k+1}} a_{;k+2}}.$$

We have $\mathfrak{p} \in \mathcal{E}_{\alpha_{k+1}}[E_{\alpha_{k+1}} a_{;k+2}]$ and $\mathfrak{p} \sqsubset E_{\alpha_{k+1}} a_{;k+2}$, so m and $\mathfrak{p}$ satisfy (14.2.5). We deduce that $\mathbf{Smp}_{\mathcal{G}}$ is a subclass of $\mathbf{Ne}$.

Conversely, consider $b \in \mathbf{Ne}$ and set $c := \pi_{\mathcal{G}}[b]$. So there are $i_1, i_2 \in \mathbb{N}$ and $(g, h) \in \mathcal{E}'_{\alpha_{i_1}} \times \mathcal{E}'_{\alpha_{i_2}}$ with $\Psi_{i_1, g_1}(b) < c < \Psi_{i_2, g_2}(b)$. Let $i := \max(i_1 + 1, i_2 + 1)$. By Lemma 14.2.3, there exist $d_1, d_2 \in \{1/2, 2\}$ with $\Psi_{i_1, g_1} < \Psi_{i, H_{d_1}}$ and $\Psi_{i_2, g_2} < \Psi_{i, H_{d_2}}$, whence $\Psi_{i, H_{d_1^{-1}}}(b) < c < \Psi_{i, H_{d_2}}(b)$. Since $\Phi_{;i}$ is strictly monotone, we get $c_{;i} - \varphi_i \asymp b_{;i} - \varphi_i$. The numbers $\varepsilon_i (c_{;i} - \varphi_i)$ and $\varepsilon_i (b_{;i} - \varphi_i)$ are monomials, so $c_{;i} - \varphi_i = b_{;i} - \varphi_i$. Therefore $b = c \in \mathbf{Smp}_{\mathcal{G}}$. □

In view of Theorem 14.2.4, Lemma 14.2.3, and Proposition 10.2.3, we have the following parametrization of $\mathbf{Ne}$:

$$\forall z \in \mathbf{No}, \quad \Xi_{\mathbf{Ne}} z = \{L, \Psi_{\mathbb{N}, \mathcal{H}} \Xi_{\mathbf{Ne}} z_L \mid \Psi_{\mathbb{N}, \mathcal{H}} \Xi_{\mathbf{Ne}} z_R, R\}.$$

We conclude this section with a few remarkable identities for $\Xi_{\mathbf{Ne}}$.

Lemma 14.2.5. *If Σ is nested, then for $i \in \mathbb{N}$ and $a, b \in \mathbf{Ne}$, we have $a \sqsubseteq b \Longleftrightarrow a_{;i} \sqsubseteq b_{;i}$.*

Proof. By [11, Lemma 4.5] and since the function $\Phi_{i;}$ is strictly monotone, it is enough to prove that $\forall a, b \in \mathbf{Ne}, a \sqsubseteq b \Longleftarrow a_{;i} \sqsubseteq b_{;i}$. By induction, we may also restrict to the case when $i = 1$. So assume that $a_{;1} \sqsubseteq b_{;1}$. Recall that $L_{\alpha_0} a_{;1} \prec \mathcal{E}_{\alpha_0} b_{;1}$ by Lemma 14.1.9. Since $a_{;1}, b_{;1} \in \mathbf{No}_{\succ, \alpha_0}$, we deduce with Lemma 13.1.25 that $E_{\alpha_0}^{a_{;1}} \sqsubseteq E_{\alpha_0}^{b_{;1}}$. It follows using the decomposition lemmas that $a \sqsubseteq b$. □

Proposition 14.2.6. *If Σ is nested, then we have $\mathbf{Ne} = (\mathbf{Ne}_{\nearrow 1})_{1;} = \varphi_0 + \varepsilon_0 \mathrm{e}^{\psi_0} (E_{\alpha_0}^{\mathbf{Ne}_{\nearrow 1}})^{\iota_0}$.*

Proof. We have $\mathbf{Ne} \subseteq (\mathbf{Ne}_{\nearrow 1})_{1;}$ by definition of $\mathbf{Ne}$. So we only need to prove that $(\mathbf{Ne}_{\nearrow 1})_{1;} \subseteq \mathbf{Ne}$. Consider $b \in \mathbf{Ne}_{\nearrow 1}$. Since Σ is nested, the number $a := \varphi_0 + \varepsilon_0 \, \mathrm{e}^{\psi_0} (E_{\alpha_0} b)$ is Σ-admissible, so we need only justify that $E_{\alpha_0} b \in \mathbf{Mo}_{\alpha_0} \setminus L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$. Since a is Σ-admissible, we have $\varphi_1 \lhd \sharp_{\alpha_0}(b)$. But b is $\Sigma_{\nearrow 1}$-nested, so $b = \varphi_1 +\!\!\!\!+\, \tau$ for a certain term τ. We deduce that $b = \sharp_{\alpha_0}(b) \in \mathbf{No}_{\succ, \alpha_0}$, whence $E_{\alpha_0} b \in \mathbf{Mo}_{\alpha_0}$.

Assume for contradiction that $E_{\alpha_0}^b \in L_{<\alpha_0} \mathbf{Mo}_{\alpha_0 \omega}$ and write $E_{\alpha_0}^b = L_\gamma \, \mathfrak{a}$ where $\mathfrak{a} \in \mathbf{Mo}_{\alpha_0 \omega}$ and $\gamma < \alpha_0$. Note that $\gamma \neq 0$: otherwise φ_i and ψ_i would be zero for all $i \geqslant 1$, thereby contradicting Definition 14.1.1(e). By Corollary 13.1.6, we must have $\alpha_0 = \omega^{\mu + 1}$ for a certain ordinal μ and $\gamma = \omega^\mu n$ for a certain $n \in \mathbb{N}^>$. Consequently, $b = L_{\alpha_0} \mathfrak{a} - n \in \mathbf{Mo} - n$. If $\varphi_1 \neq 0$, then the condition $\varphi_1 \lhd \sharp_{\alpha_0}(b)$ implies $\varphi_1 = b$, which leads to the contradiction that $b_{1;2} = 0 \notin \mathbf{No}^{>, \succ}$. If $\varphi_1 = 0$, then $\mathbf{Ne}_{\nearrow 1} \subseteq \varepsilon_1 \mathbf{Mo}$, whence $n = 0$: a contradiction. □

Corollary 14.2.7. *If Σ is nested, then for $z \in \mathbf{No}$, we have*

$$\Xi_{\mathbf{Ne}} \, z = \varphi_0 + \varepsilon_0 \, \mathrm{e}^{\psi_0} \, (E_{\alpha_0}^{\Xi_{\mathbf{Ne}_{\nearrow 1}} \sigma_{;1} z})^{\iota_0}.$$

Corollary 14.2.8. *If Σ is nested and $k \in \mathbb{N}$, then*

$$\Xi_{\mathbf{Ne}} = \Phi_{k;} \circ \Xi_{\mathbf{Ne}_{\nearrow k}} \circ H_{\sigma_{;k}}.$$

Proposition 14.2.9. *Assume that Σ is nested with $(\varphi_0, \varepsilon_0, \psi_0, \iota_0) = (0, 1, 0, 1)$, assume that $\alpha_0 \in \omega^{\mathbf{On} + 1}$ and write $\beta := (\alpha_0)_{/\omega}$. Consider the coding sequence Σ' with $(\varphi_i', \varepsilon_i', \psi_i', \iota_i', \alpha_i') = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)$ for all $i \in \mathbb{N}$, with the only exception that*

$$\varphi_1' = \varphi_1 - n.$$

If $\psi_1 < 0$, or $\psi_1 = 0$ and $\iota_1 = -1$, then Σ' is nested and we have

$$\Xi_{\mathbf{Ne}'} = L_{\beta n} \circ \Xi_{\mathbf{Ne}},$$

where $\mathbf{Ne}'$ is the class of $\Sigma^{[n]}$-nested numbers.

Proof. Assume that $\psi_1 < 0$, or $\psi_1 = 0$ and $\iota_1 = -1$. In particular, if a is Σ-admissible, then $a_{;1} - \varphi_1 \prec 1$, so $a_{;1} - \varphi_1 \prec \operatorname{supp} \varphi_1'$. For $b \in \mathbf{No}^{>, \succ}$, it follows that $E_{\alpha_0}(b - n)$ is $\Sigma^{[n]}$-admissible if and only if $E_{\alpha_0} b$ is Σ-admissible. Let $\mathbf{Ad}'_{\nearrow i}$ be the class of $\Sigma'_{\nearrow i}$-admissible numbers, for each $i \in \mathbb{N}$. We have $L_{\beta n} \mathbf{Ad} = \mathbf{Ad}'$ by the previous remarks, and Σ' is admissible. For $i > 1$, we have $\Sigma'_{\nearrow i} = \Sigma_{\nearrow i}$, so

$$\mathbf{Ad}'_{\nearrow i} = \mathbf{Ad}_{\nearrow i} \supseteq \varphi_i' + \varepsilon_i' \, \mathrm{e}^{\psi_i'} \, (E_{\alpha_i'} \mathbf{Ad}'_{\nearrow i+1})^{\iota_i'}.$$

Moreover, $\mathbf{Ad}'_{\nearrow 1} = \mathbf{Ad}_{\nearrow 1} - n$, so

$$\begin{aligned} \mathbf{Ad}' &\supseteq L_{\beta n} \mathbf{Ad} \supseteq L_{\beta n} E_{\alpha_0} \mathbf{Ad}_{\nearrow 1} = L_{\beta n} E_{\alpha_0} (\mathbf{Ad}'_{\nearrow 1} + n) = E_{\alpha_0^{[n]}} \mathbf{Ad}'_{\nearrow 1} \\ \mathbf{Ad}'_{\nearrow 1} &\supseteq \varphi_1 - n + \varepsilon_1 \, \mathrm{e}^{\psi_1} \, (E_{\alpha_1} \mathbf{Ad}_{\nearrow 2})^{\iota_1} = \varphi_1' + \varepsilon_1' \, \mathrm{e}^{\psi_1'} \left(E_{\alpha_1^{[n]}} \mathbf{Ad}'_{\nearrow 2} \right)^{\iota_1'}. \end{aligned}$$

So Σ' is nested. We deduce that $L_{\beta n} \mathbf{Ne} = \mathbf{Ne}'$, that is, we have a strictly increasing bijection $L_{\beta n}$: $\mathbf{Ne} \longrightarrow \mathbf{Ne}'$. It is enough to prove that for $a, b \in \mathbf{Ne}$ with $a \sqsubseteq b$, we have $L_{\beta n} a \sqsubseteq L_{\beta n} b$. Proceeding by induction on n, we may assume without loss of generality that $n = 1$. By [12, identity (6.3)], the function L_β has the following equation on $\mathbf{Mo}_{\alpha_0}$:

$$\forall \mathfrak{a} \in \mathbf{Mo}_{\alpha_0}, \quad L_\beta \, \mathfrak{a} = \left\{ L_\beta \, \mathfrak{a}_L^{\mathbf{Mo}_{\alpha_0}} \,\middle|\, L_\beta \, \mathfrak{a}_R^{\mathbf{Mo}_{\alpha_0}}, \mathfrak{a} \right\}_{\mathbf{Mo}_{\alpha_0}}.$$

So it is enough to prove that $L_\beta \, b < a$. Note that $L_\beta \, b = E_{\alpha_0}^{b_{;1} - 1}$ and $a = E_{\alpha_0}^{a_{;1}}$ where $b_{;1} - \varphi_1$, $a_{;1} - \varphi_1 \prec 1$. So $b_{;1} - a_{;1} \prec 1$, whence $b_{;1} - 1 < a_{;1}$. This concludes the proof. □

14.2.3 Pre-nested and nested numbers

Let $a \in \mathbf{No}$ be a number. We say that a is *pre-nested* if there exists an infinite path P in a without any bad index for a. In that case, Lemma 14.1.2 yields a coding sequence Σ_P which is admissible due to the fact that $a \in (L \mid R)$ with the notations from Section 7.3.2. By Theorem 14.1.15, we get a smallest $k \in \mathbb{N}$ such that $(\Sigma_P)_{\nearrow k}$ is nested. If $k = 0$, then we say that a is *nested*. In that case, Theorem 14.2.4 ensures that the class $\mathbf{Ne}$ of Σ_P-nested numbers forms a surreal substructure, so a can uniquely be written as $a = \Xi_{\mathbf{Ne}}(c)$ for some surreal parameter $c \in \mathbf{No}$.

One may wonder whether it could happen that $k > 0$. In other words: do there exist pre-nested numbers that are not nested? For this, let us now describe an example of an admissible sequence Σ^* such that the class $\mathbf{Ne}_{\Sigma^*}$ of Σ^*-nested numbers contains a smallest element b. This number b is pre-nested, but cannot be nested by Theorem 14.2.4. Note that our example is "transserial" in the sense that it does not involve any hyperexponentials.

Example 14.2.10. Let $\Sigma = (\varphi_i, \varepsilon_i, 0, 1, 1)_{i \in \mathbb{N}}$ be a nested sequence with $\varepsilon_1 = -1$. Let a be the simplest Σ-nested number. We define a coding sequence $\Sigma^* = (\varphi_i^*, \varepsilon_i^*, 0, 1, 1)_{i \in \mathbb{N}}$ by

$$\begin{aligned} \varepsilon_0^* &:= -1 \\ \varphi_0^* &:= \mathrm{e}^{\varphi_1 - \frac{1}{2}\mathrm{e}^{a_{;2}}} \\ (\varphi_i^*, \varepsilon_i^*) &:= (\varphi_i, \varepsilon_i) \qquad \text{for all } i > 0. \end{aligned}$$

Note that

$$a_{;1} = \varphi_1 - \mathrm{e}^{a_{;2}} = \varphi_1 + \varepsilon_1 \, \mathrm{e}^{a_{;2}},$$

where $\mathrm{e}^{a_{;2}}$ is an infinite monomial, so $b := \varphi_0^* - \mathrm{e}^{a_{;1}}$ is Σ^*-nested. In particular, the sequence Σ^* is admissible.

Assume for contradiction that there is a Σ^*-nested number c with $c < b$. Since $\varepsilon_0^* = \varepsilon_1^* = -1$, we have $c_{;2} < b_{;2}$. Recall that $c_{;2}$ and $b_{;2}$ are purely large, so $\mathrm{e}^{c_{;2}} \prec \mathrm{e}^{b_{;2}} = \mathrm{e}^{a_{;2}}$. In particular

$$\mathrm{e}^{c_{;1}} = \mathrm{e}^{\varphi_1 - \mathrm{e}^{c_{;2}}} \succcurlyeq \mathrm{e}^{\varphi_1 - \frac{1}{2}\mathrm{e}^{a_{;2}}} = \varphi_0^*,$$

which contradicts the assumption that c is Σ^*-nested. We deduce that b is the minimum of the class $\mathbf{Ne}_{\Sigma^*}$ of Σ^*-nested numbers. In view of Theorem 14.2.4, the sequence Σ^* cannot be nested.

The above examples shows that there exist admissible sequences that are not nested. Let us now construct an admissible sequence $\Sigma^{\varnothing}$ such that the class $\mathbf{Ne}_{\Sigma^{\varnothing}}$ of $\Sigma^{\varnothing}$-nested numbers is actually empty.

Example 14.2.11. We use the same notations as in Example 14.2.10. Define $(\varphi_0^{\varnothing}, \varepsilon_0^{\varnothing}) := (\mathrm{e}^b, 1)$ and set $(\varphi_i^{\varnothing}, \varepsilon_i^{\varnothing}) := (\varphi_{i-1}^*, \varepsilon_{i-1}^*)$ for all $i > 0$. We claim that the coding sequence $\Sigma^{\varnothing} := (\varphi_i^{\varnothing}, \varepsilon_i^{\varnothing}, 0, 1, 1)_{i \in \mathbb{N}}$ is admissible. In order to see this, let $\psi := {}^1\!/_2 \, \mathrm{e}^{b_{;1}}$. Then

$$\mathrm{e}^{\varphi_1^{\varnothing} + \varepsilon_1 \psi} = \mathrm{e}^{\varphi_0^* + \varepsilon_0^* \psi} \prec \mathrm{e}^{\varphi_0^* + \varepsilon_0^* \mathrm{e}^{b_{;1}}} = \mathrm{e}^b.$$

Since $\varphi_1^{\varnothing} + \varepsilon_1 \psi$ is $(\Sigma^{\varnothing})_{\nearrow 1}$-admissible (i.e. Σ^*-admissible), we deduce that $\mathrm{e}^b + \mathrm{e}^{\varphi_1^{\varnothing} + \varepsilon_1 \psi}$ is $\Sigma^{\varnothing}$-admissible, whence $\Sigma^{\varnothing}$ is admissible. Assume for contradiction that $\mathbf{Ne}_{\Sigma^{\varnothing}}$ is non-empty, and let $\mathrm{e}^b + \mathfrak{m} \in \mathbf{Ne}_{\Sigma^{\varnothing}}$. Then $\log \mathfrak{m}$ is Σ^*-nested, so $\log \mathfrak{m} \geqslant b$, whence $\mathfrak{m} \succcurlyeq \mathrm{e}^b$: a contradiction.

Chapter 15
Hyperserial representations

Traditional transseries in x can be regarded as infinite expressions that involve x, real constants, infinite summation, exponentiation and logarithms. It is convenient to regard such expressions as infinite labelled trees. In this section, we show that surreal numbers can be represented similarly as infinite expressions in ω that also involve hyperexponentials and hyperlogarithms. One technical difficulty is that the most straightforward way to do this leads to ambiguities in the case of nested numbers. These ambiguities can be resolved by associating a surreal number to every infinite path in the tree. In view of the results from Section 7.3.2, this will enable us to regard any surreal number as a unique hyperseries in ω.

Remark 15.0.1. In the case of ordinary transseries, our notion of tree expansions below is slightly different from the notion of tree representations that was used in [60, 92]. Nevertheless, both notions coincide modulo straightforward rewritings.

15.1 Introductory example

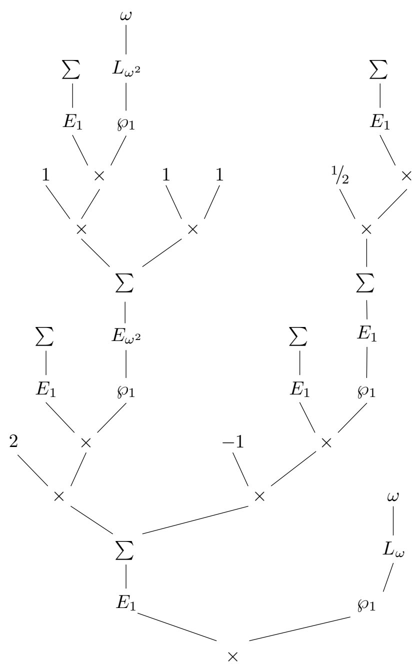
Let us consider the monomial $\mathfrak{m} = \exp(2\, E_\omega\, \omega - \sqrt{\omega} + L_{\omega+1}\, \omega)$ from Example 13.1.3. We may recursively expand $\mathfrak{m}$ as

$$\mathfrak{m} = \mathrm{e}^{2 E_{\omega^2}^{L_{\omega^2}\omega+1} - E_1^{\frac{1}{2} L_1 \omega}} (L_\omega\, \omega).$$

In order to formalize the general recursive expansion process, it is more convenient to work with the unsimplified version of this expression

$$\mathfrak{m} = \mathrm{e}^{2 \cdot \mathrm{e}^0 \cdot \left(E_{\omega^2}^{1 \cdot \mathrm{e}^0 \cdot (L_{\omega^2} \omega)^1 + 1 \cdot 1} \right)^1 + (-1) \cdot \mathrm{e}^0 \cdot \left(E_1^{1/2 \cdot \mathrm{e}^0 \cdot (L_1 \omega)^1} \right)^1} (L_\omega\, \omega)^1.$$

Introducing $\wp_c: x \longmapsto x^c$ as a notation for the "power" operator, the above expression may naturally be rewritten as a tree:



In the next subsection, we will describe a general procedure to expand surreal monomials and numbers as trees.

15.2 Tree expansions

In what follows, a *tree* T is a set of nodes N_T together with a function that associates to each node $\nu \in N_T$ an *arity* $\ell_\nu \in \mathbf{On}$ and a sequence $(\nu[\alpha])_{\alpha<\ell_\nu} \in N_T^{\ell_\nu}$ of *children*; we write $C_\nu := \{\nu[\alpha] : \alpha < \ell_\nu\}$ for the set of children of ν. Moreover, we assume that N_T contains a special element ρ_T, called the *root* of T, such that for any $\nu \in N_T$ there exist a unique h (called the *height* of ν and also denoted by h_ν) and unique nodes $\nu_0, \ldots, \nu_h$ with $\nu_0 = \rho_T$, $\nu_h = \nu$, and $\nu_i \in C_{\nu_{i-1}}$ for $i = 1, \ldots, h$. The height h_T of the tree T is the maximum of the heights of all nodes; we set $h_T := \omega$ if there exist nodes of arbitrarily large heights.

Given a class $\mathbf{\Sigma}$, an $\mathbf{\Sigma}$*-labelled tree* is a tree together with a map $\lambda: N_T \longrightarrow \mathbf{\Sigma}; \nu \longmapsto \lambda_\nu$, called the *labelling*. Our final objective is to express numbers using $\mathbf{\Sigma}$-labelled trees, where

$$\mathbf{\Sigma} \;:=\; \mathbb{R}^{\neq} \cup \{\omega, \textstyle\sum, \times, \wp_{-1}, \wp_1\} \cup L_{\omega^{\mathbf{On}}} \cup E_{\omega^{\mathbf{On}}}.$$

Instead of computing such expressions in a top-down manner (from the leaves until the root), we will compute them in a bottom-up fashion (from the root until the leaves). For this purpose, it is convenient to introduce a separate formal symbol $?_c$ for every $c \in \mathbf{On}$, together with the extended signature

$$\mathbf{\Sigma}^\sharp \;:=\; \mathbf{\Sigma} \cup \{?_c : c \in \mathbf{No}\}.$$

We use $?_c$ as a placeholder for a tree expression for c whose determination is postponed to a later stage.

Consider a $\mathbf{\Sigma}^\sharp$-labelled tree T and a map $v: N_T \longrightarrow \mathbf{On}$. We say that v is an *evaluation* of T if for each node $\nu \in N_T$ one of the following statements holds:

E1. $\lambda_\nu \in \mathbb{R}^{\neq} \cup \{\omega\}$, $\ell_\nu = 0$, and $v(\nu) = \lambda_\nu$;

E2. $\lambda_\nu = \sum$, the family $(v(\nu[\alpha]))_{\alpha<\ell_\nu}$ is well based and $v(\nu) = \sum_{\alpha<\ell_v} v(\nu[\alpha])$;

E3. $\lambda_\nu = \times$, $\ell_\nu = 2$, and $v(\nu) = v(\nu[0])\, v(\nu[1])$;

E4. $\lambda_\nu = \wp_\iota$, $\iota \in \{-1, 1\}$, $\ell_\nu = 1$, and $v(\nu) = v(\nu[0])^\iota$;

E5. $\lambda_\nu = L_{\omega^\mu}$, $\ell_\nu = 1$, and $v(\nu) = L_{\omega^\mu}\, v(\nu[0])$;

E6. $\lambda_\nu = E_{\omega^\mu}$, $\ell_\nu = 1$, and $v(\nu) = E_{\omega^\mu}\, v(\nu[0])$;

E7. $\lambda_\nu = ?_\alpha$, $\ell_\nu = 0$, and $v(\nu) = \alpha$.

We call $v(\rho_T)$ the *value* of T via v. We say that $a \in \mathbf{No}$ is *a value* of T if there exists an evaluation of T with $a = v(\rho_T)$.

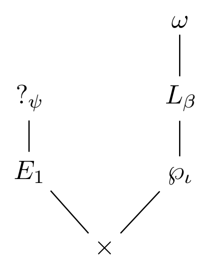
Lemma 15.2.1.

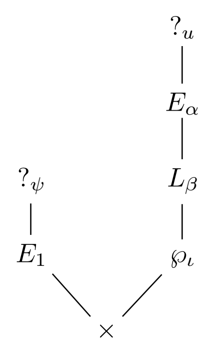
a) If T has finite height, then there exists at most one evaluation of T.

b) Let v and v' be evaluations of T with $v(\rho_T) = v'(\rho_T)$. Then $v = v'$.

Proof. This is straightforward, by applying the rules **E1–E7** recursively (from the leaves to the root in the case of (*a*) and the other way around for (*b*)). □

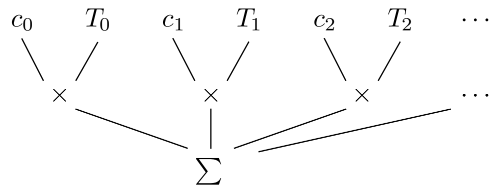
Although evaluations with a given end-value are unique for a fixed tree T, different trees may produce the same value. Our next aim is to describe a standard way to expand numbers using trees. Let us first consider the case of a monomial $\mathfrak{m} \in \mathbf{Mo}$. If $\mathfrak{m} = 1$, then the *standard monomial expansion* of $\mathfrak{m}$ is the $\mathbf{\Sigma}^\sharp$-labelled tree T with $N_T = \{\rho_T\}$ and $\lambda_{\rho_T} = 1$. Otherwise, we may write $\mathfrak{m} = \mathrm{e}^\psi\, (L_\beta\, g)^\iota$ with $g = \omega$ or $g = E_\alpha^u$. Depending on whether $g = \omega$ or $g = E_\alpha^u$, we respectively take

$T :=$ 

or $T :=$ 

and call T the *standard monomial expansion* of $\mathfrak{m}$. Let us next consider a general number $a \in \mathbf{No}$ and let $\ell \in \mathbf{On}$ be the ordinal size of its support. Then we may write $a = \sum_{\alpha < \ell} c_\alpha \mathfrak{m}_\alpha$ for a sequence $(c_\alpha)_{\alpha<\ell} \in (\mathbb{R}^{\neq})^\ell$ and a $\prec$-decreasing sequence $(\mathfrak{m}_\alpha)_{\alpha<\ell} \in \mathbf{Mo}^\ell$. For each $\alpha < \ell$, let T_α be the standard monomial expansion of $\mathfrak{m}_\alpha$. Then we define the $\mathbf{\Sigma}^\sharp$-labelled tree

$T :=$



and call it the *standard expansion* of a. Note that the height of T is at most 6, there exists a unique evaluation $v: N_T \longrightarrow \mathbf{No}$ of T, and $v(\rho_T) = a$.

Now consider two trees T and T' with respective labellings $\lambda: N_T \longrightarrow \mathbf{\Sigma}^\sharp$ and $\lambda': N_{T'} \longrightarrow \mathbf{\Sigma}^\sharp$. We say that T' *refines* T if $N_{T'} \supseteq N_T$ and there exist evaluations $v: N_T \longrightarrow \mathbf{No}$ and $v': N_{T'} \longrightarrow \mathbf{No}$ such that $v(\nu) = v'(\nu)$ for all $\nu \in N_T$ and $\lambda_\nu = \lambda'_\nu$ whenever $\lambda_\nu \notin ?_{\mathbf{No}}$. Now assume that $v(\rho_T) = a$ for some evaluation $v: N_T \longrightarrow \mathbf{No}$. Then we say that T is a *tree expansion* of a if for every $\nu \in N_T$ with $\lambda_\nu = \sum$, the subtree T' of T with root ν refines the standard expansion of $v(\nu)$. In particular, a tree expansion T of a number $a \in \mathbf{No}$ with $\lambda_{\rho_T} \notin ?_{\mathbf{No}}$ always refines the standard expansion of a.

Lemma 15.2.2. *Any $a \in \mathbf{No}$ has a unique tree expansion with labels in $\mathbf{\Sigma}$.*

Proof. Given $n \in \mathbb{N}$, we say that an $\mathbf{\Sigma}^\sharp$-labelled tree T is *n-settled* if $\lambda_\nu \notin ?_{\mathbf{No}}$ for all nodes $\nu \in N_T$ of height $<n$. Let us show how to construct a sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathbf{\Sigma}^\sharp$-labelled tree expansions of a such that the following statements hold for each $n \in \mathbb{N}$:

- **S1.** T_n is an n-settled and of finite height;
- **S2.** $v_n(\rho_{T_n}) = a$ for some (necessarily unique) evaluation $v_n: N_{T_n} \longrightarrow \mathbf{No}$ of T_n;
- **S3.** If $n > 0$, then T_n refines T_{n-1};
- **S4.** If T is a tree expansion of a with labels in $\mathbf{\Sigma}$, then T refines T_n.

We will write $\lambda_n: N_{T_n} \longrightarrow \mathbf{\Sigma}^\sharp$ for the labelling of T_n.

We take T_0 such that $N_{T_0} = \{\rho_{T_0}\}$ and $\lambda_{\rho_{T_0}} = ?_a$. Setting $v_0(\rho_{T_0}) := a$, the conditions **S1**, **S2**, **S3**, and **S4** are naturally satisfied.

Assume now that T_n has been constructed and let us show how to construct T_{n+1}. Let S be the subset of N_{T_n} of nodes ν of level n with $v_n(\nu) \in ?_{\mathbf{No}}$. Given $\nu \in S$, let T_ν be the standard expansion of $v_n(\nu)$ and let v_ν be the unique evaluation of T_ν. We define T_{n+1} to be the tree that is obtained from T_n when replacing each node $\nu \in S$ by the tree T_ν.

Since each tree T_ν is of height at most 6, the height of T_{n+1} is finite. Since T_{n+1} is clearly $(n+1)$-settled, this proves **S1**. We define an evaluation $v_{n+1}: N_{T_{n+1}} \longrightarrow \mathbf{\Sigma}^\sharp$ by setting $v_{n+1}(\sigma) = v_n(\sigma)$ for any $\sigma \in N_{T_n}$ and $v_{n+1}(\sigma) = v_\nu(\sigma)$ for any $\nu \in S$ and $\sigma \in N_{T_\nu}$ (note that v_{n+1} is well defined since $v_\nu(\rho_{T_\nu}) = (\lambda_n)_\nu = v_n(\nu)$ for all $\nu \in S$). We have $v_{n+1}(\rho_{T_{n+1}}) = v_n(\rho_{T_n}) = a$, so **S2** holds for v_{n+1}. By construction, $N_{T_{n+1}} \supseteq N_{T_n}$ and the evaluations v_n and v_{n+1} coincide on N_{T_n}; this proves **S3**. Finally, let T be a tree expansion of a with labels in $\mathbf{\Sigma}$ and let v be the unique evaluation of T with $v(\rho_T) = a$. Then T refines T_n, so v coincides with v_n on N_{T_n}. Let $\nu \in S$. Since T is a tree expansion of a, the subtree T' of T with root ν refines T_ν, whence $N_T \supseteq N_{T_\nu}$. Moreover, $v(\nu) = v_{n+1}(\nu)$, so v coincides with v_ν on T_ν. Altogether, this shows that T refines T_{n+1}.

Having completed the construction of our sequence, we next define a $\mathbf{\Sigma}$-labelled tree T_∞ and a map $v_\infty: N_{T_\infty} \longrightarrow \mathbf{No}$ by taking $N_{T_\infty} = \bigcup_{n\in\mathbb{N}} N_{T_n}$ and by setting $(\lambda_\infty)_\nu := (\lambda_n)_\nu$ and $v_\infty(\nu) = v_n(\nu)$ for any $n \in \mathbb{N}$ and $\nu \in N_{T_n}$ such that $(\lambda_n)_\nu \notin ?_{\mathbf{No}}$. By construction, we have $v_\infty(\rho_{T_\infty}) = a$ and T_∞ refines T_n for every $n \in \mathbb{N}$.

We claim that T_∞ is a tree expansion of a. Indeed, consider a node $\nu \in N_{T_\infty}$ of height n with $\lambda_\nu = \sum$. Then $\nu \in N_{T_{n+1}}$ and $(\lambda_{n+1})_\nu = \sum$, since T_{n+1} is $(n+1)$-settled. Consequently, the subtree of T_{n+1} with root ν refines the standard expansion of $v_{n+1}(\nu)$. Since T_∞ refines T_{n+1}, it follows that the subtree of T_∞ with root ν also refines the standard expansion of $v_\infty(\nu) = v_{n+1}(\nu)$. This completes the proof of our claim.

It remains to show that T_∞ is the unique tree expansion of a with labels in $\mathbf{\Sigma}$. So let T be any tree expansion of a with labelling $\lambda: N_T \longrightarrow \mathbf{\Sigma}$. For every $n \in \mathbb{N}$, it follows from **S4** that $N_T \supseteq N_{T_n}$. Moreover, since T_n is n-settled, λ coincides with both λ_n and λ_∞ on those nodes in N_{T_n} that are of height $<n$. Consequently, $N_T \supseteq N_{T_\infty}$ and λ coincides with λ_∞ on N_{T_∞}. Since every node in N_T has finite height, we conclude that $T = T_\infty$. □

15.3 Hyperserial descriptions

From now on, we only consider tree expansions with labels in $\mathbf{\Sigma}$, as in Lemma 15.2.2. Given a class $\mathbf{Ne}$ of nested numbers as in Section 7.3.2, it can be verified that every element in $\mathbf{Ne}$ has the same tree expansion. We still need a notational way to distinguish numbers with the same expansion.

Let $a \in \mathbf{No}$ be a pre-nested number. By Theorem 14.1.15, we get a smallest $k \in \mathbb{N}$ such that $(\Sigma_P)_{\nearrow k}$ is nested. Hence $a_{P,k} \in \mathbf{Ne}$ for the class $\mathbf{Ne}$ of $(\Sigma_P)_{\nearrow k}$-nested numbers. Theorem 14.2.4 implies that there exists a unique number c with $a_{P,k} = \Xi_{\mathbf{Ne}}(\sigma_{;k}\, c)$. We call c the *nested rank* of a and write $\xi_a := c$. By Corollary 14.2.8, we note that $\xi_{u_{P,i}} = \sigma_{;i}\, \xi_a$ for all $i \in \mathbb{N}$. Given an arbitrary infinite path P in a number $a \in \mathbf{No}$, there exists a $k > 0$ such that $P_{\nearrow k}$ has no bad indexes for $a_{P,k}$ (modulo a further increase of k, we may even assume $a_{P,k}$ to be nested). Let $\sigma_{P,k} := \operatorname{sign}(r_{P,0} \cdots r_{P,k-1})\, \iota_{P,0} \cdots \iota_{P,k-1} \in \{-1, 1\}$. We call $\xi_P := \sigma_{P,k}\, \xi_{u_{P,k}}$ the *nested rank* of P, where we note that the value of $\sigma_{P,k}\, \xi_{u_{P,k}}$ does not depend on the choice of k.

Let T be the tree expansion of a number $a \in \mathbf{No}$ and let $v: N_T \longrightarrow \mathbf{No}$ be the evaluation with $a = v(\rho_T)$. An *infinite path* in T is a sequence $\nu_0, \nu_1, \ldots$ of nodes in N_T with $\nu_0 = \rho_T$ and $\nu_{i+1} \in C_{\nu_i}$ for all $i \in \mathbb{N}$. Such a path induces an infinite path P in a: let $i_1 < i_2 < \cdots$ be the indexes with $\lambda_{\nu_{i_k}} = \sum$; then we take $\tau_{P,k} = v(\nu_{i_k+1})$ for each $k \in \mathbb{N}$. It is easily verified that this induces a one-to-one correspondence between the infinite paths in T and the infinite paths in a. We call $\xi_\nu := \xi_P$ the *nested rank* of the infinite path $\nu = (\nu_n)_{n \in \mathbb{N}}$ in T. Denoting by I_T the set of all infinite paths in T, we thus have a map $\xi: I_T \longrightarrow \mathbf{No}; \nu \mapsto \xi_\nu$. We call (T, ξ) the *hyperserial description* of a.

We are now in a position to prove the final theorem of this paper.

Theorem 15.3.1. *Every surreal number has a unique hyperserial description. Two numbers with the same hyperserial description are equal.*

Proof. Consider two numbers $a, a' \in \mathbf{No}$ with the same hyperserial description (T, ξ) and let $v, v': N_T \longrightarrow \mathbf{No}$ be the evaluations of T with $v(\rho_T) = a$ and $v'(\rho_T) = a'$. We need to prove that $a = a'$. Assume for contradiction that $a \neq a'$. We define an infinite path $\nu_0, \nu_1, \ldots$ in T with $v(\nu_n) \neq v'(\nu_n)$ for all n by setting $\nu_0 := \rho_T$ and $\nu_{n+1} := \nu_n[m]$, where $m \in \mathbb{N}$ is minimal such that $v(\nu_n[m]) \neq v'(\nu_n[m])$. (Note that such a number m indeed exists, since otherwise $v(\nu_n) = v'(\nu_n)$ using the rules **E1–E7**.) This infinite path also induces infinite paths P and P' in a and a' with $a_{P,n} = v(\nu_{i_n})$ and $a_{P',n} = v'(\nu_{i_n})$ for a certain sequence $i_1 < i_2 < \cdots$ and all $n \in \mathbb{N}$. Let $n > 0$ be such that $P_{\nearrow n}$ and $P'_{\nearrow n}$ have no bad indexes for $a_{P,n}$ and $a_{P',n}$. The way we chose $\nu_0, \nu_1, \ldots$ ensures that the coding sequences associated to the paths $P_{\nearrow n}$ and $P'_{\nearrow n}$ coincide, so they induce the same nested surreal substructure $\mathbf{Ne}$. It follows that $v(\nu_{i_n}) = a_{P,n} = \Xi_{\mathbf{Ne}}(\sigma_{;n}\, \xi_\nu) = a_{P',n} = v'(\nu_{i_n})$, which contradicts our assumptions. We conclude that a and a' must be equal. □

Conclusion and further research

The presentation of surreal numbers as hyperseries opens several problems on which I started to work during my PhD. I will now, in a more colloquial tone, describe the main ones and propose a few research questions related to this thesis.

1 Defining the derivation and composition law on No

Before the year 2022 when the manuscript was finished, a substantial amount of additional material was considered for this thesis, most of which now consists in manuscripts in preparation. The main goal in this respect is to define the natural derivation $\partial\colon \mathbf{No} \longrightarrow \mathbf{No}$ and composition law $\circ$: $\mathbf{No} \times \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}$ on surreal numbers by relying extensively on their hyperseries representation. More precisely, I have proofs of the two following results:

Theorem 1.1. [Work in progress] *There is a unique strongly linear derivation $\partial\colon \mathbf{No} \longrightarrow \mathbf{No}$ satisfying:*

SD1. $\mathrm{Ker}(\partial) = \mathbb{R}$.

SD2. *For all $a \in \mathbf{No}^{>,\succ}$ and $f \in \mathbb{L}$, we have $\partial(f \circ a) = \partial(a) \cdot (f' \circ a)$.*

SD3. *If $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$ is an admissible sequence and $a \in \mathbf{No}$ is Σ-nested, then writing*

$$\begin{aligned} \mathfrak{l}_i &:= \ell_{(0,\alpha_i)} \circ E_{\alpha_i}^{a;i+1}, \quad \textit{and} \\ \mathfrak{n}_i &:= \mathrm{e}^{\psi_i} (E_{\alpha_i}^{a;i+1})^{\iota_i}. \end{aligned}$$

for all $i \in \mathbb{N}$, we have

$$\partial(a) = \sum_{i \in \mathbb{N}} \left(\prod_{k<i} \varepsilon_k \, \iota_k \, \mathfrak{n}_k \, \mathfrak{l}_k \right) (\partial(\varphi_i) + \varepsilon_i \, \partial(\psi_i)). \tag{1.1}$$

Moreover, the ordered valued differential field $(\mathbf{No}, +, \times, <, \prec, \partial)$ is an elementary extension of the field of log-exp *transseries.*

The formula (1.1) corresponds to the simplest expression for which the derivation can satisfy the other conditions.

Theorem 1.2. [Work in progress] *There is a unique function $\circ\colon \mathbf{No} \times \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}$ satisfying the following:*

SC1. *For all $\xi \in \mathbf{No}$, the function $\mathbf{No} \longrightarrow \mathbf{No}; a \mapsto a \circ \xi$ is a strongly linear morphism of rings.*

SC2. *For all $\xi \in \mathbf{No}^{>,\succ}$ and $f \in \mathbb{L}$, we have $(f \circ \omega) \circ \xi = f \circ \xi$.*

SC3. *For all $a \in \mathbf{No}$ and $\xi, \zeta \in \mathbf{No}^{>,\succ}$, we have $(a \circ \zeta) \circ \xi = a \circ (\zeta \circ \xi)$.*

SC4. *For all $a \in \mathbf{No}^{>,\succ}$ and $\xi, \zeta \in \mathbf{No}^{>,\succ}$, we have $\xi < \zeta \Longrightarrow a \circ \xi < a \circ \zeta$.*

SC5. *For all $a \in \mathbf{No}$, $\xi \in \mathbf{No}^{>,\succ}$ and $\delta \in \mathbf{No}$ with $\delta \prec \xi$ and $(\partial(\mathfrak{m}) \circ \xi) \, \delta \prec \mathfrak{m} \circ \xi$ whenever $\mathfrak{m} \in \operatorname{supp} a$, we have*

$$a \circ (\xi + \delta) = \sum_{k \in \mathbb{N}} \frac{\partial^k(a) \circ \xi}{k!} \, \delta^k.$$

SC6. *If $\Sigma = (\varphi_i, \varepsilon_i, \psi_i, \iota_i, \alpha_i)_{i \in \mathbb{N}}$ is a nested sequence and $\xi \in \mathbf{No}^{>,\succ}$ is such that*

$$\Sigma \circ \xi := (\varphi_i \circ \xi, \varepsilon_i, \psi_i \circ \xi, \iota_i, \alpha_i)_{i \in \mathbb{N}} \tag{1.2}$$

is also a nested sequence, then writing Ξ_Σ *and* $\Xi_{\Sigma\circ\xi}$ *for the parametrizations of the classes of* Σ*-nested and* $(\Sigma\circ\xi)$*-nested numbers respectively, we have*

$$\forall z\in\mathbf{No}, (\Xi_\Sigma\, z)\circ\xi=\Xi_{\Sigma\circ\xi}\, z. \tag{1.3}$$

Moreover, the structure $(\mathbf{No}^{>,\succ},\circ,\omega,<)$ *is a bi-ordered group in which any two strictly positive elements are conjugate.*

The formula (1.3) corresponds to the simplest value for which the composition law can satisfy the other conditions. It should be noted that the versions of these conjectures where **No** is replaced by its subfield $\tilde{\mathbb{L}}\circ\omega$ are already proved in [10]. In fact, this subfield will constitute the base case of an inductive proof of the second conjecture (granted, this base case excludes nested numbers and thus evacuates the crucial parts **SD3** and **SC6** of the conjectures).

1.1 Surreal numbers as a hyperserial field

In order to define $(\partial,\circ)$ and prove the results, a first step is to derive further properties of the structure of **No** as a hyperserial field. The main ingredients are contained in this manuscript (and in [13]): the well-nestedness of all surreal numbers (Theorem 13.2.7), the eventual nestedness for all admissible sequences (Theorem 14.1.15), and the nature of surreal substructure for classes of Σ-nested numbers in the case when Σ is nested (Theorem 14.2.4). Using this, one can show that **No** is the increasing union of confluent hyperserial subfields $\mathbb{T}_\gamma,\gamma\in\mathbf{On}$ of force $(\mathbf{On},\mathbf{On})$, where each $\mathbb{T}_{\gamma+1}$ is obtained by adding to $\mathbb{T}_\gamma$ all nested numbers whose coding sequence is nested and takes values in $\mathbb{T}_\gamma$, and then closing under hyperexponentials. We take unions of monomial groups at non-zero limit stages γ, and start with $\mathbb{T}_0=\tilde{\mathbb{L}}\circ\omega$.

This also applies to subfields of **No** of smaller force. Fix $\nu\in\mathbf{On}^>$ and set $\lambda:=\omega^\nu$. We have a "force ν version" $\mathbf{No}_\lambda$ of **No**. We can define an embedding $\mathbf{No}_\lambda\longrightarrow\mathbf{No}; a\mapsto a^{\uparrow\lambda}$ of force ν whose effect on the hyperserial description of an $a\in\mathbf{No}_\lambda$ is to replace each occurring leaf ω by the hyperserial description of the number

$$E_\lambda^\omega=E_{\lambda\omega}^{L_{\lambda\omega}\omega+1}.$$

This embedding can be seen as a composition on the right with the hyperexponential of strength λ. Such embeddings have the important feature of simplifying the behavior of derivations and compositions, i.e. derivations and compositions on the right on $\mathbf{No}_\lambda^{\uparrow\lambda}:=\{a^{\uparrow\lambda}:a\in\mathbf{No}_\lambda\}$ are easy to describe in terms of their operator support and relative support respectively, as per Section 1.3.2 (these properties are more accurately stated by considering *near* supports and *near* relative supports, see [10, Section 1.6]).

1.2 The surreal derivation

It is possible to define ∂ by using path derivatives as in [92, Section 4.1]. This is probably the most sensible method in order to do so. The idea here is the same as in [92]: given $a\in\mathbf{No}$ every element P in the set $\mathcal{P}_a$ of finite and maximal paths P in a with $P(|P|)\notin\mathbb{R}^{\neq}$ contributes exactly one term $\partial(P)\in\mathbb{R}^{\neq}\mathbf{Mo}$ to the derivative $\partial(a)$ of a, in that $\partial(a)$ can be defined as the sum of the well-based family $(\partial(P))_{P\in\mathcal{P}_a}$. Proving that this family is well-based can be done by reducing to the case when a is Σ-nested for a nested sequence Σ, and then using the properties of nested sequences (in particular Lemma 14.2.1) and compositions with hyperexponentials in order to isolate from one another the contributions $\partial(P)$ for distinct paths $P\in\mathcal{P}_a$.

1.3 The surreal composition law

The definition of the composition law is much more technical. This is in part because a combinatorial description of composition laws, in the same vein as the method of path derivatives, seems too complex to be achieved. That leaves us with **SC1–SC6** as sole guides in order to define $\circ$, and it must be shown that they suffice. Understanding when (and why) the sequence in (1.2) should be nested, and dealing with the well-basedness of complicated families involved in the Taylor series in **SC5** proves particularly difficult. In that respect, many useful tools can be found in the present thesis, in particular in Chapter 2, as well as in [10].

2 Model theory of ordered structures with composition

In view of Results 1.1 and 1.2, we have a first-order structure

$$(\mathbf{No}, +, \times, \partial, \circ, <, \prec)$$

of a peculiar type which, in view of its interesting closure properties (conjugacy equations, algebraic differential equations, set-wise order saturation...), deserves to be studied. This feeling is bolstered by the model theoretic tameness of its reducts: e.g. the first-order theories of $(\mathbf{No}, <)$, $(\mathbf{No}, +, <)$, $(\mathbf{No}, +, \times, <)$, $(\mathbf{No}, +, \times, <, \prec, \partial)$ are all model complete and decidable. Even so, the literature on the model theory ordered structures with composition laws is scarce. How are we to start studying this rich structure?

2.1 Growth order groups

One route is to restrain ourselves to studying a small part of the language which includes the composition law. Accordingly, the natural candidate is the bi-ordered group $(\mathbf{No}^{>,\succ}, \circ, \omega, <)$. The algebraic theory of non-commutative, linearly (bi-)ordered groups is involved, in comparison to the commutative case, as constructing and classifying extensions of such groups frequently leads to open problems and dead-ends (see [53]). However, the ordered group $(\mathbf{No}^{>,\succ}, \circ, \omega, <)$ has specific properties which are not often considered by group theorists, but are related to properties of ordered groups of unary germs definable in o-minimal expansions of real-closed fields, as I will now explain. In particular, the basic inequality

$$a \circ b > b \circ a$$

is valid whenever $a, b > \omega$ and a lies above all compositional iterates $b^{[n]}, n \in \mathbb{N}$ of b. This can be stated as a first-order sentence in the language $\{\cdot, 1, <\}$ of ordered groups. Indeed it can be shown that the centralizer $\mathcal{C}(b) := \{c \in \mathbf{No}^{>,\succ} : c \circ b = b \circ c\}$ of b can be described using real compositional iterates $b^{[r]}, r \in \mathbb{R}$ of b, hence it is mutually cofinal with $\{b^{[n]} : n \in \mathbb{N}\}$ (see [10, Section 10.3] in the case of $\tilde{\mathbb{L}}$). This leads to the following axiom satisfied by $(\mathbf{No}^{>,\succ}, \circ, \omega, <)$:

GOG1. $\forall x \forall y ((x, y > 1 \wedge x > \mathcal{C}(y)) \Longrightarrow x \cdot y < y \cdot x)$.

Along with this, this group satisfies the first-order property **GOG2** below which implies that the relation $a \Supset b \Longleftrightarrow \max(a, a^{\mathrm{inv}}) > \mathcal{C}(b)$ behaves similarly to a dominance relation on Abelian ordered groups (in fact, here the relation is given by the Archimedean valuation on ordered groups, see [51, Section 4.1]):

GOG2. $\forall x \forall y \forall z \forall t ((x > y > 1 \wedge z \in \mathcal{C}(y)) \Longrightarrow \exists t (t \in \mathcal{C}(x) \wedge t > z)$.

I call *growth order groups* those ordered groups (including all Abelian ordered groups) which satisfy **GOG1** and **GOG2**. So $(\mathbf{No}^{>,\succ}, \omega, \circ, <)$ and $(\tilde{\mathbb{L}}^{>,\succ}, \ell_0, \circ, <)$ are growth order groups. In fact, further important and universal first-order properties, such as the commutativity of centralizers, require additional axioms in order to be necessary, but I will not go into such details here. Many growth order groups should originate from tame expansions of the real ordered field:

Conjecture 2.1. *Let $\mathcal{R} = (\mathbb{R}, +, \times, <, \ldots)$ be an o-minimal expansion of the real ordered field. Let $\mathcal{G}$ denote the group of germs at $+\infty$ of $\mathcal{R}$-definable functions $\mathbb{R} \longrightarrow \mathbb{R}$ which tend to $+\infty$ at $+\infty$, ordered by comparison of germs at $+\infty$. Then $\mathcal{G}$ is a growth order group.*

I expect that the non-commutative valuation theory of a growth ordered group retains certain features of the valuation theory of Abelian ordered groups (see [4, Section 2.4]). Just as valuation theory gives tools to obtain asymptotic expansions of regular growth rates of an additive nature, I expect that growth rates of functions definable in certain o-minimal structures can be decomposed as non-commutative compositions of simpler growth rates.

Using real iteration, one can construe the group $\mathbb{R}^{>}$ as a function group on $\mathbf{No}^{>\omega}$, thus yielding a surreal substructure $\mathbf{Smp}_{\mathbb{R}^{>}}$ whose elements are simplest in their convex equivalence class in the compositional sense (just as monomials in $\mathbf{Mo}$ are simplest in the additive sense). Using Conway brackets, it is then possible to define non-commutative transfinite compositions

$$\bigodot_{\gamma < \lambda} \mathfrak{p}_{\gamma}^{[r_{\gamma}]} \tag{2.1}$$

of well-based families $\big(\mathfrak{p}_\gamma^{[r_\gamma]}\big)_{\gamma<\lambda}$ in $((\mathbf{Smp}_{\mathbb{R}^>})^{[\mathbb{R}]}, \Subset)$. By convention, I define these as trailing on the left. We should then have

Conjecture 2.2. *For each $a \in \mathbf{No}^{>,\succ}$, there is a unique well-based family $\big(\mathfrak{p}_\gamma^{[r_\gamma]}\big)_{\gamma<\lambda}$ in $\big((\mathbf{Smp}_{\mathbb{R}^>})^{[\mathbb{R}^{\neq}]}, \Subset\big)$ with*

$$a = \bigodot_{\gamma<\lambda} \mathfrak{p}_\gamma^{[r_\gamma]}.$$

I plan to illustrate these ideas first by proving a case of the Conjecture 2.1, namely when the expansion of $\mathcal{R}$ with the exponential function is levelled in the sense of [79]. I will also show that Conjecture 2.2 is valid in finitely nested hyperseries and log-exp transseries, where all terms $\mathfrak{p}_\gamma^{[r_\gamma]}$ lie in the corresponding fields, and where r_0 must be an integer if $\mathfrak{p}_0 = \mathrm{e}^x$ in the case of log-exp transseries. In those cases, the element (2.1) can be defined without relying on the existence of a composition law on $\mathbf{No}$. Thus this work can be started right away.

2.2 A model theoretic approach to genetic definitions

Genetic, or recursive definitions are, in the language of the thesis, cut equations of a particular nature. They are sound and uniquely define surreal-valued functions by way of the principle of definition by induction. I was impressed by unpublished notes of Antongiulio Fornasiero [50] who derived in a concise way surprisingly strong properties of functions that can be defined in a recursive way over $\mathbf{No}$.

Of particular importance to us is the possibility of deriving intermediate value properties [50, Definition 3.3] for such functions. Indeed, intermediate value properties (IVP) for unary terms in the corresponding first-order language completely axiomatize the first-order theories of $(\mathbf{No}, +, <)$, $(\mathbf{No}, +, \times, <)$, $(\mathbf{No}, +, \times, <, \prec, \partial)$ over simple finite fragments of those theories (namely and respectively: linearly ordered Abelian groups, ordered domains, and Liouville-closed H-fields with small derivation). Moreover, in view of van der Hoeven's IVP conjecture [4, Conjecture 4.3] for $\mathbf{Hy} \simeq \mathbf{No}$, it would be interesting to see if such a result could be derived from intrinsic properties of surreal numbers, rather than from possibly difficult computations on hyperseries. We are far from being able to prove such things, since there is no known genetic definition of a derivation (let alone a composition law) on $\mathbf{No}$. It seems well in the realm of possibility to me that no such definition should exist.

Still, an equally interesting problem concerns the possibility of interpreting first-order languages and realizing models of corresponding theories within $\mathbf{No}$. More precisely, given a first-order language $\mathcal{L}$ with $<$ as the only relation symbol, and an $\mathcal{L}$-theory T of dense linear orders without endpoints (DLOWs), when and how (and in what order) can one define, in a recursive way, interpretations of the function symbols $\underline{f_i}, i \in I$ in $\mathcal{L}$ as functions $f_i, i \in I$ on $\mathbf{No}$ and its Cartesian powers, in such a way that $(\mathbf{No}, <, (f_i)_{i\in I})$ be a model of T? When doing so, what is the complete theory $\mathrm{Th}(\mathbf{No})$ which we obtain?

The known examples of ordered Abelian groups, ordered rings and real-closed exponential fields seem to work exceptionally well in the sense that the corresponding recursive definitions (as usual a', a'', b', b'' range in a_L, a_R, b_L and b_R respectively, see Chapters 8 and 11)

$$\begin{aligned} a+b &= \{a'+b, a+b' \mid a+b'', a''+b\}, \\ a\,b &= \{a'\,b+a\,b'-a'\,b', a''\,b+a\,b''-a''\,b'' \mid a''\,b+a\,b'-a''\,b', a'\,b+a\,b''-a'\,b''\}, \text{ and} \\ \exp(a) &= \left\{0,\ [a-a']_{\mathbb{N}} \exp a',\ [a-a'']_{2\mathbb{N}+1} \exp a'' \;\middle|\; \frac{\exp a''}{[a-a'']_{2\mathbb{N}+1}},\ \frac{\exp a'}{[a'-a]_{\mathbb{N}}}\right\} \end{aligned}$$

encompass only a very small part of the resulting theory $\mathrm{Th}(\mathbf{No})$, but still end up producing *the* good complete theory containing the axioms used in the recursive definition. For instance, it is remarkable that the theory of DLOWs with a binary operation that is strictly increasing in both variable suffices to obtain as a result the theory of divisible Abelian linearly ordered groups!

I have found conditions on such languages and theories which, when satisfied should make such a recursive interpretation over $\mathbf{No}$ possible. These conditions are extremely restrictive, still I believe that the game is worth the sacrifice.

3 Analytico-geometric interpretation of surreal numbers

As told in the introduction of the thesis, this work is part of a program whose origins are questions about the growth rates of concrete, very regular, and commonplace real-valued functions. Yet, after only a reasonable amount of pages, these grounded questions have sprouted a class-sized array of ordinal indexed hyper-fast growing functions, the first of which, already, would but raise eyebrows among analysts and geometers; a stack of “fields-with-no-escape” containing arbitrarily long series and infinitely deep vertical expansions thereof; full binary trees of fixed points for well-behaved operators; and so on… Maybe it is now time to look back?

3.1 Real hyperexponentiation

One long standing open question in o-minimality is the existence of an o-minimal expansion of the real ordered exponential field which defines a transexponential function. As we have seen in Chapter 12, Abel's equation

$$y \circ (\omega + 1) = \mathrm{e}^{\omega} \circ y \tag{3.1}$$

naturally arises when studying surreal-valued functions with transexponential asymptotics. In fact, the simplicity theorem [15, Theorem 21] can be strengthened by discarding the condition b), i.e. Abel's equation.

Thus, functions like Kneser's [66] solution E of Abel's equation on $\mathbb{R}^{\geqslant}$ are natural candidates for such o-minimal investigations. However, despite interesting unique properties of E [29, Proposition 1], it is unclear whether it is more interesting than other similar solutions (e.g. [95]), see [39, Chapter 8]. More generally, there doesn't seem to be known non-trivial model theoretic constraints of a solution of Abel's equations for it to generate an o-minimal expansion.

If there are no such conditions, then our hope is that the calculus of hyperseries on $\mathbf{No}$ faithfully represents asymptotic properties of real-valued germs which are solutions of (3.1), and that it can help in understanding them. The use of formal series in order to understand functions with transexponential growth rates is convincingly illustrated in [83]. Padgett's first-order theory T_{transexp} for $(\mathbb{R}, +, \times, E)$ in an expanded language $\mathcal{L}_{\text{transexp}}$ is indeed sufficient to order the field $\mathcal{H}_{\text{transexp}}$ of germs of unary terms in $\mathcal{L}_{\text{transexp}}$ [83, Theorem 1.1.5]. This is shown by embedding such germs into fields of formal series. Let $\exp_{\omega}$ extend E_{ω} to $\mathbf{No}^{\geqslant 0}$ via the restricted analytic function method (see Section 2.4) applied to the analytic function E. Then $(\mathbf{No}, +, \times, \exp_{\omega})$ is a natural model of T_{transexp}, which raises questions as to the compatibility between $\mathbf{No}$ and $\mathcal{H}_{\text{transexp}}$. There should exist a natural inclusion

$$\begin{aligned} \mathrm{ev}_{\omega} : \mathcal{H}_{\text{transexp}} &\longrightarrow \mathbf{No} \\ t(x) &\longmapsto t(\omega). \end{aligned}$$

We make the following conjectures, on which we have worked with good progress together with Adele Padgett and Elliot Kaplan:

Conjecture 3.1. *The function* ev_{ω} *is a well-defined* $\mathcal{L}_{\text{transexp}}$*-embedding.*

Conjecture 3.2. *The function* ev_{ω} *commutes with the derivations and composition laws on* $\mathcal{H}_{\text{transexp}}$ *and* $\mathbf{No}$.

In Conjecture 3.2, one can replace $\mathbf{No}$ with $\tilde{\mathbb{L}}$ or some set-sized subfields thereof, for which the derivation and composition are already defined in [10] and known to be well-behaved. A positive answer to Conjecture 3.2 would have consequences for the differential algebra of $\mathcal{H}_{\text{transexp}}$:

Conjecture 3.3. *The field* $\mathcal{H}_{\text{transexp}}$ *is an ω-free H-field with small derivation.*

The property of ω-freeness is a very robust property pertaining to the behavior of differential polynomials (see [4, Section 11.7]). Finally, we expect that, like the Hardy field $\mathcal{H}_{\text{an,exp}}$ of the real-exponential field with restricted analytic functions [49], the field $\mathcal{H}_{\text{transexp}}$ sits nicely within $\mathbf{No}$:

Conjecture 3.4. *The set* $\mathrm{ev}_{\omega}(\mathcal{H}_{\text{transexp}}) \subseteq \mathbf{No}$ *is initial, i.e. downward closed in* $(\mathbf{No}, \sqsubset)$.

3.2 Nested germs

A unique feature of surreal numbers is that they naturally contain nested numbers. Les us draw, one last time, our favourite nested expansion

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}},$$

which we recall only specifies a proper class $\mathbf{Ne} \simeq \mathbf{No}$ of numbers. It should be of little surprise to learn that these nested numbers will exactly be the solutions of the functional equation

$$y = \sqrt{\omega} + \exp(y \circ \log \omega), \qquad \text{with the condition} \qquad y \sim \sqrt{\omega}. \tag{3.2}$$

This equation makes perfect sense for germs of real-valued functions, and can thus be solved in various classes of regularity. Such equations and solutions have barely been studied in the past (with the notable exception of [61]), so simple questions remain open. Écalle suggested to us that the geometric relevance of nested numbers is questionable. Therefore it would be interesting to test whether problems arise when considering germs with the corresponding nested expansions, with the help of the functional equation.

Among the least degrees of regularity for germs, we can consider van den Dries' notion of Hardian germs, i.e. germs lying in a Hardy field. Two Hardian solutions of (3.2) living in a common Hardy field must be, in particular, comparable. If surreal numbers really are deeply connected to growth rates of regular functions, then one would expect that the existence of a very large class of solutions of (3.2) in **No** is reflected in properties of solutions sets in Hardy fields. It is sensible, when considering germs of real-valued functions, to restrict ourselves, at most, to the field $\mathbf{No}(\omega_1)$ of surreal numbers with countable birth day (under the continuum hypothesis, this field can be represented as a Hardy field: see [5, p. 11]). Still, the existence of as many as $|\mathbf{No}(\omega_1)|$ comparable solutions of a functional equations is puzzling.

In contrast, the set of surreal solutions $E_\omega^{\omega+r}, r \in \mathbb{R}$ of Abel's equation (3.1) is only parametrized by real numbers. Yet given a Hardian solution y of (3.1), all other Hardian solutions $y \circ \varphi$, most of which lie in distinct Hardy fields, are parametrized by functions φ with $\varphi \circ (x+1) = \varphi + 1$, most of which are non-Hardian and pairwise incomparable. It is conceivable that (3.2) is much more compatible with a linear ordering of its solutions than Abel's equation, so that its solution set in a common Hardy field may be parametrized by a linearly ordered set of germs, possibly of size $|\mathbf{No}(\omega_1)|$. Or, the amount of solutions in **No** may be an artifact of algebraic and model theoretic properties of the functional equation, without analytic meaning. In any case, an aspect of the correspondence between numbers and regular growth rates can be reduced to the following question, which I plan to investigate:

Question. What linear orderings can be represented by the set of solutions of (3.2) lying in a common Hardy field? In particular, does (3.2) have a Hardian solution?

Here, it would be more daring to state a definite conjecture, but is it ever in bad taste to end with a question?

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Glossary

$\mathcal{H}_{\mathcal{R}}$	Hardy field of definable germs in $\mathcal{R}$	8
$\mathbb{T}_{\mathrm{LE}}$	field of logarithmic-exponential transseries	14
GC	(set theory) axiom of global choice	25
LOS	(set theory) limitation of size	25
$\mathbf{Y}^X$	class of functions $X \longrightarrow \mathbf{Y}$	26
$>$	reverse ordering of $<$	27
$\leqslant$	large ordering corresponding to $<$	27
$\min \mathbf{A}$	minimum of $\mathbf{A}$	27
On	class of ordinals	28
$\boldsymbol{\nu} \leqslant \mathbf{On}$	$\boldsymbol{\nu} \in \mathbf{On}$ or $\boldsymbol{\nu} = \mathbf{On}$	28
$\operatorname{ord}(X, <)$	order type of $(X, <)$	28
$\mathbf{On}_{\mathrm{Lim}}$	class of limit ordinals	28
κ^+	successor cardinal	28
$\mathbf{M}^{>\mathbf{X}}$	$\{a \in \mathbf{M} : a > \mathbf{X}\}$	28
$\mathbf{M}^{\geqslant\mathbf{X}}$	$\{a \in \mathbf{M} : a \geqslant \mathbf{X}\}$	28
$\mathbf{M}^{>x}$	$\{a \in \mathbf{M} : a > x\}$	28
$\mathbf{M}^{\geqslant x}$	$\{a \in \mathbf{M} : a \geqslant x\}$	28
$\mathbf{M}^{>}$	$\{a \in \mathbf{M} : a > 1\}$	28
$\mathbf{M}^{\geqslant}$	$\{a \in \mathbf{M} : a \geqslant 1\}$	28
$\mathbf{M}^{\neq}$	$\{a \in \mathbf{M} : a \neq 1\}$	28
$x \prec 1$	x is infinitesimal	29
$\mathbf{D}^{\prec}$	class of infinitesimal elements in $\mathbf{D}$	29
∂	a derivation	29
∂^k	k-fold iterate of ∂	29
$x^{(k)}$	k-th iterate derivative of x	29
$\mathfrak{S}^{\infty}$	the class $\bigcup_{n \in \mathbb{N}} \mathfrak{S}^n$	38
$x^{\mathbf{G}}$	multiplicative copy of $\mathbf{G}$	39
$\operatorname{supp} f$	set of elements l with $f(l) \neq 0$	39
$H[\mathbf{L}, \mathbf{G}]$	Hahn product of $\mathbf{G}$ to the power $\mathbf{L}$	39
$\sum_{i \in \mathbf{I}} f_i$	sum of the family $(f_i)_{i \in \mathbf{I}}$	40
$\sum F$	sum of the family F	40
$\mathfrak{M}, \mathfrak{N}, \ldots$	monomial groups of $\mathbb{R}[[\mathfrak{M}]], \mathbb{R}[[\mathfrak{N}]], \ldots$	44
$\mathbb{R}[[\mathfrak{M}]]$	field of well-based series with real coefficients over $\mathfrak{M}$	44
$\mathbb{R}\,\mathfrak{M}$	class of terms in $\mathbb{R}[[\mathfrak{M}]]$	44
$\mathfrak{d}_s$	dominant monomial $\max \operatorname{supp} s$ of s	44
τ_s	dominant term $\mathfrak{d}_s\, s_{\mathfrak{d}_s}$ of s	44
$f_{\succ \mathfrak{m}}$	truncation $\sum_{\mathfrak{n} \succ \mathfrak{m}} f_{\mathfrak{n}}\, \mathfrak{n}$ of f	44
$s_{\succ}$	$s_{\succ 1}$	44
$s_{\asymp}$	$s_1 \in \mathbb{R}$	44
$s_{\prec}$	$s - s_{\succ} - s_1 \in \mathbb{S}^{\prec}$	44
$t \lhd s$	$t \neq s \wedge \operatorname{supp} t \succ s - t$	44
$\unlhd$	large ordering corresponding to $\lhd$	44
$s \mathbin{+\!\!\!\!+} t$	sum $s + t$ when $\operatorname{supp} s \succ t$	44
$f \prec g$	$\mathbb{R}^{>} \lvert f \rvert < \lvert g \rvert$	45
$s \preccurlyeq t$	$\exists r \in \mathbb{R}^{>}, \lvert s \rvert < r\, \lvert t \rvert$	45
$s \asymp t$	$s \preccurlyeq t$ and $t \preccurlyeq s$	45
$\mathbb{S}^{\preccurlyeq}$	class of series s with $s \preccurlyeq 1$	45
$\mathbb{S}_{\succ}$	class of series $s \in \mathbb{S}$ with $\operatorname{supp} s \succ 1$	45
$\mathbb{S}^{\prec}$	class of series $s \in \mathbb{S}$ with $s \prec 1$	45
$\mathbb{S}^{>,\succ}$	class of series $s \in \mathbb{S}$ with $s \geqslant 0$ and $s \succ 1$	45
$\mathbb{R}[[x^{\mathbb{R}}]]$	field of real-powered series	46
$\mathbb{R}[[x^{\mathbb{Q}}]]$	field of rational-powered series	46
$\mathbb{x}_{[i]},\ \mathbb{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$	x_i	47
$\lvert v \rvert,\ v \in \mathbb{N}^n$	$\sum_{i=1}^n v_{[i]}$	47
s^+	$\max(s, s^{-1})$	49
s^-	$\min(s, s^{-1})$	49

$s \lll t$	$(s^+)^n < t^+$ for all $n \in \mathbb{N}^>$	49
$s \asymp\!\asymp t$	$t^+ < (s^+)^m < (t^+)^n$ for some $m, n \in \mathbb{N}^>$	49
$s \underline{\lll} t$	$s \lll t$ or $s \asymp\!\asymp t$	49
$\operatorname{supp} \Phi$	support of the function Φ	50
$\operatorname{supp}_{\odot} \Phi$	relative support of the function Φ	50
$\mathbf{D}[[z_1, \ldots, z_n]]$	ring of power series in n variables over $\mathbf{D}$	55
$\mathbf{D}[[z]]$	ring of power series in one variable over $\mathbf{D}$	55
P'	derivative of the power series P	55
$P^{(n)}$	n-th iterated derivative of the power series P	55
$P \circ Q$	composite power series	55
$\operatorname{Conv}(P)$	class of series ε where $P(\varepsilon)$ is defined	55
$\operatorname{Conv}(f)_s$	class of series t with $f(t) = \tilde{f}_s(t - s)$	59
$f \upharpoonright \mathbf{X}$	restriction of the function f to the class $\mathbf{X}$	59
$\frac{\partial^{\lvert v \rvert}}{\partial x^v}$	$\frac{\partial^{\lvert v \rvert}}{\partial^{v[1]} x_1 \cdots \partial^{v[n]} x_n}$	62
$v!, v \in \mathbb{N}^n$	$v_{[1]}! \cdots v_{[n]}!$	62
$\bar{f}$	extension of the real-analytic function f	62
$\mathbb{R}\{z_1, \ldots, z_n\}$	power series which converge on a neighborhood of $[-1, 1]^n \subseteq \mathbb{R}$	62
$\mathbb{R}_{\mathrm{an}}$	real ordered field with restricted analytic functions	63
$\mathbb{S}_{\mathrm{an}}$	field of well-based series with restricted analytic functions	63
T_{an}	elementary theory of $\mathbb{R}_{\mathrm{an}}$	63
$\mathbb{R}_{\mathrm{an,exp}}$	real ordered exponential field with restricted analytic functions	63
$T_{\mathrm{an,exp}}$	first order theory of $\mathbb{R}_{\mathrm{an,exp}}$	63
$\binom{r}{k}, r \in \mathbb{R}, k \in \mathbb{N}$	$\frac{r\,(r-1) \cdots (r-k)}{k!}$	64
s^r	$s > 0$ to the power $r \in \mathbb{R}$	64
$L(1 + \varepsilon)$	Taylor expansion of log at $1 + \varepsilon$	69
$E(\varepsilon)$	Taylor expansion of exp at ε	69
$\log$	(natural or transserial) logarithm	70
$\log_n$	n-fold iterate of log	70
$\mathfrak{L}_{\log}$	group of logarithmic transmonomials	72
$\mathbb{T}_{\log}$	field of logarithmic transseries	72
$\mathbb{L}$	field of logarithmic hyperseries	85
μ_-	$\mu = \mu_- + 1$ if μ is a successor, $\mu = \mu_-$ if μ is a limit	85
$\beta_{/\omega}, \beta = \omega^\mu$	ω^{μ_-}	85
$\operatorname{supp} \gamma$	$\{\omega^\eta : \eta \in \mathbf{On} \wedge \gamma_\eta \neq 0\}$	86
$\gamma <_o \rho$	$\gamma < \operatorname{supp} \rho$	86
$\gamma \leqslant_o \rho$	$\gamma < (\operatorname{supp} \sigma)\, \omega$	86
$\gamma_{\geqslant \omega^\eta}$	unique ordinal with $\gamma_{\geqslant \omega^\eta} \geqslant_o \omega^\eta$ and $\exists \iota < \omega^\eta, \gamma = \gamma_{\geqslant \omega^\eta} + \iota$	86
$\gamma_{> \omega^\eta}$	unique ordinal with $\gamma_{> \omega^\eta} >_o \omega^\eta$ and $\exists \iota \leqslant \omega^{\eta+1}, \gamma = \gamma_{> \omega^\eta} + \iota$	86
ℓ_γ	formal hyperlogarithm $\ell_\gamma \in \mathbb{L}$ of strength γ	86
$\mathfrak{L}_{<\alpha}$	group of logarithmic monomials of strength α	86
$\mathbb{L}_{<\alpha}$	field of logarithmic series of strength α	86
$\mathfrak{L}$	group of logarithmic (hyper)monomials	86
∂	derivation on $\mathbb{L}$	87
$f^{(k)}$	k-th derivative $\partial^k(f)$ of $f \in \mathbb{L}$	87
$g^{\uparrow \gamma}$	for $g \in \mathbb{L}_{[\gamma, \alpha)}$, unique series with $g = (g^{\uparrow \gamma}) \circ \ell_\gamma$	88
L_{ω^μ}	hyperlogarithm of force μ	88
$\mathbf{DD}_\mu$	axiom of domain definition	88
$\mathfrak{M}_{\omega^\mu}$	class of $L_{<\omega^\mu}$-atomic elements	89
$\mathbf{FE}_\mu$	axiom of functional equations	89
$\mathbf{A}_\mu$	axiom of asymptotics	89
$\mathbf{M}_\mu$	axiom of monotonicity	89
$\mathbf{R}_\mu$	axiom of regularity	89
$\mathbf{P}_\mu$	axiom of transfinite products	90
$\mathfrak{d}_\alpha$	projection $\mathbb{T}^{>, \succ} \longrightarrow \mathfrak{M}_\alpha$	90
$\mathcal{E}_\alpha[s]$	class of series t with $\exists \gamma < \alpha, L_\gamma(t) \asymp L_\gamma(s)$	90
$\mathcal{A}_f$	analytic function induced by $f \in \mathbb{L}$	94
$\mathcal{T}_f(t, \delta), f \in \mathbb{L}$	$\sum_{k \in \mathbb{N}^>} \frac{f^{(k)} \circ t}{k!} \delta^k$	95
$\mathbb{T}_{(<\boldsymbol{\mu})}$	hyperexponential closure of $\mathbb{T}$ of force $\boldsymbol{\mu}$	108
$L_{[\gamma, \beta)}$	$s \longmapsto \prod_{\gamma \leqslant \sigma < \beta} \ell_\sigma \circ s \colon \mathbb{T}^{>, \succ} \longrightarrow \mathbb{T}^{>, \succ}$	109
$L_\beta^{\uparrow \gamma}$	$\mathbb{T}^{>, \succ} \longrightarrow \mathbb{T}^{>, \succ}; s \longmapsto \ell_\beta^{\uparrow \gamma} \circ s$	109
$L_{[\gamma, \beta)}^{\uparrow \gamma}$	$s \longmapsto \prod_{\gamma \leqslant \sigma < \beta} \ell_\sigma^{\uparrow \gamma} \circ s \colon \mathbb{T}^{>, \succ} \longrightarrow \mathbb{T}^{>, \succ}$	109
E_γ	(partially defined) functional inverse of $L_\gamma \colon \mathbb{T}^{>, \succ} \longrightarrow \mathbb{T}^{>, \succ}$	109
$\mathcal{L}_1[s]$	class of series t with $s - t \preccurlyeq 1$	109

$\mathbb{T}_{\succ,\omega^\eta}$	class of ω^η-truncated series	113
$\mathcal{L}_\alpha[s]$	$\{t\in s+\mathbb{T}^{\prec}: t=s \text{ or } (t\neq s \text{ and } s<L_\alpha^{\uparrow\gamma}(\lvert t-s\rvert^{-1}) \text{ for some } \gamma<\alpha)\}$	113
$\sharp_{\omega^\eta}(s)$	unique ω^η-truncated series in $\mathcal{L}_{\omega^\eta}[s]$	115
$s=_\beta t$	$\sharp_\beta(s)=\sharp_\beta(t)$	116
$s<_\beta t$	$\sharp_\beta(s)<\sharp_\beta(t)$	116
$\mathbf{T}_\mu(\mathbb{T})$	class of series $\varphi\in\mathbb{T}_{\succ,\omega^\mu}\setminus L_{\omega^\mu}(\mathbb{T}^{>,\succ})$	122
$\mathfrak{l}[\mathrm{e}_\beta^\varphi]$	formal adjunction of $\mathfrak{l}\circ E_\beta(\varphi)$ to a field	123
$\mathfrak{L}_{<\beta}[\mathrm{e}_\beta^{\mathbf{T}}]$	group of formal products $\prod_{\varphi\in\mathbf{T}}\mathfrak{t}_\varphi[\mathrm{e}_\beta^\varphi]$, $\mathfrak{t}_\varphi\in\mathbb{L}_{<\theta}$	123
$\operatorname{hsupp}\mathfrak{t}$	$\{\varphi\in\mathbf{T}:\mathfrak{t}_\varphi\neq 1\}$	123
$\varphi_{\mathfrak{t}}$	$\max\operatorname{hsupp}\mathfrak{t}$	123
$\gamma_{\mathfrak{t}}$	$\min\{\gamma<\theta:(\mathfrak{t}_{\varphi_{\mathfrak{t}}})_\gamma\neq 0\}$	123
$\mathfrak{M}_{(\mu)}$	monomial group for $\mathbb{T}_{(\mu)}$	123
$\mathbb{T}_{(\mu)}$	smallest extension fo $\mathbb{T}$ with $\mathbb{T}^{>,\succ}\subseteq L_{\omega^\mu}(\mathbb{T}^{>,\succ})$	123
HF1 – **HF7**	axioms for hyperserial fields	139
$(\mathbb{T},\circ_{\mathbb{T}})\subseteq(\mathbb{U},\circ_{\mathbb{U}})$	$\mathbb{T}$ is a hyperserial subfield of $\mathbb{U}$	144
$\tilde{\mathbb{L}}$	field of finitely nested hyperseries	146
$\mathbf{No}$	class of surreal numbers	153
$\operatorname{bd}(a)$	domain of a as a sign sequence	153
$x[\alpha]$	α-th term in the sign sequence of x	155
$a\sqsubset b$	a is strictly simpler than b	155
$x_\sqsubset$	set of strictly simpler numbers	155
$\operatorname{ot}(X,<_X)$	order type	155
(a_L,a_R)	canonical cut representation of a	156
$\alpha\dot{+}\beta$	ordinal sum	156
$\alpha\dot{\times}\beta$	ordinal product	156
$\alpha^{\dot{\beta}}$	ordinal exponentiation	156
$a\dot{+}b$	concatenation sum	156
$a\dot{\times}b$	concatenation product	156
$a\dot{+}\mathbf{No}$	surreal substructure of numbers whose sign sequence begins with a	161
$a\dot{\times}\mathbf{No}$	surreal substructure of transfinite concatenations of a and $-a$	161
$\mathbf{X}^\bullet$	simplest element, or root, of $\mathbf{X}$	162
$(\mathbf{L}\mid\mathbf{R})_{\mathbf{S}}$	class of elements of $\mathbf{S}$ lying between $\mathbf{L}$ and $\mathbf{R}$	162
$\{\mathbf{L}\mid\mathbf{R}\}_{\mathbf{S}}$	root of $(\mathbf{L}\mid\mathbf{R})_{\mathbf{S}}$	162
$\Xi_{\mathbf{S}}$	defining surreal isomorphism of $\mathbf{S}$	162
$(x_L^{\mathbf{S}},x_R^{\mathbf{S}})$	canonical representation of x in $\mathbf{S}$	163
$x_\sqsubset^{\mathbf{S}}$	set of elements in $\mathbf{S}$ that are strictly simpler than x	163
$\mathbf{S}^{\sqsupseteq x}$	class of numbers $y\in\mathbf{S}$ with $x\sqsubseteq y$	163
$\Xi_{\mathbf{S}}^{\mathrm{inv}}$	functional inverse of $\Xi_{\mathbf{S}}:\mathbf{No}\longrightarrow\mathbf{S}$	163
$\Xi_{\mathbf{T}}^{\mathbf{S}}$	canonical isomorphism $\mathbf{S}\longrightarrow\mathbf{T}$	163
$\mathbf{S}\prec\mathbf{T}$	imbrication $\Xi_{\mathbf{S}}\mathbf{T}$ of $\mathbf{T}$ into $\mathbf{S}$	163
$\mathbf{No}(\alpha)$	set of surreal numbers x with $\ell(x)<\alpha$	164
$\sup_{\mathbf{X},\sqsubset}X$	supremum of the $\sqsubset$-chain X in $(\mathbf{X},\sqsubset)$	164
$\mathbf{Inc}$	example of a surreal substructure defined as a tree	165
$\mathbf{Hull}_{\mathbf{Y}}(\mathbf{X})$	convex hull of $\mathbf{X}$ in $\mathbf{Y}$	165
$\mathbf{No}^{\preccurlyeq}$	class of finite surreals	166
$\mathbf{No}^{>}$	surreal substructure of strictly positive surreals	166
$\mathbf{No}^{>,\succ}$	surreal substructure of positive infinite surreals	166
$\mathbf{No}^{\prec}$	surreal substructure of infinitesimals	167
$\mathbf{No}^{\prec\operatorname{supp}\varphi}$	surreal substructure of numbers x with $x\prec\operatorname{supp}\varphi$	168
$\dot{\omega}^{\cdot}$	Conway's ω-map $\Xi_{\mathbf{Mo}}$	170
$\mathbf{Mo}$	surreal substructure of monomials	170
$\mathbf{\Pi}[x]$	equivalence class of x for $\mathbf{\Pi}$	171
$\mathbf{Smp}_{\mathbf{\Pi}}$	class of $\mathbf{\Pi}$-simple elements	171
$=_{\mathbf{\Pi}},<_{\mathbf{\Pi}},\leqslant_{\mathbf{\Pi}}$	$\mathbf{\Pi}$-induced relations	171
$\pi_{\mathbf{\Pi}}$	projection $\mathbf{S}\twoheadrightarrow\mathbf{Smp}_{\mathbf{\Pi}}$ onto roots of equivalence classes	171
$\mathbf{\Pi}[\mathbf{X}]$	union of $\mathbf{\Pi}[x]$ for $x\in\mathbf{X}$	172
$\Pi[x]$	distinguished cofinal and coinitial subset of $\mathbf{\Pi}[x]$	173
$\mathbf{Oz}$	class $\mathbf{No}_\succ+\mathbb{Z}$ of Conway integers	173
$\mathbf{\Pi}\trianglelefteq\mathbf{\Pi}'$	$\mathbf{\Pi}$ is a refinement of $\mathbf{\Pi}'$	174
$\langle X\rangle$	function group generated by X	175
$\mathcal{G}[x]$	convex hull in $\mathbf{S}$ of orbit under $\mathcal{G}$	175
$\mathbf{Smp}_{\mathcal{G}}$	surreal substructure of $\mathcal{G}$-simple elements	175
$<_{\mathcal{G}},=_{\mathcal{G}},\leqslant_{\mathcal{G}}$	$<_{\mathbf{\Pi}_{\mathcal{G}}},=_{\mathbf{\Pi}_{\mathcal{G}}},\leqslant_{\mathbf{\Pi}_{\mathcal{G}}}$	175
$Y\trianglelefteq X$	X is pointwise cofinal with respect to Y	175
$X\lesseqgtr Y$	X and Y are mutually pointwise cofinal	175

T_c	translation by c	176
$\mathcal{T}$	group of real translations	176
H_s	homothety by s	176
$\mathcal{H}$	group of positive real homotheties	176
$\asymp, \prec, \preccurlyeq$	valuation-theoretic asymptotical relations	176
$\mathfrak{d}$	dominant monomial projection $\pi_{\mathcal{H}}$ onto **Mo**	176
P_s	power by s	176
$\mathcal{P}$	group of positive real powers	176
$\mathcal{E}^*$	group of (finite) iterations of exp and log	179
$\mathcal{E}$	group generated by $\exp_n \circ \mathcal{H} \circ \log_n$ for $n \in \mathbb{N}$	179
$\lambda.$	Berarducci-Mantova's λ-map $\Xi_{\mathbf{La}}$	179
$\mathbf{K}$	surreal substructure of κ-numbers	179
$\kappa.$	Kuhlmann-Matusinski's κ-map $\Xi_{\mathbf{K}}$	179
T_r	translation $a \longmapsto a + r$	190
H_s	homothety $a \longmapsto s\,a$	190
P_s	power function $a \longmapsto a^s$	190
$\mathcal{T}$	function group $\{T_r : r \in \mathbb{R}\}$	190
$\mathcal{H}$	function group $\{H_s : s \in \mathbb{R}^>\}$	190
$\mathcal{P}$	function group $\{P_s : s \in \mathbb{R}^>\}$	190
$\mathcal{E}'$	function group $\langle E_n\, H_s\, L_n : n \in \mathbb{N}, s \in \mathbb{R}^> \rangle$	190
$\mathcal{E}^*$	function group $\{E_n, L_n : n \in \mathbb{N}\}$	190
$\mathcal{E}'_\alpha$	function group $\langle E_\gamma\, \mathcal{H}\, L_\gamma : \gamma < \alpha \rangle$	190
$\mathcal{E}^*_\alpha$	function group $\langle E_{<\alpha}, \mathcal{P} \rangle$	190
$\mathcal{L}'_\alpha$	function group $L_\alpha\, \mathcal{E}'_\alpha\, E_\alpha$	190
$\mathcal{L}^*_\alpha$	function group $L_\alpha\, \mathcal{E}^*_\alpha\, E_\alpha$	190
$\mathbf{Mo}'_\alpha$	structure of $\mathcal{E}'_\alpha$-simple elements	191
$\mathbf{Mo}^*_\alpha$	structure of $\mathcal{E}^*_\alpha$-simple elements	191
$\mathbf{Tr}'_\alpha$	structure of $\mathcal{L}'_\alpha$-simple elements	191
$\mathbf{Tr}^*_\alpha$	structure of $\mathcal{L}^*_\alpha$-simple elements	191
$\tau_{P,i}$	value $\tau_{P,i} = P(i)$ of the path P at $i < 1 + \lvert P \rvert$	209
$\mathfrak{m}_{P,i}$	dominant monomial of $\tau_{P,i}$	209
$r_{P,i}$	constant coefficient of $\tau_{P,i}$	209
$\lvert P \rvert$	length of a path $P \in (\mathbb{R}^{\neq}\,\mathbf{Mo})^{1+\lvert P \rvert}$	209
$P * Q$	concatenation of paths	210
$\blacktriangleleft_{\mathrm{BM}}$	Berarducci and Mantova's nested truncation relation	212
$\mathbf{Ad}$	class of admissible numbers	222
$\mathbf{Ad}_{\nearrow k}$	class of $\Sigma_{\nearrow k}$-admissible numbers	222
$\mathbf{Ne}_{\nearrow k}$	class of $\Sigma_{\nearrow k}$-nested numbers	223
$\mathbf{Ne}$	class of Σ-nested numbers	223

ECOLE
DOCTORALE

Titre : Hyperséries et nombres surréels

Mots clés : nombres surréels, théorie des modèles, transséries

Résumé : Les *hyperséries* sont des transséries généralisées construites à partir d'exponentielles e^x et de logarithmes $\log x$ d'une variable positive et infiniment grande x, ainsi qu'à partir d'*itérateurs transfinis* $\mathrm{e}_{\omega^\alpha}$ et ℓ_{ω^α} de e^x et $\log x$ respectivement, pour tout ordinal α. Par exemple, les éléments $\mathrm{e}_1, \mathrm{e}_\omega, \ldots$ peuvent être vus comme des avatars formels de solutions régulières (en particulier monotones et analytiques) de l'équation d'Abel

$$E_{\omega^{\alpha+1}}(r+1) = E_{\omega^\alpha}(E_{\omega^{\alpha+1}}(r))$$

pour $r \in \mathbb{R}$ suffisamment grand, où $E_1 = \exp$ est la fonction exponentielle réelle. De telles fonctions, exotiques en apparence, sont particulièrement intéressantes du fait qu'il est possible d'effectuer un "calcul hypersériel" simple et bien défini avec leur contrepartie formelles $\mathrm{e}_{\omega^\alpha}, \ell_{\omega^\alpha}$. Selon ces règles de calcul, il doit être possible de définir de façon naturelle des dérivations et lois de composition sur les hyperséries, de sorte qu'il en résulte des structures modérées des points de vue de la géométrie et de la théorie des modèles.

L'objectif de cette thèse est de définir une structure de corps d'hyperséries sur le corps $\mathbf{No}$ des *nombres surréels* de Conway. Nous prouvons que tout nombre surréel est représenté canoniquement par une hypersérie dans laquelle le nombre $\omega \in \mathbf{No}$ joue le rôle de la variable infinie x.

A cette fin, nous montrons comment définir les itérateurs transfinis L_{ω^α} et E_{ω^α} sur des corps généraux d'hyperséries, et nous prouvons que ces opérateurs peuvent être définis de façon naturelle sur les nombres surréels. Nous introduisons ensuite un moyen de représenter chaque nombre surréel comme une série formelle en ω impliquant les opérateurs L_{ω^α} et E_{ω^α}, des coefficients réels, et des sommes transfinies.

Title : Hyperseries and surreal numbers

Keywords : surreal numbers, model theory, transseries

Abstract : *Hyperseries* are generalized transseries that involve exponentials e^x, logarithms $\log x$ of a positive infinite variable x, as well as so-called transfinite iterators $\mathrm{e}_{\omega^\alpha}$ and ℓ_{ω^α} of e^x and $\log x$ respectively, for any ordinal index α. For instance, the elements $\mathrm{e}_1, \mathrm{e}_\omega, \ldots$ can be construed as a formal counterparts to regular (e.g. monotonous and analytic) solutions of Abel's equation

$$E_{\omega^{\alpha+1}}(r+1) = E_{\omega^\alpha}(E_{\omega^{\alpha+1}}(r))$$

for large enough $r \in \mathbb{R}$, where $E_1 = \exp$ is the real exponential function. Such seemingly exotic functions are of particular interest because their formal counterparts $\mathrm{e}_{\omega^\alpha}, \ell_{\omega^\alpha}$ are amenable to a simple "hyperserial calculus" according to which derivations and compositions of hyperseries are naturally defined, with tame properties.

The goal of the thesis is to define a structure of field of hyperseries on Conway's field $\mathbf{No}$ of *surreal numbers*. We show that each surreal number can be canonically represented as a hyperseries in which the number $\omega \in \mathbf{No}$ takes the role of the positive infinite variable x.

To that end, we show how transfinite iterators L_{ω^α} and E_{ω^α} can be defined on general fields of formal hyperseries, and we show that these functions can be defined in a natural way on surreal numbers. We then introduce a way to represent each surreal number as a formal series in ω involving the operators L_{ω^α} and E_{ω^α}, real numbers, and transfinite sums.

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