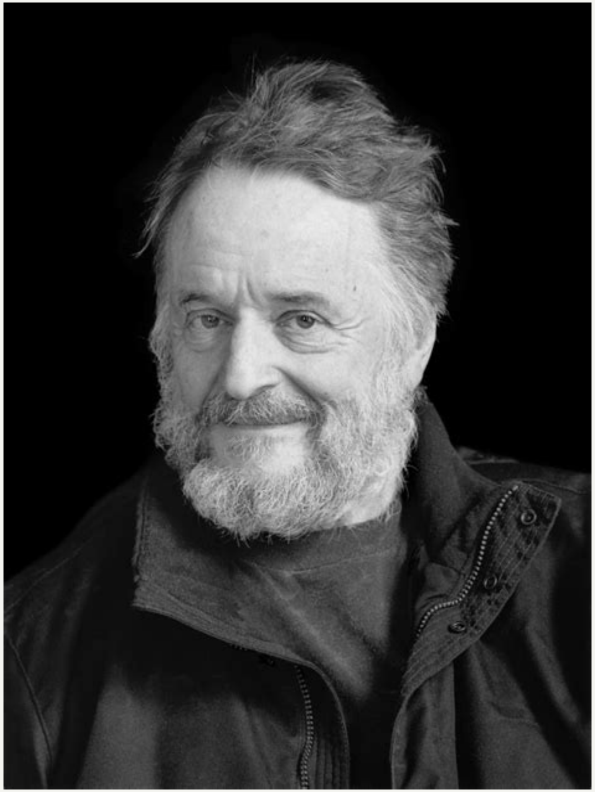
Du Bois Reymond *à la* Conway (the game)

by Vincent Bagayoko (IMJ-PRG, Paris)

Colloquium Logicum, ÖAW Vienna, 10-09-24



Germs and numbers

Germs / Hardy fields	Numbers
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b,\, g(t)=f(t)\}$	

Germs and numbers

Germs / Hardy fields	**Numbers**
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b,\, g(t)=f(t)\}$	cuts $\{L \mid R\}$ where $L<R$ for sets L, R of previously defined numbers

Germs and numbers

Germs / Hardy fields	**Numbers**
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. Germ: $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b, g(t)=f(t)\}$	cuts $\{L \mid R\}$ where $L<R$ for sets L,R of previously defined numbers
$[f]\leqslant[g]$ if $f(t)<g(t)$ for all sufficiently large t	$\{L \mid R\}\leqslant\{L' \mid R'\}$ if $L<\{L' \mid R'\}$ and $\{L \mid R\}<R'$

Germs and numbers

Germs / Hardy fields	**Numbers**
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. Germ: $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b, g(t)=f(t)\}$	cuts $\{L \mid R\}$ where $L<R$ for sets L,R of previously defined numbers
$[f]\leqslant[g]$ if $f(t)<g(t)$ for all sufficiently large t	$\{L \mid R\}\leqslant\{L' \mid R'\}$ if $L<\{L' \mid R'\}$ and $\{L \mid R\}<R'$
pointwise sum, product, derivation, integration, composition of germs, taking limits, solving algebraic, differential, functional equations...	

Germs and numbers

Germs / Hardy fields	**Numbers**
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. Germ: $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b,g(t)=f(t)\}$	cuts $\{L \mid R\}$ where $L<R$ for sets L,R of previously defined numbers
$[f]\leqslant[g]$ if $f(t)<g(t)$ for all sufficiently large t	$\{L \mid R\}\leqslant\{L' \mid R'\}$ if $L<\{L' \mid R'\}$ and $\{L \mid R\}<R'$
pointwise sum, product, derivation, integration, composition of germs, taking limits, solving algebraic, differential, functional equations...	inductively defined sum and product inductively defined infinite sums

Germs and numbers

Germs / Hardy fields	Numbers
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. Germ: $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b,g(t)=f(t)\}$	cuts $\{L \mid R\}$ where $L<R$ for sets L,R of previously defined numbers
$[f]\leqslant[g]$ if $f(t)<g(t) \quad \text{for all sufficiently large } t$	$\{L \mid R\}\leqslant\{L' \mid R'\}$ if $L<\{L' \mid R'\} \quad \text{and} \quad \{L \mid R\}<R'$
pointwise sum, product, derivation, integration, composition of germs, taking limits, solving algebraic, differential, functional equations...	inductively defined sum and product inductively defined infinite sums
Given a differential subfield $\mathcal{H}$ (i.e. a Hardy field) and countable subsets $A,B\subseteq\mathcal{H}$ with $A<B$, existence of an $f\in\mathcal{H}^*\supseteq\mathcal{H}$ with $A<f<B.$	given sets A,B of numbers, existence of a simplest number $\{A \mid B\}$ with $A<\{A \mid B\}<B.$

Germs and numbers

Germs / Hardy fields	Numbers
$f:(a,+\infty)\longrightarrow\mathbb{R}$, class $\mathcal{C}^k$ on $(a_k,+\infty)$ for all $k\in\mathbb{N}$. Germ: $[f]=\{g:(a,+\infty):\exists b\in\mathbb{R},\forall t>b, g(t)=f(t)\}$	cuts $\{L \mid R\}$ where $L<R$ for sets L,R of previously defined numbers
$[f]\leqslant[g]$ if $f(t)<g(t)$ for all sufficiently large t	$\{L \mid R\}\leqslant\{L' \mid R'\}$ if $L<\{L' \mid R'\}$ and $\{L \mid R\}<R'$
pointwise sum, product, derivation, integration, composition of germs, taking limits, solving algebraic, differential, functional equations...	inductively defined sum and product inductively defined infinite sums
Given a differential subfield $\mathcal{H}$ (i.e. a Hardy field) and countable subsets $A,B\subseteq\mathcal{H}$ with $A<B$, existence of an $f\in\mathcal{H}^*\supseteq\mathcal{H}$ with $A<f<B$.	given sets A,B of numbers, existence of a simplest number $\{A \mid B\}$ with $A<\{A \mid B\}<B$.
In Hardy fields, germs $>\mathbb{R}$ are strictly increasing, have Taylor expansions and inverses	...

In fact, the class $\mathbf{No}$ of all numbers should be a differential field of surreal valued germs:

Van der Hoeven's conjecture

In fact, the class $\mathbf{No}$ of all numbers should be a differential field of surreal valued germs:

Conjecture. *There are a derivation*

$$\partial : \mathbf{No} \longrightarrow \mathbf{No}$$

and a composition law

$$\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$$

such that for each $a \in \mathbf{No}$, *the function* $\hat{a} : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *behaves like a germ in a Hardy field.*

In fact, the class $\mathbf{No}$ of all numbers should be a differential field of surreal valued germs:

Conjecture. *There are a derivation*

$$\partial : \mathbf{No} \longrightarrow \mathbf{No}$$

and a composition law

$$\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$$

such that for each $a \in \mathbf{No}$*, the function* $\hat{a} : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *behaves like a germ in a Hardy field.*

For instance, each $\hat{a}$ should be strictly monotonous and differentiable with $\hat{a}' = \widehat{\partial(a)}$, we should have Taylor expansions, and for fixed $\xi \in \mathbf{No}^{>\mathbb{R}}$, the function $a \mapsto a \circ \xi$ should be an endomorphism of $(\mathbf{No}, +, \cdot, <)$.

In fact, the class $\mathbf{No}$ of all numbers should be a differential field of surreal valued germs:

Conjecture. *There are a derivation*

$$\partial : \mathbf{No} \longrightarrow \mathbf{No}$$

and a composition law

$$\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$$

such that for each $a \in \mathbf{No}$, the function $\hat{a} : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ behaves like a germ in a Hardy field.

For instance, each $\hat{a}$ should be strictly monotonous and differentiable with $\hat{a}' = \widehat{\partial(a)}$, we should have Taylor expansions, and for fixed $\xi \in \mathbf{No}^{>\mathbb{R}}$, the function $a \mapsto a \circ \xi$ should be an endomorphism of $(\mathbf{No}, +, \cdot, <)$.

I have found a truly marvelous proof of that conjecture, which the margins of this short talks are too narrow to contain.

In fact, the class $\mathbf{No}$ of all numbers should be a differential field of surreal valued germs:

Conjecture. *There are a derivation*

$$\partial : \mathbf{No} \longrightarrow \mathbf{No}$$

and a composition law

$$\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$$

such that for each $a \in \mathbf{No}$, *the function* $\hat{a} : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *behaves like a germ in a Hardy field.*

For instance, each $\hat{a}$ should be strictly monotonous and differentiable with $\hat{a}' = \widehat{\partial(a)}$, we should have Taylor expansions, and for fixed $\xi \in \mathbf{No}^{>\mathbb{R}}$, the function $a \mapsto a \circ \xi$ should be an endomorphism of $(\mathbf{No}, +, \cdot, <)$.

I have found a truly marvelous proof of that conjecture, which the margins of this short talks are too narrow to contain.

Let's play a game instead.

Rules of the game:

"I have a slide for that one"

cui	dec	flip	ite	iwin	nop	them	ulose	vrai

Rules of the game:

- You "give" me a number a, such that you think the composition law $\circ$ could not be defined on $\{a\} \times \mathbf{No}^{>\mathbb{R}}$.

  You may give me a cut, an algebraic expression, some number you know how to present because of your own knowledge of surreal numbers[2].

"I have a slide for that one"

cui	dec	flip	ite	iwin	nop	them	ulose	vrai

2. Caveat: I may not be able to turn a sign sequence into the way I represent numbers on the spot, so don't be too mean with that.

Cut to the chase

Rules of the game:

- You "give" me a number a, such that you think the composition law $\circ$ could not be defined on $\{a\} \times \mathbf{No}^{>\mathbb{R}}$.

  You may give me a cut, an algebraic expression, some number you know how to present because of your own knowledge of surreal numbers[3].

- My goal is to convince you that I know what the function $\hat{a} : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}\,;\ \xi \mapsto a \circ \xi$ should be.

"I have a slide for that one"

cui	dec	flip	ite	iwin	nop	them	ulose	vrai

---

3. Caveat: I may not be able to turn a sign sequence into the way I represent numbers on the spot, so don't be too mean with that.

Cut to the chase

Rules of the game:

- You "give" me a number a, such that you think the composition law $\circ$ could not be defined on $\{a\} \times \mathbf{No}^{>\mathbb{R}}$.

  You may give me a cut, an algebraic expression, some number you know how to present because of your own knowledge of surreal numbers[4].

- My goal is to convince you that I know what the function $\hat{a} : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}\,;\, \xi \mapsto a \circ \xi$ should be.

If you win, I'll buy you a drink before the end of times.

"I have a slide for that one"

cui	dec	flip	ite	iwin	nop	them	ulose	vrai

---

4. Caveat: I may not be able to turn a sign sequence into the way I represent numbers on the spot, so don't be too mean with that.

Reals and small ordinals

The ordered field of surreal numbers contains a canonical copy of the ordered field of real numbers:

- surreal numbers $\{L \mid R\}$ where L, R are hereditarily finite are dyadic rationals
- If (L, R) is a Dedekind cut of dyadic numbers, then $\{L \mid R\}$ is the corresponding real.

The ordered field of surreal numbers contains a canonical copy of the ordered field of real numbers:

- surreal numbers $\{L \mid R\}$ where L, R are hereditarily finite are dyadic rationals
- If (L, R) is a Dedekind cut of dyadic numbers, then $\{L \mid R\}$ is the corresponding real.

The law $a \mapsto a \circ \xi$ should be $\mathbb{R}$-linear, so $\hat{r}$ should be the constant function $\xi \mapsto r$ for all $r \in \mathbb{R} \subseteq \mathbf{No}$.

Reals and small ordinals

The ordered field of surreal numbers contains a canonical copy of the ordered field of real numbers:

- surreal numbers $\{L \mid R\}$ where L, R are hereditarily finite are dyadic rationals
- If (L, R) is a Dedekind cut of dyadic numbers, then $\{L \mid R\}$ is the corresponding real.

The law $a \mapsto a \circ \xi$ should be $\mathbb{R}$-linear, so $\hat{r}$ should be the constant function $\xi \mapsto r$ for all $r \in \mathbb{R} \subseteq \mathbf{No}$.

$\mathbf{No}$ also contains the ordered semi-ring $\mathbf{On}$ of ordinal numbers under the commutative natural/Hessenberg arithmetic.

- Ordinal numbers are surreal numbers of the form $\{L \mid \varnothing\}$.
- each $\alpha \in \mathbf{On}$ is iductively identified with $\{\{\beta : \beta < \alpha\} \mid \varnothing\}$.

Reals and small ordinals

The ordered field of surreal numbers contains a canonical copy of the ordered field of real numbers:

- surreal numbers $\{L \mid R\}$ where L, R are hereditarily finite are dyadic rationals
- If (L, R) is a Dedekind cut of dyadic numbers, then $\{L \mid R\}$ is the corresponding real.

The law $a \mapsto a \circ \xi$ should be $\mathbb{R}$-linear, so $\hat{r}$ should be the constant function $\xi \mapsto r$ for all $r \in \mathbb{R} \subseteq \mathbf{No}$.

$\mathbf{No}$ also contains the ordered semi-ring $\mathbf{On}$ of ordinal numbers under the commutative natural/Hessenberg arithmetic.

- Ordinal numbers are surreal numbers of the form $\{L \mid \varnothing\}$.
- each $\alpha \in \mathbf{On}$ is iductively identified with $\{\{\beta : \beta < \alpha\} \mid \varnothing\}$.

For instance $\omega = \{\mathbb{N} \mid \varnothing\}$ is the simplest positive infinite number. It should correspond to the simplest germ that tends to $+\infty$, i.e. to the identity function $\hat{\omega} = \xi \mapsto \xi$.

Reals and small ordinals

The ordered field of surreal numbers contains a canonical copy of the ordered field of real numbers:

- surreal numbers $\{L \mid R\}$ where L, R are hereditarily finite are dyadic rationals
- If (L, R) is a Dedekind cut of dyadic numbers, then $\{L \mid R\}$ is the corresponding real.

The law $a \mapsto a \circ \xi$ should be $\mathbb{R}$-linear, so $\hat{r}$ should be the constant function $\xi \mapsto r$ for all $r \in \mathbb{R} \subseteq \mathbf{No}$.

$\mathbf{No}$ also contains the ordered semi-ring $\mathbf{On}$ of ordinal numbers under the commutative natural/Hessenberg arithmetic.

- Ordinal numbers are surreal numbers of the form $\{L \mid \varnothing\}$.
- each $\alpha \in \mathbf{On}$ is iductively identified with $\{\{\beta : \beta < \alpha\} \mid \varnothing\}$.

For instance $\omega = \{\mathbb{N} \mid \varnothing\}$ is the simplest positive infinite number. It should correspond to the simplest germ that tends to $+\infty$, i.e. to the identity function $\hat{\omega} = \xi \mapsto \xi$.

So elements of $\mathbb{R}(\omega)$ should act as the corresponding rational functions. What else?

Of course.

We already know how to deal with ordinals $<\omega^\omega$, since they lie in $\mathbb{R}(\omega)$. For $\alpha \geqslant \omega^\omega$, I have an inductive method:

Of course.

We already know how to deal with ordinals $<\omega^\omega$, since they lie in $\mathbb{R}(\omega)$. For $\alpha \geqslant \omega^\omega$, I have an inductive method:

- If α is not additively indecomposable, then its germ $\hat{\alpha}$ is the sum of germs of its (Cantor normal form) summands.

Ordinals

Of course.

We already know how to deal with ordinals $<\omega^\omega$, since they lie in $\mathbb{R}(\omega)$. For $\alpha \geqslant \omega^\omega$, I have an inductive method:

- If α is not additively indecomposable, then its germ $\hat{\alpha}$ is the sum of germs of its (Cantor normal form) summands.
- If α is additively indecomposable ordinal but not multiplicatively indecomposable, then $\hat{\alpha}$ is the product of germs of its factors.

Of course.

We already know how to deal with ordinals $<\omega^\omega$, since they lie in $\mathbb{R}(\omega)$. For $\alpha \geqslant \omega^\omega$, I have an inductive method:

- If α is not additively indecomposable, then its germ $\hat{\alpha}$ is the sum of germs of its (Cantor normal form) summands.
- If α is additively indecomposable ordinal but not multiplicatively indecomposable, then $\hat{\alpha}$ is the product of germs of its factors.
- If $\alpha = \omega^{\omega^\mu} > \omega$ is not an ε-number, then I express $\hat{\alpha}$ as $\exp \circ \widehat{\omega^\mu}$. Know what $\exp$ is? nope.

Of course.

We already know how to deal with ordinals $<\omega^\omega$, since they lie in $\mathbb{R}(\omega)$. For $\alpha \geqslant \omega^\omega$, I have an inductive method:

- If α is not additively indecomposable, then its germ $\hat{\alpha}$ is the sum of germs of its (Cantor normal form) summands.
- If α is additively indecomposable ordinal but not multiplicatively indecomposable, then $\hat{\alpha}$ is the product of germs of its factors.
- If $\alpha = \omega^{\omega^\mu} > \omega$ is not an ε-number, then I express $\hat{\alpha}$ as $\exp \circ \widehat{\omega^\mu}$. Know what $\exp$ is? nope.
- If α is an ε-number, then there is some ordinal η such that α is an fixed point of the ω-base exponentiation of (higher order) η but not $\eta + 1$.

  Suppose that α is the β-th such fixed point. We then[9] define α using $\hat{\beta}$ and the ω^η-th $\mathrm{iterate}$ $\exp_{\omega^\eta}$ of the exponential…

9. the way to do so is complicated enough that I don't want to expand on this here. If η is a non-zero limit, then this is simply $\hat{\alpha} = \exp_{\omega^\eta} \circ (\widehat{\omega \dot{+} \beta})$ where $\omega \dot{+} \beta$ is the ordinal sum of ω and β.

Of course.

We already know how to deal with ordinals $<\omega^\omega$, since they lie in $\mathbb{R}(\omega)$. For $\alpha \geqslant \omega^\omega$, I have an inductive method:

- If α is not additively indecomposable, then its germ $\hat{\alpha}$ is the sum of germs of its (Cantor normal form) summands.
- If α is additively indecomposable ordinal but not multiplicatively indecomposable, then $\hat{\alpha}$ is the product of germs of its factors.
- If $\alpha = \omega^{\omega^\mu} > \omega$ is not an ε-number, then I express $\hat{\alpha}$ as $\exp \circ \widehat{\omega^\mu}$. Know what $\exp$ is? nope.
- If α is an ε-number, then there is some ordinal η such that α is an fixed point of the ω-base exponentiation of (higher order) η but not $\eta+1$.

  Suppose that α is the β-th such fixed point. We then[10] define α using $\hat{\beta}$ and the ω^η-th $\mathrm{iterate}$ $\exp_{\omega^\eta}$ of the exponential...

10. the way to do so is complicated enough that I don't want to expand on this here. If η is a non-zero limit, then this is simply $\hat{\alpha} = \exp_{\omega^\eta} \circ (\widehat{\omega \dotplus \beta})$ where $\omega \dotplus \beta$ is the ordinal sum of ω and β.

Types of cuts

Certain positive numbers $\mathfrak{m}$ called **monomials** are additively indecomposable into simpler terms: they are the simplest elements of their Archimedean class $\left\{b > 0 : \exists n \in \mathbb{N}^{>0}, \frac{1}{n}\,\mathfrak{m} < b < n\,\mathfrak{m}\right\}$.

Certain positive numbers $\mathfrak{m}$ called **monomials** are additively indecomposable into simpler terms: they are the simplest elements of their Archimedean class $\left\{b > 0 : \exists n \in \mathbb{N}^{>0}, \frac{1}{n}\mathfrak{m} < b < n\,\mathfrak{m}\right\}$.

If $(\mathfrak{m}_\gamma)_{\gamma<\eta}$ is a strictly decreasing sequence of monomials and $(r_\gamma)_{\gamma<\eta} \in \mathbb{R}^\eta$, then one defines inductively the sum

$$\sum_{\gamma<\eta} r_\gamma\,\mathfrak{m}_\gamma = \left\{\sum_{\gamma<\rho} r_\gamma\,\mathfrak{m}_\gamma + q\,\mathfrak{m}_\rho : q \in (-\infty, r_\rho) \;\middle|\; \sum_{\gamma<\rho} r_\gamma\,\mathfrak{m}_\gamma + q\,\mathfrak{m}_\rho : q \in (r_\rho, +\infty)\right\}.$$

Certain positive numbers $\mathfrak{m}$ called **monomials** are additively indecomposable into simpler terms: they are the simplest elements of their Archimedean class $\{b > 0 : \exists n \in \mathbb{N}^{>0}, \frac{1}{n}\mathfrak{m} < b < n\,\mathfrak{m}\}$.

If $(\mathfrak{m}_\gamma)_{\gamma<\eta}$ is a strictly decreasing sequence of monomials and $(r_\gamma)_{\gamma<\eta} \in \mathbb{R}^\eta$, then one defines inductively the sum

$$\sum_{\gamma<\eta} r_\gamma\,\mathfrak{m}_\gamma = \left\{ \sum_{\gamma<\rho} r_\gamma\,\mathfrak{m}_\gamma + q\,\mathfrak{m}_\rho : q \in (-\infty, r_\rho) \mid \sum_{\gamma<\rho} r_\gamma\,\mathfrak{m}_\gamma + q\,\mathfrak{m}_\rho : q \in (r_\rho, +\infty) \right\}.$$

Theorem. [Conway, '76] *Every surreal number can be expressed uniquely in such a way. So numbers are identified with functions* $\mathbf{Monomials} \ni \mathfrak{m} \longmapsto r \in \mathbb{R}$ *whose support is anti-well-ordered.*

Types of cuts

Certain positive numbers $\mathfrak{m}$ called **monomials** are additively indecomposable into simpler terms: they are the simplest elements of their Archimedean class $\left\{b>0:\exists n\in\mathbb{N}^{>0},\frac{1}{n}\mathfrak{m}<b<n\,\mathfrak{m}\right\}$.

If $(\mathfrak{m}_\gamma)_{\gamma<\eta}$ is a strictly decreasing sequence of monomials and $(r_\gamma)_{\gamma<\eta}\in\mathbb{R}^\eta$, then one defines inductively the sum

$$\sum_{\gamma<\eta} r_\gamma\,\mathfrak{m}_\gamma=\left\{\sum_{\gamma<\rho} r_\gamma\,\mathfrak{m}_\gamma+q\,\mathfrak{m}_\rho:q\in(-\infty,r_\rho)\mid\sum_{\gamma<\rho} r_\gamma\,\mathfrak{m}_\gamma+q\,\mathfrak{m}_\rho:q\in(r_\rho,+\infty)\right\}.$$

Theorem. [Conway, '76] *Every surreal number can be expressed uniquely in such a way. So numbers are identified with functions* $\mathbf{Monomials}\ni\mathfrak{m}\longmapsto r\in\mathbb{R}$ *whose support is anti-well-ordered.*

Claim: I can define each $\widehat{\mathfrak{m}_\gamma}$ in so that for all $\xi\in\mathbf{No}^{>\mathbb{R}}$, the following number is well-defined:

$$\hat{a}(\xi)\quad:=\quad\mathfrak{n}\longmapsto\sum_{\gamma<\eta} r_\gamma\,\widehat{\mathfrak{m}_\gamma}(\xi)(\mathfrak{n}).$$

Have you stumbled upon $\omega^{\frac{1}{\omega}} = \left\{\mathbb{N} \mid \left\{\omega^{\frac{1}{n}} : n \in \mathbb{N}^{>0}\right\}\right\}$? You must have a retorse mind.

Have you stumbled upon $\omega^{\frac{1}{\omega}} = \left\{ \mathbb{N} \mid \left\{ \omega^{\frac{1}{n}} : n \in \mathbb{N}^{>0} \right\} \right\}$? You must have a retorse mind.

Or are you curious about the inverse number of ω^{ω}? What a pleasant, curious and onpen person.

Logarithm

Have you stumbled upon $\omega^{\frac{1}{\omega}} = \left\{\mathbb{N} \mid \left\{\omega^{\frac{1}{n}} : n \in \mathbb{N}^{>0}\right\}\right\}$? You must have a retorse mind.

Or are you curious about the inverse number of ω^{ω}? What a pleasant, curious and onpen person.

Logarithm

Have you stumbled upon $\omega^{\frac{1}{\omega}} = \left\{\mathbb{N} \mid \left\{\omega^{\frac{1}{n}} : n \in \mathbb{N}^{>0}\right\}\right\}$? You must have a retorse mind.

Or are you curious about the inverse number of ω^{ω}? What a pleasant, curious and onpen person.

Anyway, this ought to be the simplest function that tends to $+\infty$ slower than all positive power functions: a logaritm?

Logarithm

Have you stumbled upon $\omega^{\frac{1}{\omega}} = \left\{ \mathbb{N} \mid \left\{ \omega^{\frac{1}{n}} : n \in \mathbb{N}^{>0} \right\} \right\}$? You must have a retorse mind.

Or are you curious about the inverse number of ω^ω? What a pleasant, curious and onpen person.

Anyway, this ought to be the simplest function that tends to $+\infty$ slower than all positive power functions: a logaritm?

Unfotunately, I don't know how to justify that the simplest choice for $\omega^{1/\omega}$ is the inverse of $\exp$, nor do I know a genetic equation for $\log$. But yes, $\log = \exp^{\text{inv}}$.

Logarithm

Have you stumbled upon $\omega^{\frac{1}{\omega}} = \left\{\mathbb{N} \mid \left\{\omega^{\frac{1}{n}} : n \in \mathbb{N}^{>0}\right\}\right\}$? You must have a retorse mind.

Or are you curious about the inverse number of ω^{ω}? What a pleasant, curious and onpen person.

Anyway, this ought to be the simplest function that tends to $+\infty$ slower than all positive power functions: a logaritm?

Unfotunately, I don't know how to justify that the simplest choice for $\omega^{1/\omega}$ is the inverse of $\exp$, nor do I know a genetic equation for $\log$. But yes, $\log = \exp^{\mathrm{inv}}$.

This function was defined by Gonshor ('86).

Exponential

We have $\omega^\omega = \{\omega^{\mathbb{N}} \mid \varnothing\}$. What is the simplest strictly increasing function that grows faster than all polynomials?

$$f(\{L \mid R\}) \;=\; \{\{L \mid R\}^{\mathbb{N}}, f(L) \mid f(R)\}.$$

We have $\omega^{\omega} = \{\omega^{\mathbb{N}} \mid \varnothing\}$. What is the simplest strictly increasing function that grows faster than all polynomials?

$$f(\{L \mid R\}) \; = \; \{\{L \mid R\}^{\mathbb{N}}, f(L) \mid f(R)\}.$$

…but also has Taylor expansions and behaves like a germ in a Hardy field?

Exponential

We have $\omega^\omega = \{\omega^{\mathbb{N}} \mid \varnothing\}$. What is the simplest strictly increasing function that grows faster than all polynomials?

$$f(\{L \mid R\}) \;=\; \{\{L \mid R\}^{\mathbb{N}}, f(L) \mid f(R)\}.$$

…but also has Taylor expansions and behaves like a germ in a Hardy field?

Hardy-type asymptotics entail that the simplest choice for $\widehat{\omega^\omega}(\xi + \zeta)$ is $\widehat{\omega^\omega}(\xi)\,\widehat{\omega^\omega}(\zeta)$.

We have $\omega^\omega = \{\omega^{\mathbb{N}} \mid \varnothing\}$. What is the simplest strictly increasing function that grows faster than all polynomials?

$$f(\{L \mid R\}) \; = \; \{\{L \mid R\}^{\mathbb{N}}, f(L) \mid f(R)\}.$$

…but also has Taylor expansions and behaves like a germ in a Hardy field?

Hardy-type asymptotics entail that the simplest choice for $\widehat{\omega^\omega}(\xi + \zeta)$ is $\widehat{\omega^\omega}(\xi)\,\widehat{\omega^\omega}(\zeta)$.

Gonshor ('86) defined the exponential function as follows, for $\xi = \{L \mid R\}$

$$\exp(\xi) = \left\{ \exp(l) \sum_{i \leqslant n} \frac{(\xi - l)^i}{i!} \mid \frac{\exp(r)}{} \right\}$$

where l, r, i range in L, R and $\mathbb{N}$ respectively.

Nested cuts I

You seem to have found a nested number. Well done! those are the worst.

Nested cuts I

You seem to have found a nested number. Well done! those are the worst.

Each monomial $\mathfrak{m} \in \mathbf{No}$ (do you know about the Hahn series deccomposition yet?) can be written as $\mathfrak{m} = \mathrm{e}^{\varphi}$ for $\varphi = \log \mathfrak{m}$ a surreal number of a special type.

Nested cuts I

You seem to have found a nested number. Well done! those are the worst.

Each monomial $\mathfrak{m} \in \mathbf{No}$ (do you know about the Hahn series deccomposition yet?) can be written as $\mathfrak{m} = \mathrm{e}^{\varphi}$ for $\varphi = \log \mathfrak{m}$ a surreal number of a special type.

If one is lucky, the number φ is "simpler" in some sense than $\mathfrak{m}$, so I can claim inductively that I know what $\hat{\varphi}$ is. And then $\hat{\mathfrak{m}} = \exp \circ \hat{\varphi}$. With a bit less luck, going through monomials in φ and the inductively so, we end up finding only simpler numbers after some time.

You seem to have found a nested number. Well done! those are the worst.

Each monomial $\mathfrak{m} \in \mathbf{No}$ (do you know about the Hahn series deccomposition yet?) can be written as $\mathfrak{m} = \mathrm{e}^{\varphi}$ for $\varphi = \log \mathfrak{m}$ a surreal number of a special type.

If one is lucky, the number φ is "simpler" in some sense than $\mathfrak{m}$, so I can claim inductively that I know what $\hat{\varphi}$ is. And then $\hat{\mathfrak{m}} = \exp \circ \hat{\varphi}$. With a bit less luck, going through monomials in φ and the inductively so, we end up finding only simpler numbers after some time.

But that may not happen. One may not reach any simpler number.

Theorem. [Berarducci-Mantova, 2017] *If the process continues indefinitely, then at some stage, problematic monomials* $\mathfrak{n}$ *thus appearing have the form*

$$\mathfrak{n} = \mathrm{e}^{\varphi_1 \pm \mathrm{e}^{\varphi_2 \pm \mathrm{e}^{\varphi_3 \pm \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}$$

for some strictly simpler $\varphi_1, \ldots, \varphi_n, \ldots$.

Nested cuts I

You seem to have found a nested number. Well done! those are the worst.

Each monomial $\mathfrak{m} \in \mathbf{No}$ (do you know about the Hahn series deccomposition yet?) can be written as $\mathfrak{m} = \mathrm{e}^{\varphi}$ for $\varphi = \log \mathfrak{m}$ a surreal number of a special type.

If one is lucky, the number φ is "simpler" in some sense than $\mathfrak{m}$, so I can claim inductively that I know what $\hat{\varphi}$ is. And then $\hat{\mathfrak{m}} = \exp \circ \hat{\varphi}$. With a bit less luck, going through monomials in φ and the inductively so, we end up finding only simpler numbers after some time.

But that may not happen. One may not reach any simpler number.

Theorem. [Berarducci-Mantova, 2017] *If the process continues indefinitely, then at some stage, problematic monomials* $\mathfrak{n}$ *thus appearing have the form*

$$\mathfrak{n} = \mathrm{e}^{\varphi_1 \pm \mathrm{e}^{\varphi_2 \pm \mathrm{e}^{\varphi_3 \pm \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}}$$

for some strictly simpler $\varphi_1, \ldots, \varphi_n, \ldots$.

With van der Hoeven, we extended this to all numbers using hyperexponentials and their inverses.

So how do I deal with nested monomials? "An" example is

$$\mathfrak{n} = \mathrm{e}^{\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{1}$$

Suffices to deal with $\mathfrak{n}_i = \mathrm{e}^{\sqrt{\log_i \omega} + \mathrm{e}^{\sqrt{\log_{i+1} \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$ for some $i \in \mathbb{N}$.

So how do I deal with nested monomials? "An" example is

$$\mathfrak{n} = \mathrm{e}^{\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{2}$$

Suffices to deal with $\mathfrak{n}_i = \mathrm{e}^{\sqrt{\log_i \omega} + \mathrm{e}^{\sqrt{\log_{i+1} \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$ for some $i \in \mathbb{N}$.

For $\xi \in \mathbf{No}^{>\mathbb{R}}$, what should $\tilde{\mathfrak{n}}_i(\xi)$ be? The answer comes from a simplicity heuristic (a choice) and Taylor expansions. Rough idea:

So how do I deal with nested monomials? "An" example is

$$\mathfrak{n} = \mathrm{e}^{\sqrt{\omega}+\mathrm{e}^{\sqrt{\log \omega}+\mathrm{e}^{\sqrt{\log\log \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{3}$$

Suffices to deal with $\mathfrak{n}_i = \mathrm{e}^{\sqrt{\log_i \omega}+\mathrm{e}^{\sqrt{\log_{i+1} \omega}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$ for some $i \in \mathbb{N}$.

For $\xi \in \mathbf{No}^{>\mathbb{R}}$, what should $\tilde{\mathfrak{n}}_i(\xi)$ be? The answer comes from a simplicity heuristic (a choice) and Taylor expansions. Rough idea:

- Replace $\mathfrak{n}_i$ with $\mathfrak{n}_i \circ \exp_\omega(\omega) = \mathfrak{n}_i \circ \varepsilon_0$ (met the iterates already?). This is simply the simplest number expanding as $\mathrm{e}^{\sqrt{\log_i \varepsilon_0}+\mathrm{e}^{\sqrt{\log_{i+1} \varepsilon_0}+\mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$. Also replace ξ with $\log_\omega(\xi)$.

Nested cuts II

So how do I deal with nested monomials? "An" example is

$$\mathfrak{n} = \mathrm{e}^{\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log\log\omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{4}$$

Suffices to deal with $\mathfrak{n}_i = \mathrm{e}^{\sqrt{\log_i \omega} + \mathrm{e}^{\sqrt{\log_{i+1}\omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$ for some $i \in \mathbb{N}$.

For $\xi \in \mathbf{No}^{>\mathbb{R}}$, what should $\tilde{\mathfrak{n}}_i(\xi)$ be? The answer comes from a simplicity heuristic (a choice) and Taylor expansions. Rough idea:

- Replace $\mathfrak{n}_i$ with $\mathfrak{n}_i \circ \exp_\omega(\omega) = \mathfrak{n}_i \circ \varepsilon_0$ (met the iterates already?). This is simply the simplest number expanding as $\mathrm{e}^{\sqrt{\log_i \varepsilon_0} + \mathrm{e}^{\sqrt{\log_{i+1}\varepsilon_0} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$. Also replace ξ with $\log_\omega(\xi)$.
- It can be shown that $\xi = \varphi + \varepsilon$ where $\varphi \neq 0$ is a truncation of ξ as a series, and there is a simplest monomial $\hat{\mathfrak{n}}_i(\varphi)$ expanding as $\mathrm{e}^{\sqrt{\log_i \hat{\varepsilon_0}(\varphi)} + \mathrm{e}^{\sqrt{\log_{i+1}\hat{\varepsilon_0}(\varphi)} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$.

Nested cuts II

So how do I deal with nested monomials? "An" example is

$$\mathfrak{n} = \mathrm{e}^{\sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log \log \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}} \tag{5}$$

Suffices to deal with $\mathfrak{n}_i = \mathrm{e}^{\sqrt{\log_i \omega} + \mathrm{e}^{\sqrt{\log_{i+1} \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$ for some $i \in \mathbb{N}$.

For $\xi \in \mathbf{No}^{>\mathbb{R}}$, what should $\tilde{\mathfrak{n}_i}(\xi)$ be? The answer comes from a simplicity heuristic (a choice) and Taylor expansions. Rough idea:

- Replace $\mathfrak{n}_i$ with $\mathfrak{n}_i \circ \exp_\omega(\omega) = \mathfrak{n}_i \circ \varepsilon_0$ (met the iterates already?). This is simply the simplest number expanding as $\mathrm{e}^{\sqrt{\log_i \varepsilon_0} + \mathrm{e}^{\sqrt{\log_{i+1} \varepsilon_0} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$. Also replace ξ with $\log_\omega(\xi)$.
- It can be shown that $\xi = \varphi + \varepsilon$ where $\varphi \neq 0$ is a truncation of ξ as a series, and there is a simplest monomial $\hat{\mathfrak{n}_i}(\varphi)$ expanding as $\mathrm{e}^{\sqrt{\log_i \hat{\varepsilon_0}(\varphi)} + \mathrm{e}^{\sqrt{\log_{i+1} \hat{\varepsilon_0}(\varphi)} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}$.
- Choosing i large enough, one can insure that φ is longest, such that the Taylor expansion of $\hat{\mathfrak{n}_i}(\varphi)$ with radius ε converges formally. We thus set $\hat{\mathfrak{n}_i}(\xi) := \sum_{k \in \mathbb{N}} \frac{\hat{\mathfrak{n}_i}^{(k)}(\varphi)}{k!} \varepsilon^k$.

Hyperexponentials I

Did you suggest ε_0? Or some nastier ε-number α? I will explain the ε_0 case and say how to obtain the general one.

Hyperexponentials I

Did you suggest ε_0? Or some nastier ε-number α? I will explain the ε_0 case and say how to obtain the general one.

I **claim** that $\hat{\varepsilon_0}$ is a surreal version of Abel/Kneser/Ecalle's ω-th iterate E of $\exp$, a real-analytic function on $\mathbb{R}^{\geqslant 0}$ satisfying $E(t+1) = \exp(E(t))$ for all $t \geqslant 0$.

Did you suggest ε_0? Or some nastier ε-number α? I will explain the ε_0 case and say how to obtain the general one.

I **claim** that $\hat{\varepsilon_0}$ is a surreal version of Abel/Kneser/Ecalle's ω-th iterate E of $\exp$, a real-analytic function on $\mathbb{R}^{\geqslant 0}$ satisfying $E(t+1)=\exp(E(t))$ for all $t\geqslant 0$.

Indeed since $\varepsilon_0=\{\{\omega,\exp(\omega),\exp(\exp(\omega)),\ldots\}\mid\varnothing\}$, the function $\hat{\varepsilon_0}$ should grow faster than all finite iterates of $\exp$. Hardy-type asymptotics give an approximation of

$$\hat{\varepsilon_0}'\approx\hat{\varepsilon_0}\,(\log\circ\hat{\varepsilon_0})\,(\log\circ\log\circ\hat{\varepsilon_0})\cdots=T\circ\hat{\varepsilon_0}\quad\text{for a transseries } T.$$

Hyperexponentials I

Did you suggest ε_0? Or some nastier ε-number α? I will explain the ε_0 case and say how to obtain the general one.

I **claim** that $\hat{\varepsilon_0}$ is a surreal version of ABEL/KNESER/ECALLE's ω-th iterate E of $\exp$, a real-analytic function on $\mathbb{R}^{\geqslant 0}$ satisfying $E(t+1)=\exp(E(t))$ for all $t\geqslant 0$.

Indeed since $\varepsilon_0=\{\{\omega,\exp(\omega),\exp(\exp(\omega)),\ldots\}\mid\varnothing\}$, the function $\hat{\varepsilon_0}$ should grow faster than all finite iterates of $\exp$. Hardy-type asymptotics give an approximation of

$$\hat{\varepsilon_0}'\approx\hat{\varepsilon_0}\,(\log\circ\hat{\varepsilon_0})\,(\log\circ\log\circ\hat{\varepsilon_0})\cdots=T\circ\hat{\varepsilon_0}\quad\text{for a transseries } T.$$

Now for $\varphi\in\mathbf{No}^{>\mathbb{R}}$ and $\varepsilon\prec\frac{\hat{\varepsilon_0}(\varphi)}{\hat{\varepsilon_0}'(\varphi)}$, the Taylor expansion of $\hat{\varepsilon_0}$ at φ,ε converges formally.

Hyperexponentials I

Did you suggest ε_0? Or some nastier ε-number α? I will explain the ε_0 case and say how to obtain the general one.

I **claim** that $\hat{\varepsilon_0}$ is a surreal version of Abel/Kneser/Ecalle's ω-th iterate E of $\exp$, a real-analytic function on $\mathbb{R}^{\geqslant 0}$ satisfying $E(t+1) = \exp(E(t))$ for all $t \geqslant 0$.

Indeed since $\varepsilon_0 = \{\{\omega, \exp(\omega), \exp(\exp(\omega)), \ldots\} \mid \varnothing\}$, the function $\hat{\varepsilon_0}$ should grow faster than all finite iterates of $\exp$. Hardy-type asymptotics give an approximation of

$$\hat{\varepsilon_0}' \approx \hat{\varepsilon_0}\,(\log \circ \hat{\varepsilon_0})\,(\log \circ \log \circ \hat{\varepsilon_0}) \cdots = T \circ \hat{\varepsilon_0} \quad \text{for a transseries } T.$$

Now for $\varphi \in \mathbf{No}^{>\mathbb{R}}$ and $\varepsilon \prec \frac{\hat{\varepsilon_0}(\varphi)}{\hat{\varepsilon_0}'(\varphi)}$, the Taylor expansion of $\hat{\varepsilon_0}$ at φ, ε converges formally.

So if we can define $\hat{\varepsilon_0}$ at sufficiently simple φ's, then we can extend the definition to all $\varphi + \varepsilon$ for sufficiently small ε, by

$$\hat{\varepsilon_0}(\varphi + \varepsilon) := \sum_{k \in \mathbb{N}} \frac{\hat{\varepsilon_0}^{(k)}(\varphi)}{k!}\,\varepsilon^k.$$

Hyperexponentials II

Some heuristics suggest that the simplest value for $\hat{\varepsilon_0}(\omega+1)$ is $\exp(\hat{\varepsilon_0}(\omega))$. Thus we should have $\hat{\varepsilon_0}(\xi+1)=\exp(\hat{\varepsilon_0}(\xi))$ for all $\xi\in\mathbf{No}^{>\mathbb{R}}$.

Hyperexponentials II

Some heuristics suggest that the simplest value for $\hat{\varepsilon_0}(\omega+1)$ is $\exp(\hat{\varepsilon_0}(\omega))$. Thus we should have $\hat{\varepsilon_0}(\xi+1)=\exp(\hat{\varepsilon_0}(\xi))$ for all $\xi\in\mathbf{No}^{>\mathbb{R}}$.

To define $\hat{\varepsilon}_0$ at a ξ, it suffices to define it at $\xi-n$ for some $n\in\mathbb{N}$ and then take the n-fold iterated exponential of the result. Fix $\xi\in\mathbf{No}^{>\mathbb{R}}$ and suppose $\hat{\varepsilon_0}(\zeta)$ is defined for simpler ζ.

Hyperexponentials II

Some heuristics suggest that the simplest value for $\hat{\varepsilon_0}(\omega+1)$ is $\exp(\hat{\varepsilon_0}(\omega))$. Thus we should have $\hat{\varepsilon_0}(\xi+1)=\exp(\hat{\varepsilon_0}(\xi))$ for all $\xi\in\mathbf{No}^{>\mathbb{R}}$.

To define $\hat{\varepsilon}_0$ at a ξ, it suffices to define it at $\xi-n$ for some $n\in\mathbb{N}$ and then take the n-fold iterated exponential of the result. Fix $\xi\in\mathbf{No}^{>\mathbb{R}}$ and suppose $\hat{\varepsilon_0}(\zeta)$ is defined for simpler ζ.

If that $\xi-n$ has a strict (hence simpler) truncation φ with $\xi-n-\varphi<\frac{\hat{\varepsilon_0}(\varphi)}{\hat{\varepsilon_0}'(\varphi)}$ for some $n\in\mathbb{N}$, then we define $\hat{\varepsilon_0}(\xi)$ *via* Taylor expansions.

Hyperexponentials II

Some heuristics suggest that the simplest value for $\hat{\varepsilon_0}(\omega+1)$ is $\exp(\hat{\varepsilon_0}(\omega))$. Thus we should have $\hat{\varepsilon_0}(\xi+1)=\exp(\hat{\varepsilon_0}(\xi))$ for all $\xi\in\mathbf{No}^{>\mathbb{R}}$.

To define $\hat{\varepsilon}_0$ at a ξ, it suffices to define it at $\xi-n$ for some $n\in\mathbb{N}$ and then take the n-fold iterated exponential of the result. Fix $\xi\in\mathbf{No}^{>\mathbb{R}}$ and suppose $\hat{\varepsilon_0}(\zeta)$ is defined for simpler ζ.

If that $\xi-n$ has a strict (hence simpler) truncation φ with $\xi-n-\varphi<\frac{\hat{\varepsilon_0}(\varphi)}{\hat{\varepsilon_0}'(\varphi)}$ for some $n\in\mathbb{N}$, then we define $\hat{\varepsilon_0}(\xi)$ *via* Taylor expansions.

So it suffices to define $\hat{\varepsilon_0}$ on the class $\mathbf{Tr}$ of numbers that do not have such truncations. For $\varphi,\psi\in\mathbf{Tr}$, $\varphi<\psi$, monotonicity of $\hat{\varepsilon_0}$ entails that $\mathcal{E}_n^{\pm}(\hat{\varepsilon_0}(\varphi))<\hat{\varepsilon_0}(\psi)$ where

$$\forall\zeta\in\mathbf{No}^{>\mathbb{R}},\mathcal{E}_n^{\pm}(\zeta)=\exp^{\circ n}(\log^{\circ n}(\zeta)\pm 1)$$

Some heuristics suggest that the simplest value for $\hat{\varepsilon_0}(\omega+1)$ is $\exp(\hat{\varepsilon_0}(\omega))$. Thus we should have $\hat{\varepsilon_0}(\xi+1) = \exp(\hat{\varepsilon_0}(\xi))$ for all $\xi \in \mathbf{No}^{>\mathbb{R}}$.

To define $\hat{\varepsilon}_0$ at a ξ, it suffices to define it at $\xi - n$ for some $n \in \mathbb{N}$ and then take the n-fold iterated exponential of the result. Fix $\xi \in \mathbf{No}^{>\mathbb{R}}$ and suppose $\hat{\varepsilon_0}(\zeta)$ is defined for simpler ζ.

If that $\xi - n$ has a strict (hence simpler) truncation φ with $\xi - n - \varphi < \frac{\hat{\varepsilon_0}(\varphi)}{\hat{\varepsilon_0}'(\varphi)}$ for some $n \in \mathbb{N}$, then we define $\hat{\varepsilon_0}(\xi)$ *via* Taylor expansions.

So it suffices to define $\hat{\varepsilon_0}$ on the class $\mathbf{Tr}$ of numbers that do not have such truncations. For $\varphi, \psi \in \mathbf{Tr}$, $\varphi < \psi$, monotonicity of $\hat{\varepsilon_0}$ entails that $\mathcal{E}_n^{\pm}(\hat{\varepsilon_0}(\varphi)) < \hat{\varepsilon_0}(\psi)$ where

$$\forall \zeta \in \mathbf{No}^{>\mathbb{R}}, \mathcal{E}_n^{\pm}(\zeta) = \exp^{\circ n}(\log^{\circ n}(\zeta) \pm 1)$$

This gives an inductive definition, if φ is the simplest element of $\mathbf{Tr}$ with $L < \varphi < R$, $L, R \subseteq \mathbf{Tr}$:

$$\hat{\varepsilon_0}(\varphi) = \{\exp^{\circ n}(\varphi), \mathcal{E}_n^{+}(\hat{\varepsilon_0}(l)) : n \in \mathbb{N} \wedge l \in L\} \mid \{\mathcal{E}_n^{-}(\hat{\varepsilon_0}(r)) : n \in \mathbb{N} \wedge r \in R\}.$$

Hyperexponentials II

Some heuristics suggest that the simplest value for $\hat{\varepsilon_0}(\omega+1)$ is $\exp(\hat{\varepsilon_0}(\omega))$. Thus we should have $\hat{\varepsilon_0}(\xi+1)=\exp(\hat{\varepsilon_0}(\xi))$ for all $\xi\in\mathbf{No}^{>\mathbb{R}}$.

To define $\hat{\varepsilon}_0$ at a ξ, it suffices to define it at $\xi-n$ for some $n\in\mathbb{N}$ and then take the n-fold iterated exponential of the result. Fix $\xi\in\mathbf{No}^{>\mathbb{R}}$ and suppose $\hat{\varepsilon_0}(\zeta)$ is defined for simpler ζ.

If that $\xi-n$ has a strict (hence simpler) truncation φ with $\xi-n-\varphi<\frac{\hat{\varepsilon_0}(\varphi)}{\hat{\varepsilon_0}'(\varphi)}$ for some $n\in\mathbb{N}$, then we define $\hat{\varepsilon_0}(\xi)$ *via* Taylor expansions.

So it suffices to define $\hat{\varepsilon_0}$ on the class $\mathbf{Tr}$ of numbers that do not have such truncations. For $\varphi,\psi\in\mathbf{Tr}$, $\varphi<\psi$, monotonicity of $\hat{\varepsilon_0}$ entails that $\mathcal{E}_n^{\pm}(\hat{\varepsilon_0}(\varphi))<\hat{\varepsilon_0}(\psi)$ where

$$\forall\zeta\in\mathbf{No}^{>\mathbb{R}},\mathcal{E}_n^{\pm}(\zeta)=\exp^{\circ n}(\log^{\circ n}(\zeta)\pm 1)$$

This gives an inductive definition, if φ is the simplest element of $\mathbf{Tr}$ with $L<\varphi<R$, $L,R\subseteq\mathbf{Tr}$:

$$\hat{\varepsilon_0}(\varphi)=\{\exp^{\circ n}(\varphi),\mathcal{E}_n^{+}(\hat{\varepsilon_0}(l)):n\in\mathbb{N}\wedge l\in L\}\mid\{\mathcal{E}_n^{-}(\hat{\varepsilon_0}(r)):n\in\mathbb{N}\wedge r\in R\}.$$

General case of $\exp_{\omega^\mu}$: proceed inductively, replacing $\{\exp^{\circ n}\in\mathbb{N}\}$ by $\{\exp_{\omega^\eta}^{\circ n}:n\in\mathbb{N}\wedge\eta<\mu\}$.

I win the game. **Thanks for playing along!**

Proof of van der Hoeven's conjecture. Proof by game. If the conjecture were false, then 10 minutes should be enough, to a room full of smart people, to disprove it. But they just lost the very fair game. Therefore the conjecture is true. □

You win!

Thanks for playing along (and ruining my project; saves time)!