Research statement

BY VINCENT BAGAYOKO

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In my thesis, I have shown how to represent surreal numbers, as defined by CONWAY [6], as hyperseries, which are generalizations of DAHN-GÖRING [7] and ÉCALLE's [8] transseries.

Hyperseries are generalized power series s in a variable x, which are endowed with a **hyperserial structure**: a collection of functions $s \mapsto f \circ s$ which are regular (e.g. analytic and monotone) in a formal sense, and which are themselves represented as hyperseries f. These functions include

- exponentials e^x and logarithms $\log x$,
- so-called **hyperexponentials**, e.g. a formal term e_ω^x which satisfies Abel's equation
$$\mathrm{e}_\omega^x \circ (x+1) = \mathrm{e}^x \circ \mathrm{e}_\omega^x,$$
and its functional inverse $\ell_\omega x$ with $\mathrm{e}_\omega^x \circ (\ell_\omega x) = (\ell_\omega x) \circ \mathrm{e}_\omega^x = x$,
- so-called **nested series** such as expansions with transfinite depth
$$\sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log\log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}, \tag{1}$$
which can be made rigorous sense of in hyperseries.

My thesis consisted in showing that the class **No** of surreal numbers has a natural hyperserial structure. This raised several research questions that I am now interested in, and which I summarize below.

1 Defining the derivation and composition law on No

Using the hyperserial structure on the non-Archimedean ordered field extension $\mathbf{No} \supsetneq \mathbb{R}$, it is possible to represent surreal numbers as hypereries with real coefficients. This representation gives a natural way to treat numbers as *functions* defined on $\mathbf{No}^{>\mathbb{R}}$, and to *differentiate* them as such. In other words, this gives a canonical way to define a derivation $\partial: \mathbf{No} \longrightarrow \mathbf{No}$ and a composition law $\circ: \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ on surreal numbers. I have notes on how to define these operations, that I plan to turn into a series of papers.

2 Model theory of ordered structures with composition

I want to study several problems related to the first-order structure
$$\mathcal{N} := (\mathbf{No}, +, \times, \partial, \circ, <, \prec).$$
This structure has interesting properties: closure under conjugacy equations, model completeness as an ordered valued differential field, set-wise order saturation...

Nevertheless, it is difficult to even start studying its first-order properties when taking the composition law into account, because not much is known about the model theory of ordered structures with (non-commutative) composition laws. I plan to propose interesting first-order theories accounting for such structures, and to make baby steps toward understanding $\mathcal{N}$, which I believe is a prime and natural example of those.

2.1 Growth order groups

One way to start is restrain ourselves to studying a small part of the language which includes the composition law. Accordingly, the natural candidate is the bi-ordered group $\mathcal{G} := (\mathbf{No}^{>,\succ}, \circ, \omega, <)$. This group shares many first-order properties with ordered groups of germs (at $+\infty$) of definable unary functions in o-minimal expansions of real-closed fields.

I want to draw on this connection to further the knowledge about these groups, as well as more abstract ordered groups which I call **growth order groups**. The elementary class of growth order groups in the language $\langle \cdot, 1, < \rangle$ of ordered groups is, roughly speaking, characterized by way the inequality

$$f\, g > g\, f$$

is solved in such groups. I studied in detail [3] a prominent example of growth order group, which is a small subgroup of $\mathcal{G}$.

I expect that o-minimality is a natural source of growth ordered groups. In particular, I want to prove or disprove the following conjecture:

Conjecture 2.1. *Let $\mathcal{R} = (\mathbb{R}, +, \times, <, \dots)$ be an o-minimal expansion of the real ordered field. Let $\mathcal{G}$ denote the group of germs at $+\infty$ of $\mathcal{R}$-definable functions $\mathbb{R} \longrightarrow \mathbb{R}$ which tend to $+\infty$ at $+\infty$, ordered by comparison of germs at $+\infty$. Then $\mathcal{G}$ is a growth order group.*

I can prove this in the particular case when the expansion of $\mathcal{R}$ with the exponential function is levelled in the sense of [12], and I would like to generalize this.

I expect that the non-commutative valuation theory of a growth orderd group retains certain features of the valuation theory of Abelian ordered groups. Just as valuation theory gives tools to obtain asymptotic expansions of regular growth rates of an additive nature, I expect that growth rates of functions definable in certain o-minimal structures can be decomposed as non-commutative compositions of simpler growth rates. Growth order groups are a way to make this idea precise. Another way is to find a formal framework in which transfinite non-commutative product make sense and can be made to form a growth order group. I have an ongoing project of defining such structures, and showing that certain groups of positive infinite transseries are of this form.

2.2 Model theoretic approach to recursive definitions

A crucial feature of surreal numbers is the possibility of defining operations on Cartesian powers of **No** via **recursive definitions** as per [9]. Indeed this is how the arithmetic [6] and exponential function [10] and hyperserial structure [4] on **No** were defined. It is sometimes possible [9] to show that a function with recursive definition on **No** is "tame", for instance satisfying the **intermediate value property** (IVT). This is particularly interesting because having the IVT for unary terms in a first-order language is sometimes conducive to proving existential closedness for the corresponding structure (e.g. adding the IVT for terms to linearly ordered Abelian groups, ordered domains, and Liouville-closed H-fields with small derivation gives existential closedness).

Thus it would be interesting to generalize previous constructions of opertions on **No** with a more model theoretic approach. In particular given a first-order language $\mathcal{L}$ with $<$ as the only relation symbol, and an $\mathcal{L}$-theory T of dense linear orders without endpoints, when and how (and in what order) can one define, in a recursive way, interpretations of the function symbols $\underline{f_i}, i \in I$ in $\mathcal{L}$ as functions $f_i, i \in I$ on **No** and its Cartesian powers, in such a way that $(\mathbf{No}, <, (f_i)_{i \in I})$ be a model of T? When doing so, what is the complete theory $\mathrm{Th}(\mathbf{No})$ which we obtain?

3 Hyperexponential and nested functions

3.1 Real hyperexponentiation

One long standing open question in o-minimality is the existence of a **transexponential o-minimal expansion** of the real ordered exponential field, i.e. an expansion which defines a unary function growing faster than any finite iterate of exp. Abel's equation in f

$$f(t+1) = \exp(f(t)) \quad \text{for large enough } t \tag{3.1}$$

is the simplest functional equation whose solutions in Hardy fields [5] are transexponential.

On the real-analytic side, Kneser's [11] solution E to (3.1) on $\mathbb{R}^{\geqslant}$ is a natural candidate for such o-minimal investigations. On the formal-surreal side, I hope that the calculus of hyperseries on **No** faithfully represents asymptotic properties (at $+\infty$) of E. Recently, Adele PADGETT studied in her PhD thesis [13] a first-order theory T_{transexp} for $(\mathbb{R}, +, \times, E)$ in an expanded language $\mathcal{L}_{\text{transexp}}$, and showed that the the field $\mathcal{H}_{\text{transexp}}$ of germs of unary terms in $\mathcal{L}_{\text{transexp}}$ is a Hardy field. Surreal numbers with the natural hyperexponential form a natural model of T_{transexp}. A first step towards studying the relationship between the surreal/formal model **No** and the geometric/analytic model $(\mathbb{R}, E)$ is to prove that there is a natural inclusion

$$\begin{aligned} \mathrm{ev}_\omega : \mathcal{H}_{\text{transexp}} &\longrightarrow \mathbf{No} \\ t(x) &\longmapsto t(\omega). \end{aligned}$$

Together with A. PADGETT and E. KAPLAN, we have started to work on the following conjectures:

Conjecture 3.1. *The function* ev_ω *is a well-defined* $\mathcal{L}_{\text{transexp}}$*-embedding.*

Conjecture 3.2. *The function* ev_ω *commutes with the derivations and composition laws on* $\mathcal{H}_{\text{transexp}}$ *and* **No**.

3.2 Nested germs

A unique feature of surreal numbers is that they naturally contain nested numbers, e.g. numbers which expand as

$$\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}, \tag{3.2}$$

where $\omega \in \mathbf{No}$ is a specific surreal number which plays the role of a positive infinite variable.

On the geometric/analytic side, the functional equation

$$g(t) = \sqrt{t} + \mathrm{e}^{g(t)}, \qquad \text{with the condition} \qquad g(t) \sim \sqrt{t} \text{ for large enough } t, \tag{3.3}$$

naturally generate germs which, when represented using logarithmico-exponential terms, expand in a similar way as (3.2). There are good reasons [2] to believe that the behavior of differential polynomials on these "nested germs" is exactly the same as their behavior on the corresponding nested numbers. Exploiting this, I want to study how the functional equation (3.2) can be solved in Hardy fields:

Question. What linear orderings can be represented by the set of quasi-analytic solutions of (3.3) lying in a common Hardy field?

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