Hyperseries and surreal Numbers*

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a PhD supervised by **Joris van der Hoeven** and **Françoise Point**

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I - Functions

Germs at $+\infty$

asymptotic $\equiv$ *as the variable is large enough*

We identify two functions f, g in

$$\bigcap_{n \in \mathbb{N}} \bigcup_{r \in \mathbb{R}} \mathcal{C}^n((r, +\infty), \mathbb{R}),$$

if $f(r) = g(r)$ for all sufficiently large $r \in \mathbb{R}$ (written $r \gg 1$).

$\mathcal{G}$*: differential ring of equivalence classes, called* **germs** at $+\infty$.

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Definition

Hardy fields *are differential subfields of* $\mathcal{G}$ *containing* $\mathbb{R}$*. Ordered by*

$$f < g \iff \forall r \gg 1, f(r) < g(r).$$

E.g. $\mathbb{R}((x^r)_{r \in \mathbb{R}})$, $\mathbb{R}(x, \arctan x)$, $\mathbb{R}(x, \mathrm{e}^x)$, *and* $\mathbb{R}(\log x, \log \log x)$.

I.2 - Further examples

O-minimality and Hardy fields

Let $\mathcal{R} = (\mathbb{R}, +, \times, <, \ldots)$ *be an o-minimal extension of the real ordered field. Write* $\mathcal{H}(\mathcal{R})$ *for the set of germs at* $+\infty$ *of functions* $(a, +\infty) \longrightarrow \mathbb{R}, a \in \mathbb{R}$ *that are definable in* $\mathcal{R}$. *Then* $\mathcal{H}(\mathcal{R})$ *is a Hardy field.*

In particular $\mathcal{H}(\mathbb{R}_{\exp})$ is a Hardy field.

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Maximal Hardy fields

A Hardy field $\mathcal{H}$ *is said* **maximal** *if it has no proper superset which is a Hardy field.*

Conjecture 1 (ADH): *The structure* $(\mathcal{H}, +, \times, <, \prec, \partial)$ *is elementarily equivalent to the field of log-exp transseries.*

Conjecture 2 (ADH): *For all countable subsets* $L, R \subseteq \mathcal{H}$ *with* $L < R$*, there is an* $f \in \mathcal{H}$ *with* $L < f < R$*.*

Theorem [H. Kneser - 1949]

There is a strictly increasing analytic function $E_\omega : \mathbb{R} \longrightarrow \mathbb{R}$ *which solves* **Abel's equation***:*

$$\forall r \gg 1, E_\omega(r+1) = \exp(E_\omega(r)).$$

We have $E_\omega(r) > e^{e^{e^{\cdot^{\cdot^{\cdot^{e^r}}}}}}$ for $r \gg 1$: E_ω is a **hyperexponential** function.

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Theorem [Padgett - 2022]

There is a Hardy field $\mathcal{H}$ *containing* $\mathcal{H}_{\text{an,exp}}$, *the germ of* E_ω, *its inverse* L_ω, *and which is closed under composition of germs.*

II - Series

II.1 - Hahn series

Definition: Hahn series [Hahn - 1907]

Let $(\mathfrak{M}, \times, \prec)$ *be a linearly ordered abelian group. The* **Hahn series** *field* $\mathbb{R}[[\mathfrak{M}]]$ *is the class of functions* $f : \mathfrak{M} \longrightarrow \mathbb{R}$ *whose support*

$$\operatorname{supp} f \;:=\; \{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\} \;\subseteq\; \mathfrak{M}$$

is a well-ordered <u>*subset*</u> *of* $(\mathfrak{M}, \succ)$. $\mathbb{R}[[\mathfrak{M}]]$ *is an ordered valued field.*

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is a well-ordered *__subset__* *of* $(\mathfrak{M}, \succ)$. $\mathbb{R}[[\mathfrak{M}]]$ *is an ordered valued field:*

Pointwise sum / Cauchy product:

$(f+g) = \mathfrak{m} \longmapsto f(\mathfrak{m}) + g(\mathfrak{m}) \qquad / \qquad (fg) = \mathfrak{m} \longmapsto \sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} f(\mathfrak{u})\, g(\mathfrak{v})$

Order / valuation:

$0 < f \Longleftrightarrow 0 < f(\max \operatorname{supp} f) \qquad / \qquad f \prec g \Longleftrightarrow \max \operatorname{supp} f \prec \max \operatorname{supp} g$

Identification:

Each $f \in \mathbb{R}[[\mathfrak{M}]]$ *is seen as the formal series* $f \equiv \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}}\, \mathfrak{m}$.

II.2 - Summable families

Fix a Hahn series field $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$. **Idea:** infinite pointwise sums in $\mathbb{S}$.

Definition

Let I be a set, and let $(f_i)_{i \in I} \in \mathbb{S}^I$ be a family. We say that $(f_i)_{i \in I}$ is **summable** *if*

i. The set $\bigcup_{i \in I} \operatorname{supp} f_i$ is well-ordered in $(\mathfrak{M}, \succ)$.

ii. For all $\mathfrak{m} \in \mathfrak{M}$, the set $I_{\mathfrak{m}} := \{i \in I : \mathfrak{m} \in \operatorname{supp} f_i\}$ is finite.

Then the following is a well-defined element of $\mathbb{S}$:

$$\sum_{i \in I} f_i := \sum_{\mathfrak{m} \in \mathfrak{M}} \left(\sum_{i \in I_{\mathfrak{m}}} f_i(\mathfrak{m}) \right) \mathfrak{m}.$$

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[B. Neumann - 1949] For $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $\varepsilon \prec 1$, the family $(a_n \varepsilon^n)_{n \in \mathbb{N}}$ is summable.

$$\text{e.g.} \quad \frac{1}{1+\varepsilon} = \sum_{n \in \mathbb{N}} (-1)^n \varepsilon^n.$$

II.3 - Transseries

Log-exp transseries [Dahn, Göring - 1987 and Ecalle - 1992]

Field $\mathbb{T}_{\mathrm{LE}}$ *of* **log-exp transseries***: Hahn series involving formal terms* x, $\log x$ *and* e^x *and combinations thereof.*

$$\text{e.g.}\quad f := \sum_{n=1}^{+\infty} n!\, e^{x/n} + \log_2 x + 7 + x^{-2}\log x + \sum_{p=0}^{+\infty} e^{-x^{p+1}} x^p \quad \text{is a log-exp transseries.}$$

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$$\partial f = \sum_{n=1}^{+\infty} (n-1)!\,\mathrm{e}^{x/n} + \frac{1}{x\log x} - 2x^{-3}\log x + x^{-3} + \sum_{p=0}^{+\infty} (p\,x^{p-1} - (p+1)\,x^p)\,\mathrm{e}^{-x^{p+1}}$$

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$$f \circ \log x = \sum_{n=1}^{+\infty} n!\,x^{1/n} + \log_3 x + 7 + (\log x)^{-2}\log_2 x + \sum_{p=0}^{+\infty} \mathrm{e}^{-(\log x)^{p+1}} (\log x)^p.$$

II.4 - Cuts in transseries

Cuts in $\mathbb{T}_{\mathrm{LE}}$ [van der Hoeven - 2006]

- *A* **cut** *(in $\mathbb{T}_{\mathrm{LE}}$) is an $\leqslant$-initial subset $\boldsymbol{c} \subseteq \mathbb{T}_{\mathrm{LE}}$ without supremum in $(\mathbb{T}_{\mathrm{LE}}, \leqslant)$*

- *Monotonous operations $\boldsymbol{c}_1 + \boldsymbol{c}_2, \boldsymbol{c}_1 \times \boldsymbol{c}_2, \mathrm{e}^{\boldsymbol{c}_1}$ on cuts $\boldsymbol{c}_1, \boldsymbol{c}_2$. Also $\partial(\boldsymbol{c}_1)$ for certain cuts.*
  Problem: cut operations behave quite differently as standard operations.

  "Good" way to define operations on cuts? Yes: surreal numbers.

- *Classification of cuts in $\mathbb{T}_g \subsetneq \mathbb{T}_{\mathrm{LE}}$ using the cut operations and transseries $f \in \mathbb{T}_g$.*

A) "Horizontal" cuts related to missing pseudo-limits:

$$\mathbf{L} := \{f : \exists n \in \mathbb{N}, f < \log x + \log_2 x + \cdots + \log_n x\}$$

$$\mathbf{\Lambda} := \left\{f : \exists n \in \mathbb{N}, f < \frac{1}{x} + \frac{1}{x \log x} + \cdots + \frac{1}{x \log x \cdots \log_n x}\right\}$$

$$\mathbf{\Gamma} := \mathrm{e}^{-\mathbf{L}} = \left\{f : \forall n \in \mathbb{N}, f < \frac{1}{x \log x \cdots \log_n x}\right\}.$$

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\boldsymbol{\Gamma} := \mathrm{e}^{-\mathbf{L}} &= \left\{f : \forall n \in \mathbb{N}, f < \frac{1}{x \log x \cdots \log_n x}\right\}.
\end{aligned}$$

Horizontal cuts / difference equations / Hahn series

The difference equation

$$f - f \circ \log x = \log x \qquad (\mathcal{E}_h)$$

has no solution in $\mathbb{T}_{\mathrm{LE}}$. *Any solution* f *in* $\mathbb{T} \supsetneq \mathbb{T}_{\mathrm{LE}}$ *fills* $\mathbf{L}$.

Solution: $f = \log x + \log_2 x + \cdots$ *in* $\mathbb{R}[[\mathfrak{M}]] \supseteq \mathbb{T}_{\mathrm{LE}}$. *We have* $\partial(\mathbf{L}) = \boldsymbol{\Lambda}$*; we expect that*

$$\partial(\log x + \log_2 x + \cdots) = \frac{1}{x} + \frac{1}{x \log x} + \cdots + \frac{1}{x \log x \cdots \log_n x} + \cdots$$

B) "Vertical" cuts

$$\begin{aligned} \mathbf{\Omega} \;:=\; \mathbb{T}_{\mathrm{LE}} &= \left\{f : \exists n \in \mathbb{N}, f < \mathrm{e}^{\cdot^{\cdot^{\cdot \mathrm{e}^{x}}}} \ (n \text{ times})\right\} \\ \infty &= \{f : \exists r \in \mathbb{R}, f < r\} \;=\; \{f : \forall n \in \mathbb{N}, f < \log_n x\}. \end{aligned}$$

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$$\infty = \{ f : \exists r \in \mathbb{R}, f < r \} = \{ f : \forall n \in \mathbb{N}, f < \log_n x \}.$$

Vertical cuts / conjugation equations / hyperseries

The formal version of Abel's equations

$$f \circ (x+1) = \mathrm{e}^x \circ f \qquad (\mathcal{E}_\nu^+) \qquad \textit{and} \qquad f \circ \log x = f - 1 \qquad (\mathcal{E}_\nu^-)$$

have no solution in $\mathbb{T}_{\mathrm{LE}}$*. Any solution of* $(\mathcal{E}_\nu^+)$ *(resp.* $(\mathcal{E}_\nu^-)$*) fills* $\mathbf{\Omega}$ *(resp.* ∞*).*

Solutions: *"hyperexponential"* $f_\nu^+ = \mathrm{e}_\omega$ *and "hyperlogarithm"* $f_\nu^- = \ell_\omega$*.*

Note that $\partial(\infty) = \mathbf{\Gamma}$*. Indeed we expect that*

$$\partial(\ell_\omega) = \frac{1}{x \log x \cdots \log_n x \cdots}.$$

C) "Oblique" cuts

$$\mathbf{N} := \left\{ f : \exists n \in \mathbb{N}, f < x + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot^{\sqrt{\log_n x}}}}}} \right\}.$$

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Oblique cuts / mixed equations / nested series

The equation

$$f = x + \mathrm{e}^{f \circ \log x} \qquad f \sim x \qquad (\mathcal{E}_o)$$

has no solution in $\mathbb{T}_{\mathrm{LE}}$. *Any solution* f_o *of* $(\mathcal{E}_o)$ *would fill* $\mathbf{N}$.

Solutions: *"Syntactic" solutions*

$$f_o = x + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log_2 x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}.$$

[Schmeling - 2001]: *It is consistent to consider fields of transseries containing* f_o.

A field of hyperseries [Schmeling - 2001]

Schmeling defines a Hahn series field $\mathbb{H}_{<\omega}$ *with functions* E_{ω^k}, L_{ω^k} *for all* $k \in \mathbb{N}$ *, where*

$$E_{\omega^k} \circ L_{\omega^k} = L_{\omega^k} \circ E_{\omega^k} = \mathrm{id}_{\mathbb{H}_{<\omega}}, \quad \textit{and}$$

$$E_{\omega^{k+1}}(s+1) = E_{\omega^k} \circ E_{\omega^{k+1}}(s) \quad \textit{for all } k \in \mathbb{N} \textit{ and } s \in \mathbb{H}_{<\omega}^{>\mathbb{R}}.$$

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Logarithmic hyperseries [van den Dries, van der Hoeven, Kaplan - 2019]

Field $(\mathbb{L}, {}', \circ)$ *of* **logarithmic hyperseries**, *given as* $\mathbb{L} = \mathbb{R}[[\mathfrak{L}]]$ *where*

$$\mathfrak{L} \textit{ is the group of formal products } \quad \mathfrak{l} = \prod_{\gamma < \rho} \ell_\gamma^{\mathfrak{l}_\gamma}, \ (\mathfrak{l}_\gamma)_{\gamma < \rho} \in \mathbb{R}^\rho,$$

(lexicographic order and pointwise product). Set $\ell_{\omega^\mu + \gamma} := \ell_\gamma \circ \ell_{\omega^\mu}$ *for all* $\gamma < \omega^{\mu+1}$ *and*

$$\ell_\rho' = \frac{1}{\prod_{\gamma < \rho} \ell_\gamma}$$
$$\ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} - 1.$$

Definition

A **hyperserial field** *is a Hahn series field* $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ *endowed with a function*

$$\circ : \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}, \quad \textit{called the composition law,}$$

which satisfies (among other technical details), for all $f \in \mathbb{L}$*:*

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Taylor expansions $f \circ (s + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ *for all* $s \in \mathbb{T}^{>\mathbb{R}}$*, and* $\delta \in \mathbb{T}$ *with* $\delta \prec s$*.*

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[van den Dries, van der Hoeven, Kaplan] $(\mathbb{L}, \circ)$ itself is a hyperserial field.

II.8 - Reducing to partial hyperlogarithms

Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a hyperserial field, let $f \in \mathbb{L}$ and $s \in \mathbb{T}^{>\mathbb{R}}$. We note that

1. The series $f \circ s$ is determined by the class of series $\ell_{\omega^\mu} \circ t$ for all $\mu \in \mathbf{On}$ and $t \in \mathbb{T}^{>\mathbb{R}}$.

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3. For $\mu \in \mathbf{On}$, we have the following self-contained definition of $(\mathfrak{M}_{\omega^\mu})_{\mu \in \mathbf{On}}$:

$$\begin{aligned}
\mathfrak{M}_1 &= \mathfrak{M}^{\succ 1}, \\
\mathfrak{M}_{\omega^{\mu+1}} &= \{\mathfrak{m} \in \mathfrak{M} : \forall n \in \mathbb{N}, \ell_{{\omega^\mu}_n} \circ \mathfrak{m} \in \mathfrak{M}_{\omega^\mu}\}, \quad \text{and} \\
\mathfrak{M}_{\omega^\mu} &= \bigcap_{\iota < \mu} \mathfrak{M}_{\omega^\iota} \quad \text{if } \mu > 0 \text{ is a limit.}
\end{aligned}$$

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Goal: Defining $\mathfrak{M}_\alpha$ as in 3, find conditions on partial functions

$$\breve{L}_\alpha : \mathfrak{M}_\alpha \longrightarrow \mathbb{T}; \mathfrak{a} \longmapsto \ell_\alpha \circ \mathfrak{a}, \quad \alpha = \omega^\mu, \mu \in \mathbf{On}$$

so that $(\breve{L}_\alpha)_{\alpha = \omega^\mu, \mu \in \mathbf{On}}$ determine a composition law $\circ : \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$.

Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ and let $\breve{L}_{\alpha}, \alpha \in \omega^{\mathbf{On}}$ be partial functions $\breve{L}_{\alpha} : \mathfrak{M}_{\alpha} \longrightarrow \mathbb{T}$. Consider a law

$$\mathbb{R} \times \mathfrak{M} \longrightarrow \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^{r}$$

of ordered $\mathbb{R}$-vector space on $\mathfrak{M}$, called the **real powering operation** on $\mathfrak{M}$.

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Our conditions on $\breve{L}_\alpha$ are inequalities, the identities for the logarithm, and the restricted Abel equations:

$$\breve{L}_{\omega^{\mu+1}}(\breve{L}_{\omega^\mu}(\mathfrak{a})) = \breve{L}_{\omega^{\mu+1}}(\mathfrak{a}) - 1 \qquad \text{for all } \mu \in \mathbf{On} \text{ and } \mathfrak{a} \in \mathfrak{M}_{\omega^{\mu+1}}.$$

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Our conditions on $\breve{L}_\alpha$ are inequalities, the identities for the logarithm, and the restricted Abel equations:

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Theorem

Assume that for all $\mu\in\mathbf{On}$, each $s\in\mathbb{T}^{>\mathbb{R}}$ is « sufficiently close » to an $\mathfrak{a}_s\in\mathfrak{M}_{\omega^\mu}$. Then there is a unique function $\circ:\mathbb{L}\times\mathbb{T}^{>\mathbb{R}}\longrightarrow\mathbb{T}$ with $\ell_{\omega^\mu}\circ\mathfrak{a}=\breve{L}_{\omega^\mu}(\mathfrak{a})$ for all $\mu\in\mathbf{On}$ and $\mathfrak{a}\in\mathfrak{M}_{\omega^\mu}$, and such that $(\mathbb{T},\circ)$ is a hyperserial field.

A hyperserial field $(\mathbb{T}, \circ)$ is said **hyperexponentially closed** if for all $\mu \in \mathbf{On}$, the following function is bijective:

$$L_{\omega^\mu}: \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}^{>\mathbb{R}}; s \longmapsto \ell_{\omega^\mu} \circ s.$$

II.10 - Hyperexponential closure

A hyperserial field $(\mathbb{T}, \circ)$ is said **hyperexponentially closed** if for all $\mu \in \mathbf{On}$, the following function is bijective:

$$L_{\omega^\mu} : \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}^{>\mathbb{R}}; s \longmapsto \ell_{\omega^\mu} \circ s.$$

Theorem

There is a hyperexponentially closed extension $\iota : \mathbb{T} \longrightarrow \tilde{\mathbb{T}}$ such that for each hyperexponentially closed extension $\varphi : \mathbb{T} \longrightarrow \mathbb{U}$, there is a unique embedding $\psi : \mathbb{U} \longrightarrow \tilde{\mathbb{T}}$ with $\varphi = \psi \circ \iota$.

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\iota} & \tilde{\mathbb{T}} \\ & \varphi \searrow & \downarrow \exists! \, \psi \\ & & \mathbb{U} \end{array}$$

III - Numbers

III.1 - Surreal numbers

Conway's class **No** of surreal numbers. Underlying order: lexicographically ordered complete binary tree $\{-1,1\}^{<\mathbf{On}}$ whose depths are arbitrary ordinals.

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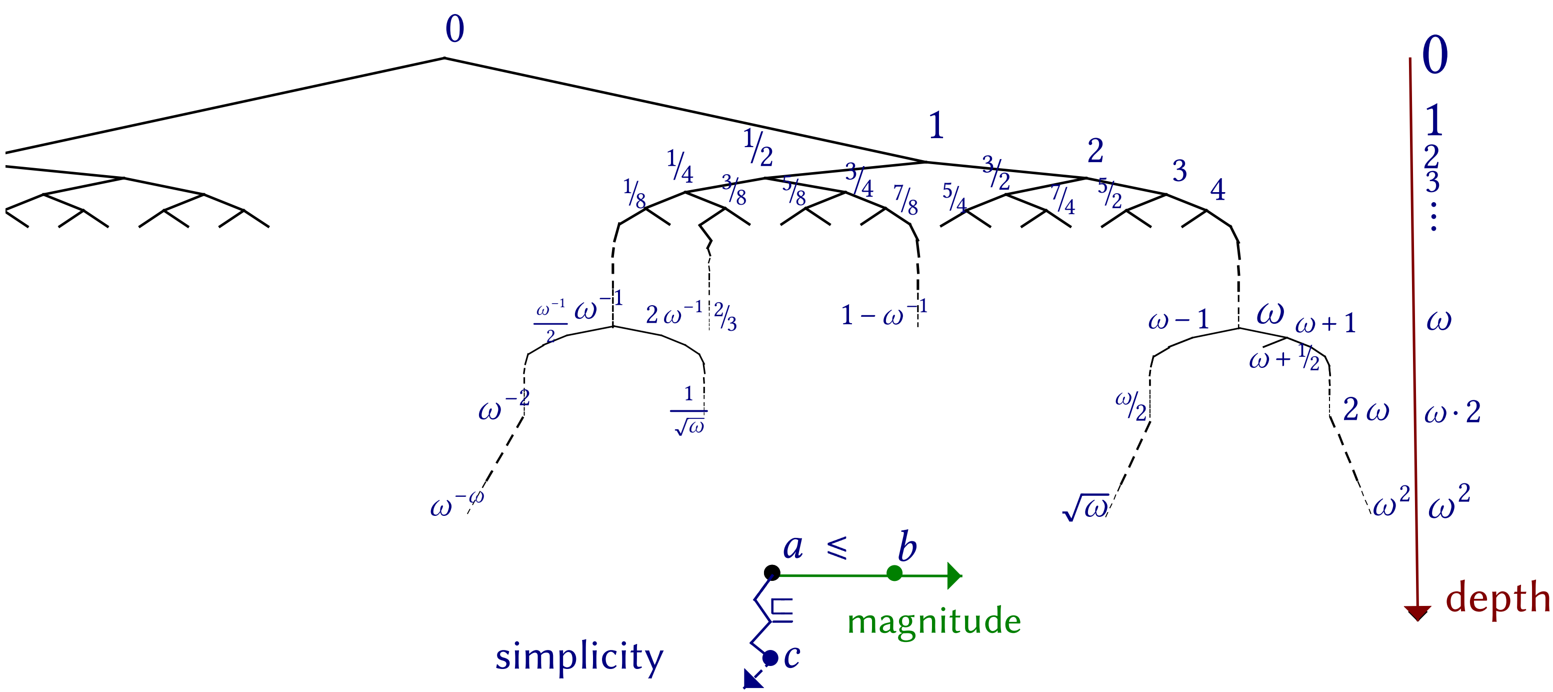

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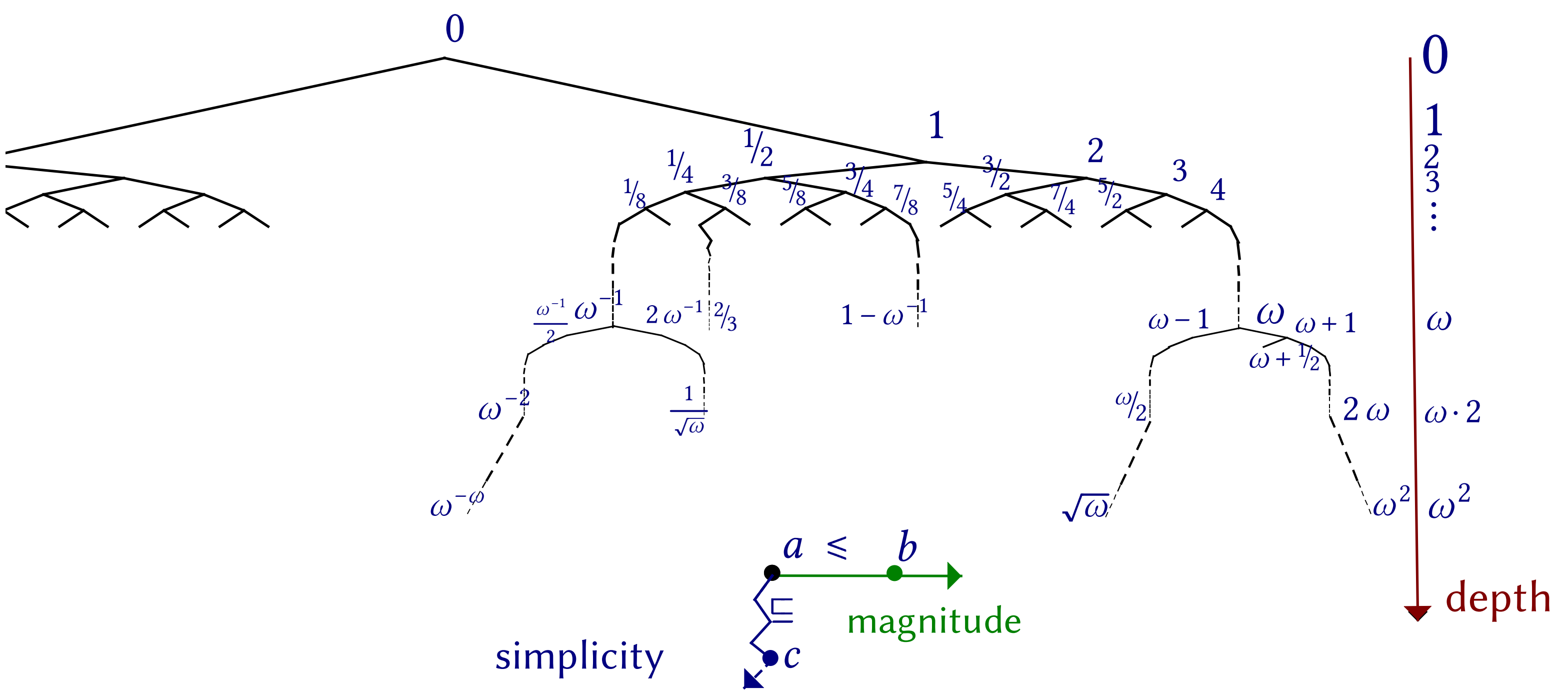

Simplicity: a is **simpler** than b, written $a \sqsubseteq b$, if there is a (descending) path from a to b in the tree.

Fundamental property of $(\mathbf{No},<,\sqsubset)$

For all <u>sets</u> of numbers L, R with $L < R$, there is a unique $\sqsubset$-minimal number $\{L \mid R\}$ with

$$L \;<\; \{L \mid R\} \;<\; R.$$

Equivalently: [order saturation + any non-empty convex subclass has a $\sqsubset$-minimum]

III.2 - Inductive definitions

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$$L < \{L \mid R\} < R.$$

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Well-founded order

For $a \in \mathbf{No}$, we have two sets $a_L := \{b \in \mathbf{No} : b < a, b \sqsubseteq a\}$ *and* $a_R := \{b \in \mathbf{No} : b > a, b \sqsubseteq a\}$.

So $a = \{a_L \mid a_R\}$. The partial order $(\mathbf{No}, \sqsubseteq)$ is well-founded $\longrightarrow$ inductive definitions.

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Surreal arithmetic [Conway - 1976]

Inductive definition of the sum $a + b$ of numbers a, b:

Inductive hypothesis: $a_L + b, a + b_L$ *and* $a + b_R, a_R + b$ *are defined.*

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By definition, $\underset{<a}{a_L} + b,\ a + \underset{<b}{b_L}$ *and* $a + \underset{>b}{b_R},\ \underset{>a}{a_R} + b$

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Inductive definition of the sum $a + b$ of numbers a, b. We have

$$a + b = \{a_L + b, a + b_L \mid a + b_R, a_R + b\}.$$

Similar equations exist for $-a, a\,b, {}^{a}/_{b}$.

III.3 - Valuation theory of **No**

Monomials [Conway - 1976]

For each $a \in \mathbf{No}^{\times}$, *there is a simplest* $\mathfrak{d}_a \in \mathbf{No}^{>}$ *with* $v\mathfrak{d}_a = va$ *(natural valuation). We set*

$$\mathbf{Mo} \ := \ \{\mathfrak{d}_a : a \in \mathbf{No}^{\times}\} \ \subseteq \ \mathbf{No}^{>0}: \quad \textit{class of } \textbf{monomials}$$

$\mathbf{Mo}$ *is a subgroup of* $(\mathbf{No}^{>0}, \times)$ *with a natural isomorphism* $(v\mathbf{No}^{\times}, <) \simeq (\mathbf{Mo}, >)$.

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$(\mathbf{No}, <)$ is saturated $\longrightarrow$ $(\mathbf{No}, +, \times, v)$ is spherically complete. If u is a pseudo-Cauchy sequence in $\mathbf{No}$, then its convex class of pseudo-limits has a $\sqsubseteq$-minimum $\operatorname{simplim} u$.

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For $f = \sum_{\gamma < \rho} f_\gamma \mathfrak{m}_\gamma \in \mathbb{R}[[\mathbf{Mo}]]$, define $\hat{f} \in \mathbf{No}$ by induction on $\rho \in \mathbf{On}$:

$$\hat{f} := \widehat{\sum_{\gamma < \beta} f_\gamma \mathfrak{m}_\gamma} + f_\beta \mathfrak{m}_\beta \qquad \text{if } \rho = \beta + 1 \text{ is a successor ordinal, and}$$

$$\hat{f} := \operatorname{simplim} \left(\widehat{\sum_{\gamma < \sigma} f_\gamma \mathfrak{m}_\gamma} \right)_{\sigma < \rho} \qquad \text{if } \rho \text{ is a limit ordinal.}$$

Then $\mathbb{R}[[\mathbf{Mo}]] \longrightarrow \mathbf{No}; f \mapsto \hat{f}$ is an isomorphism of ordered valued fields.

Exponential [Gonshor - 1986]

For $a \in \mathbf{No}$ *and* $n \in \mathbb{N}$, *set* $[a]_n := \sum_{k \leqslant n} \frac{a^k}{k!}$. *The function*

$$\boldsymbol{E_1}(a) := \left\{ \boldsymbol{E_1}(a_L)\,[a-a_L]_{\mathbb{N}}\,,\ \boldsymbol{E_1}(a_R)\,[a-a_R]_{2\mathbb{N}+1} \;\middle|\; \frac{\boldsymbol{E_1}(a_R)}{[a_R-a]_{2\mathbb{N}+1}},\ \frac{\boldsymbol{E_1}(a_L)}{[a_L-a]_{\mathbb{N}}} \right\}$$

is an isomorphism $(\mathbf{No},+,<) \longrightarrow (\mathbf{No}^{>0},\times,<)$. *Set* $E_n = E_1^{\circ n}$ *and* $L_n = E_1^{\circ(-n)}$ *for all* $n \in \mathbb{N}$.

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The exponential interacts with the structure of Hahn series field as follows:

$$\begin{aligned} \mathbf{Mo} &= E_1(\mathbb{R}[[\mathbf{Mo}^{>1}]]) \\ \forall \varepsilon \prec 1, E_1(\varepsilon) &= \sum_{k \in \mathbb{N}} \frac{1}{k!} \varepsilon^k. \end{aligned}$$

III.4 - Surreal exponentiation

Exponential [Gonshor - 1986]

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[Berarducci, S. Kuhlmann, Mantova and Matusinski - 2019]: How to use these properties to define more general exponential and logarithmic functions on fields of Hahn series.

III.5 - Transseries and cuts as numbers

[Berarducci, Mantova - 2017]: There is a unique embedding $\mathbb{T}_{\mathrm{LE}} \longrightarrow \mathbf{No}$ which commutes with transfinite sums and exponentials, and sends x to ω.

numbers as generalized transseries $\longrightarrow$ filling cuts $\boldsymbol{c} \equiv (L, R)$ in $\mathbb{T}_{\mathrm{LE}}$ with numbers $\{L \mid R\}$?

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Vertical cuts as numbers

We have $E_n(\omega) = \omega^{\cdot^{\cdot^{\omega}}}$ *(*n *times) for all* $n \in \mathbb{N}$ *. So the number*

$$\varepsilon_0 = \{\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots \mid \varnothing\}$$

is the simplest « transexponential » number. Candidate for e_{ω} *in* $\mathbf{No}$*?*

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Nested cuts as numbers

Candidate for f_o: the corresponding number

$$\left\{\omega, \omega + \mathrm{e}^{\sqrt{\log \omega}}, \omega + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log_2 \omega}}}, \ldots \;\middle|\; \ldots, \omega + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{r\sqrt{\log_2 \omega}}}, \omega + \mathrm{e}^{r\sqrt{\log \omega}}, r\omega : r > 1\right\}.$$

III.6 - Surreal substructures

Definition

A **surreal substructure** *is a class* $\mathbf{S} \subseteq \mathbf{No}$ *such that* $(\mathbf{S}, \leqslant, \sqsubseteq)$ *and* $(\mathbf{No}, \leqslant, \sqsubseteq)$ *are isomorphic. There is a unique such isomorphism* $\Xi_{\mathbf{S}} : \mathbf{No} \longrightarrow \mathbf{S}$*.*

$\longrightarrow$ each $a \in \mathbf{S}$ is determined by its « label » z in $a = \Xi_{\mathbf{S}}(z)$.

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Numerous classes involved in the study of $\mathbf{No}$ are surreal substructures:

- The class $\mathbf{No}^{>\mathbb{R}}$ of positive infinite numbers.
- The class $\mathbf{No}^{\prec}$ of infinitesimals.
- The class $\mathbf{Mo}$ of monomials.
- The class $\mathbb{R}[[\mathbf{Mo}^{>1}]] = L_1(\mathbf{Mo})$ of so-called purely infinite numbers.
- The class $\mathbf{Mo}_{\omega} = \bigcap_{n \in \mathbb{N}} E_n(\mathbf{Mo})$ of « log-atomic » numbers.

The first hyperlogarithm $L_\omega\colon \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}^{>\mathbb{R}}$ can be defined using simple rules:

Defining L_ω on No [with van der Hoeven and Mantova]

- *We must have* $L'_\omega = \frac{1}{L_0 L_1 L_2 L_3 \cdots}$, *so* $L''_\omega = -\sum_{k\in\mathbb{N}} \frac{1}{L_k L_{k+1} \cdots}$, *and so on…*

III.7 - Surreal hyperexponentiation (1)

The first hyperlogarithm $L_\omega : \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}^{>\mathbb{R}}$ can be defined using simple rules:

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- *Let* $a \in \mathbf{No}^{>\mathbb{R}}$. *We assume that* $L_\omega(a)$ *is defined, and try to define* $L_\omega(a+\varepsilon)$ *for each* $\varepsilon \prec 1$. *The family* $(L_\omega^{(k)}(a)\, \varepsilon^k)_{k>0}$ *is summable. So we set*

$$L_\omega(a+\varepsilon) \;:=\; \sum_{k \geqslant 0} \frac{L_\omega^{(k)}(a)}{k!}\, \varepsilon^k.$$

III.7 - Surreal hyperexponentiation (1)

The first hyperlogarithm $L_\omega \colon \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}^{>\mathbb{R}}$ can be defined using simple rules:

Defining L_ω on No [with van der Hoeven and Mantova]

- *We must have* $L'_\omega = \frac{1}{L_0 L_1 L_2 L_3 \cdots}$, *so* $L''_\omega = -\sum_{k \in \mathbb{N}} \frac{1}{L_k L_{k+1} \cdots}$, *and so on…*
- *Let* $a \in \mathbf{No}^{>\mathbb{R}}$. *We assume that* $L_\omega(a)$ *is defined, and try to define* $L_\omega(a+\varepsilon)$ *for each* $\varepsilon \prec 1$. *The family* $(L_\omega^{(k)}(a)\, \varepsilon^k)_{k>0}$ *is summable. So we set*

$$L_\omega(a+\varepsilon) \ := \ \sum_{k \geqslant 0} \frac{L_\omega^{(k)}(a)}{k!}\, \varepsilon^k.$$

Define L_ω *on a class* $\mathbf{S}$ *with* $\forall b \in \mathbf{No}^{>\mathbb{R}}, \exists n \in \mathbb{N}, L_n(b) =: a + \varepsilon \in \mathbf{S} + \mathbf{No}^{\prec 1}$. *Then*

$$L_\omega(b) \ = \ L_\omega(E_n(a+\varepsilon)) \underset{\text{Abel eq}}{=} n + L_\omega(a+\varepsilon) \ = \ n + \sum_{k=0}^{+\infty} \frac{L_\omega^{(k)}(a)}{k!}\, \varepsilon^k.$$

III.8 - Surreal hyperexponentiation (2)

The class $\mathbf{Mo}_\omega$ of log-atomic numbers satisfies the previous conditions. For $a, b \in \mathbf{No}^{>\mathbb{R}}$ with $a < b$, we require

$$\begin{aligned} L_\omega(a) &< L_n(a) \quad \text{and} \\ L_\omega(a) &< L_\omega(b), \end{aligned}$$

which calls for the following inductive definition of $\breve{L}_\omega$:

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$$\forall \mathfrak{a} \in \mathbf{Mo}_\omega, \breve{L_\omega}(\mathfrak{a}) := \left\{ \breve{L_\omega}(\mathfrak{a}') + \frac{1}{L_n(\mathfrak{a}')} \,\middle|\, \breve{L_\omega}(\mathfrak{a}'') - \frac{1}{L_n(\mathfrak{a}'')}, L_n(\mathfrak{a}) \right\},$$

for generic $\mathfrak{a}'$, $\mathfrak{a}''$ with $\mathfrak{a}', \mathfrak{a}'' \in \mathbf{Mo}_\omega$, $\mathfrak{a}', \mathfrak{a}'' \sqsubseteq \mathfrak{a}$, $\mathfrak{a}' < \mathfrak{a} < \mathfrak{a}''$ and $n \in \mathbb{N}$.

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Theorem

This generalizes to L_{ω^μ} for each $\mu \in \mathbf{On}$. We obtain a composition law

$$\circ : \tilde{\mathbb{L}} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}. \quad (\tilde{\mathbb{L}} \subsetneq \mathbf{No})$$

Hyperserial expansions

For $\mathfrak{m} \in \mathbf{Mo}^{\neq 1}$, *there are* $\alpha, \beta \in \mathbf{On}$, $u, \psi \in \mathbf{No}$, $\iota \in \{-1, 1\}$ *such that* $\mathfrak{m}$ *expands uniquely as*

$$\mathfrak{m} \;=\; \mathrm{e}^{\psi}\,(L_{\beta} E_{\alpha}^{u})^{\iota} \qquad (\,=\exp(\psi) \times (L_{\beta}(E_{\alpha}(u)))^{\iota}\,).$$

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Constructing paths in numbers

Fix $a_0 \in \mathbf{No}^\times$. *At each stage* i, *pick a term* $r_i \mathfrak{m}_i$ *in* a_i *seen as a Hahn series, and expand it as*

$$r_i \mathfrak{m}_i = r_i \mathrm{e}^{\psi_i} (L_{\beta_i} E_{\alpha_i}^{u_i})^{\iota_i}.$$

Then choose $a_{i+1} \in \{\psi_i, u_i\}$ *with the restrictions*

$$(\psi_i = 0 \implies a_i \neq \psi_i) \quad \textit{and} \quad (E_{\alpha_i}^{u_i} = \omega \implies a_i \neq u_i)$$

This defines a **path** $(\mathfrak{m}_i)_{i<\ell}$, $\ell \leqslant \omega$ *in* a_0 ($\ell < \omega$ *if and only if* $\mathfrak{m}_i = (L_{\beta_i}(\omega))^{\iota_i}$ *for some* i).

III.9 - Hyperserial expansions

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This defines a **path** $(\mathfrak{m}_i)_{i<\ell}$ *in* a_0. *It is* **well-nested** *if for large enough* i, *we have*

$$\beta_i = 0 \quad , \quad \mathfrak{m}_{i+1} \notin \operatorname{supp} \psi_i \quad , \quad \mathfrak{m}_{i+1} = \min \operatorname{supp} u_i \quad , \text{and} \quad r_i \in \{-1, 1\}.$$

Describing numbers as hyperseries

A) existence of infinite paths $\longrightarrow$ *yes, e.g. there are numbers* $a_0 \in \mathbf{No}$ *which expand as*

$$a_0 = \omega + E_1^{\sqrt{L_1(\omega)} + \mathrm{e}^{\sqrt{L_2(\omega)}} E_\omega^{\sqrt{L_\omega(\omega)} + \mathrm{e}^{\sqrt{L_{\omega^2}(\omega)}} E_{\omega^2}^{\cdot^{\cdot^{\cdot}}}}}$$

B) structure of infinite paths

C) mutiplicity of numbers with a given expansion

Describing numbers as hyperseries

A) existence of infinite paths

B) structure of infinite paths $\longrightarrow$ *well-nested paths*

C) mutiplicity of numbers with a given expansion

Describing numbers as hyperseries

A) existence of infinite paths

B) structure of infinite paths

C) mutiplicity of numbers with a given expansion $\longrightarrow$ *by B), infinite expansions end up as*

$$a_i \;=\; \varphi_i \pm \mathrm{e}^{\psi_i} \left(E_{\alpha_i}^{\varphi_{i+1} \pm \left(E_{\alpha_{i+1}}^{\cdot^{\cdot^{\cdot}}}\right)^{\iota_{i+1}}} \right)^{\iota_i}, \quad \textit{for large enough } i, \textit{ for some } \varphi_i\textit{'s.}$$

Theorem: *For large enough* i*, numbers which expand as* a_i *form a surreal substructure.*

Describing numbers as hyperseries

A) existence of infinite paths $\longrightarrow$ *yes*

B) structure of infinite paths $\longrightarrow$ *well-nested paths*

C) mutiplicity of numbers with a given expansion? $\longrightarrow$ *ultimately* $\simeq \mathbf{No}$

Theorem

Using this description of paths, one can represent any surreal number a *as a tree labelled by real and ordinal numbers*

Thank you!