Embedding Hardy fields with composition into transseries fields

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Operations on germs

Germs (at $+\infty$)

Identify two functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ if for large enough $t \in \mathbb{R}$, we have $f(t) = g(t)$. Equivalence classes are called **germs**. *$\mathcal{G}$ is the ring of germs with pointwise sum and product.*

Further structure for $f, g \in \mathcal{G}$.

Ordering. Write $f < g$ if $f(t) < g(t)$ for all $t \gg 1$. So $g > \mathbb{R}$ iff $\lim_{t \to +\infty} g(t) = +\infty$.

Dominance relation. Write $f \prec g$ if $r\,|f| < |g|$ for all $r \in \mathbb{R}^{>0}$.

Derivation. If $t \mapsto f(t)$ is differentiable on $(r, +\infty)$ for $r \gg 1$, then f' is the germ of the derivative.

Composition. If $g > \mathbb{R}$, then the germ $f \circ g$ of $t \mapsto f(g(t))$ only depends on f and g. We have a *composition law*

$$\circ\colon \mathcal{G} \times \mathcal{G}^{>\mathbb{R}} \longrightarrow \mathcal{G}.$$

For fixed $g \in \mathcal{G}^{>\mathbb{R}}$, the function $\mathcal{G} \longrightarrow \mathcal{G};\ f \mapsto f \circ g$ is a strictly increasing morphism of rings.

Hardy fields with composition

Hardy fields

A **Hardy field** *is a subfield of $\mathcal{G}$ which is closed under derivation of germs.*

Hardy fields with composition

A **Hardy field with composition** *is a Hardy field $\mathcal{H}$ which contains id and which is closed under compositions $(f, g) \mapsto f \circ g$ with $f \in \mathcal{H}$ and $g \in \mathcal{H}^{>\mathbb{R}}$.*

For fixed f, the map $g \mapsto f \circ g$ is either constant or strictly monotonous.

Terms and definable functions

Let $\mathcal{R} = (\mathbb{R}, +, -, \times, {}^{-1}, <, \ldots)$ be a *functional* expansion of the real ordered field in a first-order language $\mathcal{L}$. We define

- $\mathcal{H}_{\mathcal{R}}$ as the set of germs of functions $\mathbb{R} \longrightarrow \mathbb{R}$ that are *definable* with parameters in $\mathcal{R}$.
- $\mathcal{T}_{\mathcal{R}}$ as the subset of $\mathcal{H}$ of germs of unary functions $r \mapsto t(r)$ for all arity $\leqslant 1$ *terms* $t[u]$.

Those are partially ordered rings, and even $\mathcal{L}$-structures under pointwise operations.

In general they are just that, and we have $\mathcal{T}_{\mathcal{R}} \subsetneq \mathcal{H}_{\mathcal{R}}$.

If $f, g \in \mathcal{G}$ are given by $\mathcal{L}$-terms / are definable in $\mathcal{R}$, then so is $f \circ g$. So $\circ\colon \mathcal{G} \times \mathcal{G}^{>\mathbb{R}} \longrightarrow \mathcal{G}$ induces composition laws

$$\circ\colon \mathcal{H}_{\mathcal{R}} \times \mathcal{H}_{\mathcal{R}}^{>\mathbb{R}} \longrightarrow \mathcal{H}_{\mathcal{R}}.$$
$$\circ\colon \mathcal{T}_{\mathcal{R}} \times \mathcal{T}_{\mathcal{R}}^{>\mathbb{R}} \longrightarrow \mathcal{T}_{\mathcal{R}}.$$

Model theoretic perks

If $\mathrm{Th}(\mathcal{R})$ is o-minimal

Then $\mathcal{H}_{\mathcal{R}}$ is a Hardy field with composition.

$\mathcal{M} \succ \mathcal{R}$: non-standard model

ξ: element of $\mathcal{M}$ with $\xi > \mathbb{R}$.

There is a natural $\mathcal{L}$-embedding $\Psi: \mathcal{H}_{\mathcal{R}} \longrightarrow \mathcal{M}$ which sends id to ξ. This map commutes with definable functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

There is a unique $\mathcal{L}$-embedding $\Phi: \mathcal{T}_{\mathcal{R}} \longrightarrow \mathcal{M}$ with $\Phi(\mathrm{id}) = \xi$.

If $\mathrm{Th}(\mathcal{R})$ has QE and a universal axiomatization

Then any definable function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is given piecewise by (a finite list of) terms. In particular for $n = 1$, the germ of f lies in $\mathcal{T}_{\mathcal{R}}$. So $\mathcal{T}_{\mathcal{R}} = \mathcal{H}_{\mathcal{R}}$.

Both conditions are satisfied for $\mathcal{R} = \mathbb{R}_{\mathrm{an},\exp}$ [cite vdDMM].

A short-lived conjecture?

Conjecture on $\mathcal{H}^{>\mathbb{R}}$

Let $\mathcal{H}$ be a Hardy field with composition and let $f, g \in \mathcal{H}^{>\mathbb{R}}$. Then:

Conjecture 1. *For all $\delta \in \mathcal{H}$ with $\delta \prec g$ and $(f' \circ g)\,\delta \prec (f \circ g)$, we have*

$$f \circ (g + \delta) \sim f \circ g.$$

Conjecture 2. *For all $h \in \mathcal{H}^{>\mathbb{R}}$ with $h \neq \mathrm{id}$, we have*

$$(f \circ h = h \circ f \wedge g \circ h = h \circ g) \Longrightarrow f \circ g = g \circ f.$$

Conjecture 3. *If $f > \underset{\text{(iterates)}}{g^{[\mathbb{N}]}} > \mathrm{id}$, then $f \circ g > g \circ f$.*

This conjecture holds for $\mathcal{H}_{\mathrm{an,exp}}$. **Conjecture 1** holds whenever $\mathcal{H}$ contains $\exp$, but has no transexponential germ.

$\mathfrak{L}$: group of germs $\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}$ where $\mathfrak{l}_0, \ldots, \mathfrak{l}_{n-1} \in \mathbb{Z}$. The ordering $\prec$ on $\mathfrak{L}$ is lexicographic.

Logarithmic transseries

$\mathbb{T}_{\log}$ *is the field* $\mathbb{R}[[\mathfrak{L}]]$ *of Hahn series with real coefficients and monomial group* $\mathfrak{L}$*. E.g.*

$$f_0 = x + \pi (\log x)^3 + \frac{1}{\log x} + \frac{1}{2 (\log x)^2} + \frac{1}{3 (\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$

$$f_1 = x + \log x + \log_2 x + \cdots$$

We have a logarithm function $\log \left(\prod_{k<n} (\log_k x)^{\mathfrak{l}_k}\right) := \sum_{k<n} \mathfrak{l}_k \log_{k+1} x$*.*

Exponential extensions

For $n\in\mathbb{N}$, *we define* $\mathbb{T}_n=\mathbb{R}[[\mathfrak{G}_n]]$ *inductively. Set* $\mathfrak{G}_0:=\mathfrak{L}$ *and* $\mathfrak{G}_{n+1}:=\mathrm{e}^{(\mathbb{T}_n)^{\uparrow}}$ *where*

- $(\mathbb{T}_n)^{\uparrow}$ *is the subgroup of* $(\mathbb{T}_n,+,0)$ *of series with only infinite terms.*
- $\mathrm{e}^{(\mathbb{T}_n)^{\uparrow}}$ *is a multiplicative copy of* $((\mathbb{T}_n)^{\uparrow},+,<)$ *with a natural inclusion* $\mathfrak{G}_n\hookrightarrow\mathrm{e}^{(\mathbb{T}_n)^{\uparrow}};\mathfrak{m}\mapsto\mathrm{e}^{\log\mathfrak{m}}$.

The function $\log$ *extends inductively to* $\mathrm{e}^{(\mathbb{T}_n)^{\uparrow}}$ *by setting* $\log\mathrm{e}^{\varphi}:=\varphi$ *for* $\varphi\in\mathrm{e}^{(\mathbb{T}_n)^{\uparrow}}$.

Transseries with derivation and composition

Transseries, introduced by Dahn-Göring and Ecalle, can be declined in several forms:

Name	Symbol	Introduced by
grid-based	$\mathbb{T}_g$	Ecalle
LE series	$\mathbb{T}^{\mathrm{LE}}$	Ecalle, vdDries-Macintyre-Marker
EL series	$\mathbb{T}^{\mathrm{EL}} = \bigcup_{n \in \mathbb{N}} \mathbb{T}_n$	Kuhlmann-Tressl
generalized	$\mathbb{R}\langle\langle\omega\rangle\rangle$	vdHoeven, Schmeling, Berarducci-Mantova

Transseries as a differential field with composition

Each of those fields $\mathbb{T}$ is equipped with a derivation $\partial \colon \mathbb{T} \longrightarrow \mathbb{T}$ and a composition law $\circ \colon \mathbb{T} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$ whose properties mimic those of Hardy fields with composition.

$$f_0 = x + \pi (\log x) + \frac{1}{\log x} + \frac{1}{2 (\log x)^2} + \frac{1}{3 (\log x)^3} + \cdots + \frac{2 (\log x)^2}{x}$$
$$f_0' = 1 + \frac{\pi}{x} - \frac{1}{x (\log x)^2} - \frac{1}{x (\log x)^3} - \cdots - \frac{4 x (\log x) - 2 x (\log x)^2}{x^2}$$

$$f_1 = x + \log x + \log_2 x + \cdots$$
$$f_1 \circ (\log x) = \log x + \log_2 x + \cdots$$

There is a unique $\mathcal{L}_{\mathrm{an,exp}}$-embedding $\mathcal{H}_{\mathrm{an,exp}} \longrightarrow \mathbb{T}$ which sends id to x. This preserves ∂ and $\circ$.

[functions $\exp_\omega$ and $\log_\omega$, also $\exp^{[r]}$]

[Adele Padgett's Hardy field]

EL classes

Given $F \models \mathbb{R}_{\exp}$, $a \in F^{>\mathbb{R}}$, the **exp-log class** $\mathrm{EL}(a)$ *of $a \in F$ is its equivalence class for*

$$a \asymp^L b \qquad \textit{if and only if} \qquad \exists n \in \mathbb{N}, (\log^{[n]} a \sim \log^{[n]} b).$$

Assume that $\mathcal{H}$ is a Hardy field with composition with $\exp, \log \in \mathcal{H}$. Set

$$\mathcal{E} := \{\exp^{[n]} \circ (\log^{[n]} \pm 1) : n \in \mathbb{N}\} \subseteq \mathcal{H}^{>\mathbb{R}}.$$

Then each $\mathrm{EL}(f)$ is the convex hull of $\mathcal{E} \circ f = \{g \circ f : g \in \mathcal{E}\}$.

Marker-Miller: EL classes in $\mathcal{H}_{\mathrm{an,exp}}$ are parametrized by integers. Each $f \in \mathcal{H}_{\mathrm{an,exp}}$ lies in $\mathrm{EL}(\exp^{[n]})$ for a unique $n \in \mathbb{Z}$ called the ***level*** of f. Write $\lambda_n = \mathrm{EL}(f)$. Note that

$$\log \lambda_n = \lambda_{n-1}$$

for all $n \in \mathbb{Z}$.

More levels

Fix $r \in \mathbb{R}$. Write

$$\begin{aligned} \lambda_\omega &:= \mathrm{EL}(\exp_\omega) && (\text{in } \mathbb{R}_E) \\ \lambda_{-\omega} &:= \mathrm{EL}(\log_\omega) && (\text{in } \mathbb{R}_L) \\ \lambda_r &:= \mathrm{EL}(\exp^{[r]}) && (\text{in } \mathbb{R}_{\exp^{[r]}}) \end{aligned}$$

We have

$$\lambda_\omega > \lambda_{\mathbb{Z}}, \qquad \lambda_{-\omega} < \lambda_{\mathbb{Z}}, \qquad \text{and} \quad \forall r, s \in \mathbb{R}, (\lambda_r < \lambda_s \Longleftrightarrow r < s).$$

We also have levels $\lambda_{\omega-1}, \lambda_{\omega+1}$ with $\lambda_{\mathbb{Z}} < \lambda_{\omega-1} < \lambda_\omega < \lambda_{\omega+1}$, and so on...

For $\varphi, \psi \in \mathcal{H}_P^{>\mathbb{R}}$, we have

$$\mathrm{EL}(\exp_\omega \circ \varphi) < \mathrm{EL}(\exp_\omega \circ \psi) \iff \mathcal{E} \circ (\exp_\omega \circ \varphi) < \mathcal{E} \circ (\exp_\omega \circ \psi)$$
$$\iff (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \varphi < (\log_\omega \circ \mathcal{E} \circ \exp_\omega) \circ \psi$$

Set $g = \exp^{[n]} \circ (\log^{[n]} + 1) \in \mathcal{E}$. Since $\log_\omega' \approx 1/\mathrm{id}$, the mean value theorem for $\log_\omega$ gives

$$\mathrm{id} + \frac{1}{\log^{[n]} \circ \exp_\omega} < \log_\omega \circ g \circ \exp_\omega < \mathrm{id} + \frac{1}{\log^{[n+1]} \circ \exp_\omega}.$$

Thus if $\varphi + \left(\frac{1}{\log^{[\mathbb{N}]} \circ \exp_\omega \circ \varphi}\right) < \psi$, then $\exp_\omega \circ \varphi$ and $\exp_\omega \circ \psi$ lie in disinct levels. The EL class $\lambda_{1/\omega}$ of

$$\exp_\omega \circ \left(\log_\omega + \frac{1}{\log_\omega}\right)$$

is "infinitesimal", i.e. larger than λ_0 but smaller than each λ_r for $r \in \mathbb{R}^>$.

All levels

Tentative description of all possible levels in models of $\mathbb{R}_{\exp}$ using Conway's field $\mathbf{No}$ of surreal numbers:

Theorem (Berarducci-Mantova, 2015)

EL *classes in* $(\mathbf{No}, \exp)$ *are in canonical order isomorphism with* $(\mathbf{No}, <)$ *itself.*

There is an order embedding $\mathbf{No} \longrightarrow \mathbf{No}^{>\mathbb{R}}, z \mapsto \lambda_z$ *such that each surreal number lies in* $\mathrm{EL}(\lambda_z)$ *for a unique* $z \in \mathbf{No}$*.*

They defined a canonical derivation ∂ on $\mathbf{No}$ such that $(\mathbf{No}, \partial)$ is a Liouville-closed H-field. It is even (ADH) an elementary extension of $\mathbb{T}$.

Theorems

All Hardy fields closed under $\exp, \log$ *embed into* $\mathbf{No}$ *as*

ordered exponential fields ?	*Yes*
ordered valued differential fields ?	*Yes (ADH,)*
ordered valued exponential differential fields ?	*Unknown*

Goal

Embedding Hardy fields with composition into differential fields of generalized transseries with composition.

BERARDUCCI-MANTOVA defined a composition law

$$\circ\colon \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}.$$

Where $\lambda_{\mathbb{Z}} \subseteq \mathbb{R}\langle\langle\omega\rangle\rangle \subseteq \mathbf{No}$ is closed under $\log$, $\exp$, and transfinite sums.

But they showed that this composition does not extend to $\mathbf{No} \times \mathbf{No}^{>\mathbb{R}}$ in a compatible way with respect to ∂. So let us leave the surreal realm for now.

$\longrightarrow$ Can we "directly" build a field of generalized transseries involving transexponential and sublogarithmic behavior so as to include infinite and infinitesimal levels?

$\longrightarrow$ Can we do so while solving the functional equations

$$\exp_\omega \circ (\mathrm{id} + 1) = \exp \circ \exp_\omega \qquad \text{and} \qquad \log_\omega - 1 = \log_\omega \circ \log \ ?$$

$\longrightarrow$ Could we also get formal versions of the conjecture for those fields?

Let us build a structure $(\mathbb{L}, \partial, \circ)$ which contains a solution $L_\omega\, x$ to

$$L_\omega\, x - 1 = (L_\omega\, x) \circ (L_1\, x). \tag{1}$$

We first gather symbols $L_\gamma\, x, \gamma < \omega^2$ with $L_1\, x \eqqcolon \log x$ and

$$\forall m, n \in \mathbb{N}, L_{\omega m + n}\, x = (L_1\, x)^{[n]} \circ (L_\omega\, x)^{[m]}$$

Differentiating (1), we get

$$\begin{aligned}
(L_\omega\, x)' &= \frac{1}{x} \times (L_\omega\, x)' \circ (L_1\, x) \\
&= \frac{1}{x \log x} \times (L_\omega\, x)' \circ (L_2\, x) \\
&= \frac{1}{x\,(L_1\, x) \cdots (L_n\, x)} \times (L_\omega\, x)' \circ (L_{n+1}\, x) \\
&= \ldots \\
&\overset{?}{=} \prod_{n<\omega} (L_n\, x)^{-1}.
\end{aligned}$$

So one needs to have, as basic symbols, formal products

$$\mathfrak{l} := \prod_{\gamma < \omega^2} (L_\gamma x)^{\mathfrak{l}_\gamma}, \quad \text{for } (\mathfrak{l}_\gamma)_{\gamma < \omega^2} \in \mathbb{R}^{\omega^2}$$

Gathering those in a lexicographically ordered group $\mathfrak{L}_{<\omega^2}$ yields a Hahn field $\mathbb{L}_{<\omega^2} = \mathbb{R}[[\mathfrak{L}_{<\omega^2}]]$.

Thinking of $\mathfrak{l} \in \mathfrak{L}_{<\omega^2}$ as the exponential $\mathrm{e}^{\sum_{\gamma<\omega^2} \mathfrak{l}_\gamma (L_{\gamma+1} x)}$, we set

$$\mathfrak{l}' = (\log \mathfrak{l})' \, \mathfrak{l} = \left(\sum_{\gamma < \omega^2} \mathfrak{l}_\gamma \, (L_{\gamma+1} \, x)' \right) \cdot \mathfrak{l}.$$

This extends into a well-defined derivation ∂ on $\mathbb{L}_{<\omega^2}$. The same rule applies for any ordinal α instead of ω^2, yielding a field $\mathbb{L}_{<\alpha}$, and a class-sized field $\mathbb{L} := \bigcup_{\alpha \in \mathbf{On}} \mathbb{L}_{<\alpha}$.

Theorem [van den Dries, Kaplan, van der Hoeven - 2018]

There is a composition law $\circ : \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ *with* $L_{\omega^{\mu+1}} x - 1 = (L_{\omega^{\mu+1}} x) \circ (L_{\omega^{\mu}} x)$ *for all ordinals* μ. $(\mathbb{L}, \partial)$ *is an H-field with small derivation and integration. We have the chain rule for* $(\circ, \partial)$. *The formal version of* **Conjecture 1** *holds in* $\mathbb{L}$.

Hyperexponential closure

Write L_ω for the strictly increasing function $\mathbb{L}_{<\omega^2}^{>\mathbb{R}} \longrightarrow \mathbb{L}_{<\omega^2}^{>\mathbb{R}}; f \mapsto \ell_\omega \circ f$.

Its right inverse E_ω is partially defined, e.g. $E_\omega(x)$ is undefined. In order to close $\mathbb{L}_{<\omega^2}^{>\mathbb{R}}$ under E_ω, we adjoin formal monomials $\mathrm{e}_\omega^\varphi$ to $\mathfrak{L}_{<\omega^2}$, for certain $\varphi \in \mathbb{L}_{<\omega^2} \setminus L_\omega(\mathbb{L}_{<\omega^2}^{>\mathbb{R}})$.

When should $E_\omega(\varphi)$ be a new monomial $\mathrm{e}_\omega^\varphi$? **Idea**: If $E_\omega(\varphi)$ is defined and $\varepsilon \prec \frac{1}{(L_n x) \circ E_\omega(\varphi)}$ for some $n \in \mathbb{N}$, then $E_\omega(\varphi + \varepsilon)$ is defined using Taylor expansions and exponentiation.

So it is enough to add $\mathrm{e}_\omega^\varphi$ for representatives φ in each convex hull

$$\mathcal{L}(g) := \mathrm{Conv}\left(\left\{g \pm \frac{1}{(L_n x) \circ E_\omega(\varphi)} : n \in \mathbb{N}\right\}\right).$$

Those can be defined *without reference to* E_ω. For any two distinct representatives φ, ψ, the EL-classes of $\mathrm{e}_\omega^\varphi$ and e_ω^ψ should be disjoint. This determines an ordering of the extension of $\mathfrak{L}_{<\omega^2}$ by monomials $\mathrm{e}_\omega^\varphi$.

Extending derivations and compositions

Theorem [B.-vdHoeven-Kaplan]

There is a minimal extension $\tilde{\mathbb{L}}$ *of* $\mathbb{L}$, *and an extension* $\circ\colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ *of the composition law on* $\mathbb{L}$, *for which each* $\tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}; g \mapsto \ell_\gamma \circ g$ *for* $\gamma \in \mathbf{On}$ *is bijective.*

$\tilde{\mathbb{L}}$ is obtained by iteratively adjoining hyperexponentials $E_{\omega^\mu}(\varphi)$ of hyperseries φ, and taking an increasing union indexed by all ordinals.

So any $f \in \tilde{\mathbb{L}}$ has a concrete expression involving $L_\gamma\, x$'s, E_γ's, real numbers, and transfinite sums. E.g.

Work in progress [B.]

There is a derivation $\tilde{\partial}\colon \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ *such that* $(\tilde{\mathbb{L}}, \partial)$ *is an elementary extension of the field* $\mathbb{T}$ *of* $\mathrm{log\text{-}exp}$ *transseries.*

Work in progress [B.]

There is a composition law $\tilde{\circ}\colon \tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ *such that* $(\tilde{\partial}, \tilde{\circ})$ *satisfies the chain rule.*

Strongly linear algebra

Problem: making sense of those complicated transfinite sums. We use strongly linear algebra: a set of order theoretic results regarding a formal summability notion.

Idea: Hahn series fields are "formal" Banach spaces. Two illustrations (van der Hoeven):

- If $\Psi \colon \mathbb{R}[[\mathfrak{M}]] \longrightarrow \mathbb{R}[[\mathfrak{M}]]$ is strongly linear and *contracting*, i.e. $\Psi(s) \prec s$ for all $s \in \mathbb{R}[[\mathfrak{M}]]^{\neq}$, then $\mathrm{Id} + \Psi$ has a strongly linear functional inverse

$$(\mathrm{Id} + \Psi)^{[-1]}(s) = \sum_{k \in \mathbb{N}} (-1)^k \, \Psi^{[k]}(s).$$

- We have a strongly linear implicit function theorem.

The ordered group $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$

Work in progress [B.]

The class $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$ *is a linearly bi-ordered group:* $f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ *has an inverse in* $\tilde{\mathbb{L}}^{>\mathbb{R}}$ *and each function* $\tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}}; g \mapsto f \circ g$ *is strictly increasing.*

Work in progress [B.]

Any two series $f, g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ *with* $f, g > x$ *are conjugate, i.e. satisfy*

$$V \circ f = g \circ V$$

for a certain $V \in \tilde{\mathbb{L}}^{>\mathbb{R}}$*. E.g.* e^x *and* $x+1$ *are conjugate via* $(L_\omega x) \circ \mathrm{e}^x = (x+1) \circ L_\omega x$.

Real iterates: for each $f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ with $f \neq x$ there is a unique isomorphism

$$(\mathbb{R}, +, <) \longrightarrow (\mathcal{C}(f), \circ, <); r \mapsto f^{[r]}$$

with $f^{[1]} = f$.

What embeddings tell us

Work in progress [B.]

For all $f, g, \delta \in \tilde{\mathbb{L}}$ *with* $g > \mathbb{R}$*, if* $\delta \prec g$ *and* $(\mathfrak{m}' \circ g)\,\delta \prec \mathfrak{m} \circ g$ *for all* $\mathfrak{m} \in \operatorname{supp} f$*, then*

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ g}{k!} \delta^k,$$

where $f \circ g \succ (f' \circ g)\,\delta \succ (f'' \circ g)\,\delta^2 \succ \cdots$.

Work in progress [B.]

For all $f, g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ *with* $f, g \neq x$*, we have* $f \in \mathcal{C}(g) \Longleftrightarrow \mathcal{C}(f) = \mathcal{C}(g)$.

Work in progress [B.]

For all $f, g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ $f > g^{[\mathbb{N}]} > x$*, we have* $f \circ g > g \circ f$.

In particular, any Hardy field with composition which embeds into $\tilde{\mathbb{L}}$ satisfies the conjecture.

Linearly bi-ordered monoids and groups

Any Hardy field with composition $\mathcal{H}$ induces a linearly bi-ordered monoid $(\mathcal{H}^{>\mathbb{R}}, \circ, \mathrm{id}, <)$.

Proposition

Let $\mathcal{M} = (M, \cdot, 1, <)$ be a linearly bi-ordered monoid, let $f, g \in M^{>1}$ and $m, n \in \mathbb{N}^{>}$ with

$$f^m g^n = g^n f^m.$$

Then $f g = g f$. In particular if $f^m = g^n$, then $f g = g f$.

If furthermore M is a group, then given $f, g, h \in M$ with $f^m = h^m = g^n$, we have $f = h$, so we can write $f = g^{[n/m]}$ and consider it as *the* ${}^{n}/_{m}$'th fractional iterate of g.

Question 1. Is $\mathcal{M}$ always contained in a linearly bi-ordered group?

Question 2. Is $\mathcal{H}$ always contained in a Hardy field with composition and with functional inverses for all positive infinite germs? [I think not: find counter example in ADH:icm]