Research statement

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In my PhD thesis [5], I have shown how to represent Conway's surreal numbers [16] as hyperseries, which are generalisations of Dahn-Göring [17] and Écalle's transseries. Hyperseries are generalised power series s in a variable x, that are endowed with a **hyperserial structure** [13]: a collection of functions $s \mapsto f \circ s$ that are regular (e.g. analytic and monotone) in a formal sense, and which are themselves represented as hyperseries f. These functions include:

- exponentials e^x and logarithms $\log x$,
- so-called **hyperexponentials**, e.g. a formal term e_ω^x which satisfies Abel's equation

  $$\mathrm{e}_\omega^x \circ (x+1) = \mathrm{e}^x \circ \mathrm{e}_\omega^x,$$

  and its functional inverse $\ell_\omega x$ with $\mathrm{e}_\omega^x \circ (\ell_\omega x) = (\ell_\omega x) \circ \mathrm{e}_\omega^x = x$,
- so-called **nested** series such as expansions with transfinite depth

  $$\sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}, \qquad (1)$$

  which in can be made rigorous sense of in hyperseries.

My thesis consisted in showing that the class $\mathbf{No}$ of surreal numbers has a natural hyperserial structure. This has sprouted a series of research questions that are summarized below.

1 Defining the derivation and composition law

Using the hyperserial structure on the non-Archimedean ordered field $\mathbf{No}$ of surreal numbers [11], it is possible to represent surreal numbers as hyperseries with real coefficients [12]. This representation gives a natural way to treat numbers as *functions* defined on the class $\mathbf{No}^{>\mathbb{R}}$ of positive infinite numbers, and to *differentiate* them as such. In other words, this gives a canonical way to define a composition law $\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ and a derivation $\partial : \mathbf{No} \longrightarrow \mathbf{No}$ on surreal numbers. This is van der Hoeven's conjecture:

Conjecture A. [22] *There is are a derivation and a composition law on* $\mathbf{No}$ *that are compatible with its hyperserial structure.*

I have notes which define these operations and derive their main elementary properties. I plan to turn them into a series of papers, partly in joint work with J. van der Hoeven. This series recently started by a technical note [9], in which I showed how to compose numbers by monomials that are "sufficiently hyperexponential".

2 Model theory of groups of regular growth rates

I am studying several inter-related problems motivated by the investigation of the first-order structure

$$\mathcal{N} := (\mathbf{No}, +, \cdot, <, \prec, \partial, \circ).$$

This structure has interesting properties: closedness under conjugacy equations, model completeness as an ordered valued differential field [1, 2], set-wise order saturation...

Unfortunately, it is difficult to even start looking into its first-order properties when taking the composition law into account, because little is known about first-order theories of ordered structures with composition laws. My long-term plan is to build toward an understanding of some of these structures, in the hope of establishing tameness properties of $\mathcal{N}$ and other such structures.

2.1 Growth order groups

One way to begin this journey is to restrain oneself to a small reduct of $\mathcal{N}$. The natural candidate is the ordered group $\mathcal{N}_{\mathrm{og}} = (\mathbf{No}^{>\mathbb{R}}, \circ, <)$. It shares many first-order properties with ordered groups of germs at $+\infty$ of definable unary functions in o-minimal expansions of real-closed fields. Drawing on this connection, I introduced [10] an elementary class of ordered groups, called **growth order groups** (henceforth GOGs), that is intended to subsume the informal notion of group of regular growth rates, exemplified both by o-minimal germs and hyperseries. I expect that o-minimality is a natural source of GOGs. In particular, I want to prove the following:

Conjecture B. *Let $\mathcal{R} = (\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal expansion of the real ordered field. The ordered group, under composition and comparison at infinity, of germs at $+\infty$ of unary $\mathcal{R}$-definable maps $\mathbb{R} \longrightarrow \mathbb{R}$ is a growth order group.*

The special case when $\mathcal{R}$ is levelled in the sense of [27] was proved in [10].

2.2 Equations over valued groups

Growth order groups come equipped with a canonical definable valuation, which plays a prominent role in establishing their properties. An important matter to be understood regarding GOGs is the unary equation problem: given a GOG $\mathcal{G}$ and a unary term $t(y)$ in the language of groups with parameters in $\mathcal{G}$, when is there a growth order group $\mathcal{G}^*$ extending $\mathcal{G}$ such that

$$\mathcal{G} \vDash \exists y (t(y) = 1) \quad ? \tag{2.1}$$

Indeed, this question is a baby version of the search for existentially closed GOGs. Seeing $\mathcal{G}$ as a group of o-minimal germs, or formal series under composition, the equation $t(y) = 1$ translates to an intricate functional equation

$$g_1 \circ y^{\circ \alpha_1} \circ g_2 \circ y^{\circ \alpha_2} \circ \cdots \circ g_n \circ y^{\circ \alpha_n} = \mathrm{id}.$$

No practical theory of such general functional equations exists.

Ordered groups of o-minimal germs for expansions $\mathcal{R}$ of general real-closed fields may fail to be GOGs. Likewise, basic expansions of a GOG related to the unary equation problem fail to be GOGs. This calls for an extension of the class of growth ordered groups to a larger class of **valued groups** that would encompass all the groups involved in the unary equation problem for GOGs, while retaining sufficiently many properties of the canonical valuation on GOGs that equations over these valued groups be traceable. We introduced [6] these valued groups, called c-valued groups, and studied their properties. **Nearly Abelian** c-valued groups are c-valued groups in which commutators decrease in valuation. They include for instance groups of parabolic[2.1] formal series, or GOGs of parabolic o-minimal germs. Our theory is suited to studying unary equations over certain pure groups as well as exponential groups in the sense of [28, 29]. Adapting some classical valuation theoretic notions, such as residues and spherical completeness, to the case of non-Abelian groups (see also [31]), we obtained in particular:

2.1. we use parabolic as a synonym for "tangent to the identity"

Theorem 2.1. [6, Theorem 2] *Let $(\mathcal{G},\cdot,1,v)$ be a spherically complete, torsion-free, nearly Abelian c-valued group. Suppose that its (Abelian) residue groups are divisible. Then any equation*

$$g_1\,y^{\alpha_1}\cdots g_n\,y^{\alpha_n}=1$$

where $g_1,\ldots,g_n\in\mathcal{G}$ and $\alpha_1+\cdots+\alpha_n\neq 0$ has a unique solution y in $\mathcal{G}$.

To us, this would give a satisfactory partial answer to the unary equation problem restricted to non-singular equations, i.e. equations with $\alpha_1+\cdots+\alpha_n\neq 0$, *if* we had a positive answer to the following question:

Question C. *Given a c-valued group $\mathcal{G}$, is there a spherically complete c-valued group $\mathcal{G}^*$ extending $\mathcal{G}$?*

Our next goal besides answering that question is to find valued groups in which the geometry of definable sets is tame (see Section 3.4). This mainly entails solving singular equations over valued groups. We have a candidate of a good first-order theory whose models have this feature.

2.3 Model theoretic approach to recursive surreal definitions

A defining aspect of **No** is the possibility of defining operations on its Cartesian powers *via* **recursive definitions** as per [19]. Indeed this is how the arithmetic [16], exponential function [20] and hyperserial structure [11] were defined. It is sometimes possible [19] to show that a function with a recursive definition on **No** is "tame", for instance satisfying the intermediate value property (IVP). This is particularly interesting because the IVP for unary terms in a first-order language is sometimes sufficient to entail existential closedness for the corresponding structure (e.g. adding the IVP axiom scheme for unary terms to the theories of linearly ordered Abelian groups, ordered domains, or Liouville-closed H-fields with small derivation [1] yields their model companion).

Thus it would be interesting to revisit and generalise previous definitions of operations on **No** with a more model theoretic approach. In particular, given a first-order language $\mathcal{L}=\langle <,(\underline{f_i})_{i\in I}\rangle$ where each $\underline{f_i}, i\in I$ is a function symbol, and an $\mathcal{L}$-theory T of dense linear orders without endpoints, when and how (and in what order) can one recursively define interpretations of the function symbols $\underline{f_i}, i\in I$ as function f_i on Cartesian powers of **No**, in such a way that $(\mathbf{No},<,(f_i)_{i\in I})$ be a model of T? When doing so, what is the complete theory $\mathrm{Th}_{\mathcal{L}}(\mathbf{No})$ of $(\mathbf{No},<,(f_i)_{i\in I})$?

3 Lie calculus for algebras and groups of formal series

The formal realm usually serves as a collection of convenient extensions of first-order structures in which many equations or existential formulas are satisfied. Developing formal versions of classical tools in algebra, geometry and analysis is a way to build formal structures fitting our goals, for instance satisfying a given first-order theory.

3.1 Formal Lie correspondence

Together with S. L. Krapp, S. Kuhlmann, D. C. Panazzolo and M. Serra, we showed [14] that a fraction of the classical Lie theory applies to Lie algebras of generalised formal series called Noetherian series (see [21]). They come equipped with a partial ordering $\prec$ of strict dominance, and a notion of infinite sums, for which linear maps commuting with infinite sums are said *strongly linear*. They include for instance algebras of formal power series in commuting or non-commuting variables, and Hahn series fields and skew-fields. We showed:

Theorem 3.1. *Let $(\mathbb{A}, +, \cdot, \prec)$ be an algebra, in characteristic 0, of Noetherian series. Then the formal Taylor series of the exponential map induces an isomorphism between the group $\mathrm{Der}^{+}_{\prec}(\mathbb{A})$ of strongly linear derivations ∂ on $\mathbb{A}$ with $\partial(a) \prec a$ for all $a \in \mathbb{A} \setminus \{0\}$ and the group $1\text{-}\mathrm{Aut}^{+}(\mathbb{A})$ under composition of strongly linear automorphisms σ of $\mathbb{A}$ with $\sigma(a) - a \prec a$ for all $a \in \mathbb{A} \setminus \{0\}$.*

The group law on $\mathrm{Der}^{+}_{\prec}(\mathbb{A})$ is given by a formal Baker-Campbell-Hausdorff product (see [32]). There is also a formal version of the third homomorphism theorem. We call these relations the **formal Lie correspondence**. Motivated by [26], this raises the question:

Question D. Does the formal Lie correspondence extend between a Lie subalgebra of the algebra $\mathrm{Der}^{+}(\mathbb{A})$ of strongly linear derivations on $\mathbb{A}$ and the group $\mathrm{Aut}^{+}(\mathbb{A}, \prec)$ of strongly linear automorphism of $(\mathbb{A}, +, \cdot, \prec)$?

3.2 Groups with infinite linearly ordered products

In analogy with valuation theory in commutative algebra (see e.g. [25] and [23]), we expect that elements f in sufficiently large groups of o-minimal germs should have compositional asymptotic expansions

$$f \approx \mathfrak{s}_0^{[r_0]} \circ \cdots \circ \mathfrak{s}_\gamma^{[r_\gamma]} \circ \cdots, \gamma < \lambda$$

where $(\mathfrak{s}_\gamma)_{\gamma<\lambda}$ is a scale of functions that is strictly decreasing in rate of growth, $(r_\gamma)_{\gamma<\lambda}$ is a sequence of non-zero real numbers, and $\mathfrak{s}_\gamma^{[r_\gamma]}$ denotes the r_γ-th real iterate of $\mathfrak{s}_\gamma$. The formalisation of such asymptotic expansions was done in the general cases of growth order groups and valued groups, in [10, 6], in the form of a theory of scales and pseudo-Cauchy sequences. We lack a formalisation of the expansions themselves that wold be the non-commutative analogue of fields of formal series, i.e. a group of formal non-commutative series $\mathfrak{s}_0^{[r_0]} \cdots \mathfrak{s}_\gamma^{[r_\gamma]} \cdots$ whose group operation depends only on the assignment $\mathfrak{s}_\gamma \mapsto r_\gamma$:

Question E. Given a linearly ordered set $(I, <)$ and a family $(\mathcal{C}_i)_{i \in I}$ of Abelian ordered groups, under what conditions can one define a group law $*$ on the set $\mathbf{H}_{i \in I} \mathcal{C}_i$ of functions $f \in \Pi_{i \in I} \mathcal{C}_i$ with anti-well-ordered support $\mathrm{supp}\, f = \{i \in I : f(i) \neq 0_{\mathcal{C}_i}\}$, ordered lexicographically, such that the ordered group $(\mathbf{H}_{i \in I} \mathcal{C}_i, *, 1, <)$ is a growth order group?

We proposed [8] a framework for studying **infinite linearly ordered products** in general groups. We were able to show that some classical groups of formal series can be endowed with such infinite products in a canonical way, and showed that this gives a canonical representation of their elements as formal non-commutative series:

Theorem 3.2. (consequence of [8, Theorems 4 and 5]) *Let C be an ordered field, and let $\mathcal{P}_C$ be the group under composition of power series s with anti-well-ordered support, with exponents and coefficients in C, that are tangent to the identity. Then one can define transfinite linearly ordered products on $\mathcal{P}_C$, and for each $s \in \mathcal{P}_C$, there is a unique map $c : C \longrightarrow C$ with anti-well-ordered support with $s = \prod_{e \in c} (x + x^e)^{c(e)}$.*

3.3 Groups of formal series

In forthcoming work, relying on a study of Taylor expansions for functions defined on fields of formal series in progress with V. L. MANTOVA, we will show how to use infinite compositions of formal series in order to study properties of sets of formal series with composition laws.

Work in progress. *The set of positive infinite finitely nested series of* [4] *is a group. Moreover, any such series can be represented uniquely as a possibly transfinite composition as in Theorem 3.2.*

The same proof will apply to **No** provided the composition law is defined.

3.4 Conjugacy, resonance, and asymptotic differential algebra

In order to investigate tame first-order theories of nearly Abelian c-valued groups (see Section 2.2), one needs to find such groups where definable sets are simple in a geometric sense. A sound condition is that discreteness should only come from Abelian subgroups[3.1].

However, the phenomenon of **Poincaré resonance** (see [18]) in normalising formal local objects such as germs of vector fields and diffeomorphisms, precludes this. In the case of plain parabolic formal power series $s = x + s_0 + s_1 x^{-1} + s_2 x^{-2} + \cdots$ in $\mathbb{C}[\![x]\!]$, we say that there is resonance when the the shortest Laurent polynomial $p = x + p_0 + p_1 x^{-1} + p_2 x^{-2} + \cdots + p_n x^{-n}$ which is a conjugate of s is not the truncation $x + s_k x^{-k}$ of s consisting in its first two non-zero terms, but contains an additional *resonant* term $c\,x^{-n}$ which depends on a longer truncation of s. From a combinatorial standpoint, this means that the term $c\,x^{-n}$ cannot be eliminated by elementary operations of conjugation. From the standpoint of the model theory of valued groups, this entails that the set of parabolic power series that are conjugates of s has infinitely many definable connected components, which do not come from an Abelian subgroup.

For us, this means that such groups as that of parabolic power series are to be avoided. The reoccurrence of resonance was an unpleasant hurdle in studying valued groups, until we found a way to interpret it (in certain prominent cases) as a property of the underlying valued differential fields of series. Working on the Lie algebra side of the formal Lie correspondence, we were able [7] to formalise resonance for certain groups of transseries and to establish an equivalence between the non-existence of resonance and the existence of asymptotic integrals (see [1, p 327]). Our equivalence was only proved over certain fields of classical transseries. When the composition law $\circ$ and derivation ∂ are defined on **No**, we wish to extend it thus:

Conjecture F. *Let $\mathbb{S}$ be a direct limit of spherically complete differential subfields of $(\mathbf{No}, \partial)$, and suppose that the set $\mathbb{G}$ of parabolic elements in $\mathbb{S}$ is a nearly Abelian subgroup of the c-valued group of all parabolic numbers. Then conjugacy in $\mathbb{G}$ is resonance-free if and only if $\mathbb{S}$ has asymptotic integration.*

4 Hyperexponential and nested functions

Lastly, we turn to more concrete and geometric questions regarding the analytic content of the calculus of hyperseries.

4.1 Real hyperexponentiation

One long-standing open problem in o-minimality is the existence of a **transexponential o-minimal expansion** of the real ordered exponential field, i.e. an expansion which defines a unary function growing faster than all iterates of exp. Abel's equation in f

$$f(t+1) = \exp(f(t)) \quad \text{for large enough } t \in \mathbb{R} \tag{4.1}$$

3.1. in (non-Abelian) GOGs, centralisers of non-trivial elements are discrete Abelian subgroups, sometimes isomorphic to $(\mathbb{Z}, +, 0, <)$, $(\mathbb{Q}, +, 0, <)$ or $(\mathbb{R}, +, 0, <)\ldots$

is the simplest functional equation whose solutions [24] in Hardy fields [15] are transexponential.

On the real-analytic side, Kneser's solution E to (4.1) on $[0,+\infty)$ is a natural candidate for these o-minimal investigations. On the formal-surreal side, I hope that the calculus of hyperseries on **No** faithfully represents asymptotic properties of E. Recently, A. Padgett studied in her PhD thesis [30] a first-order theory T_{transexp} for $(\mathbb{R},+,\cdot,E)$, in a language $\mathcal{L}_{\text{transexp}}$, and showed that the set $\mathcal{H}_{\text{transexp}}$ of germs of unary terms in $\mathcal{L}_{\text{transexp}}$ is a Hardy field. The field $\tilde{\mathbb{L}} \subseteq \mathbf{No}$ of finitely nested hyperseries [13], with the hyperexponential function corresponding to e_ω^x is a model of T_{transexp}. A first step toward studying the relationship between the surreal/formal model $\tilde{\mathbb{L}}$ and the geometric/analytic model $(\mathbb{R},E)$ is to prove that there is a natural inclusion

$$\begin{aligned} \mathrm{ev}_\omega : \mathcal{H}_{\text{transexp}} &\longrightarrow \tilde{\mathbb{L}} \\ \mathrm{germ}(t(x)) &\longmapsto t(\omega). \end{aligned}$$

With A. Padgett and E. Kaplan, we started working on the following conjectures:

Conjecture G. *The function* ev_ω *is a well-defined* $\mathcal{L}_{\text{transexp}}$*-embedding.*

We defined [4] a derivation and composition law on $\tilde{\mathbb{L}}$. we also expect that:

Conjecture H. *The function* ev_ω *commutes with the derivation and composition laws on* $\mathcal{H}_{\text{transexp}}$ *and* $\tilde{\mathbb{L}}$.

4.2 Nested germs

A unique feature of surreal numbers is that they naturally contain nested numbers, e.g. numbers whose expansion as a hyperseries is

$$\sqrt{\omega}+\mathrm{e}^{\sqrt{\log \omega}+\mathrm{e}^{\sqrt{\log\log\omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}, \tag{4.2}$$

where $\omega \in \mathbf{No}$ is a surreal number that plays the role of a variable at infinity.

On the analytic side, the functional equation

$$g(t)=\sqrt{t}+\mathrm{e}^{g(\log(t))} \qquad \text{with} \quad g(t)\sim\sqrt{t} \quad \text{at } +\infty \tag{4.3}$$

naturally generates germs which, when represented using logarithmico-exponential terms, expand in a similar way as (4.2). There are good reasons [3] to believe that the behavior of differential polynomials on these nested germs is the same as their behavior on the corresponding nested numbers/hyperseries. Exploiting this, I want to study how the functional equation (4.3) can be solved in Hardy fields:

Question I. What linear orderings can be represented by quasi-analytic sets of solutions of (4.3) lying in a common Hardy field?

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