Ordered groups of regular growth rates

by Vincent Bagayoko

IMJ-PRG, Paris

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Furthermore the subset

$$\mathcal{G}_\mathcal{M} := \left\{ f \in \mathcal{M}_\infty : \lim_{+\infty} f = +\infty \right\}$$

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Answer: No

For any ordered group $(G, \cdot, 1, <)$*, the structure* $\mathcal{M} := (G, <, 1, (h \mapsto g\,h)_{g \in G})$ *is o-minimal and* $\mathcal{G}_{\mathcal{M}} \simeq G$*.*

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What if $\mathcal{M}$ expands a real closed field?

Given an ordered group $(\mathcal{G}, \cdot, 1, <)$ and a formula $\varphi(\overline{g}, \overline{y})$ with parameters $\overline{g} = (g_1, \ldots, g_n) \in \mathcal{G}$, when is there an extension $\mathcal{G}^* \supseteq \mathcal{G}$ that satisfies $\exists \overline{y} (\varphi(\overline{g}, \overline{y}))$?

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Problem: this turns out to be a very difficult question. It entails for instance matters of

divisibility	$g\,y^{-\alpha}=1$
conjugacy	$g_1\,y\,g_2^{-1}\,y^{-1}=1$
general functional inequalities	$g_1\,y^{\alpha_1}\cdots g_n\,y^{\alpha_n}\geqslant 1.$

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Analogy with real closed valued fields, or so-called H-closed fields (e.g. transseries): the formula

$$\exists\overline{y}\,(t(\overline{a},\overline{y})\geqslant 1) \qquad \text{(for a term } t(\overline{x},\overline{y}) \text{ in the language)}$$

could be translated into:

- semialgebraic relations on definable quotients with poorer structure (e.g. the residue field and value group, the residue field and the asymptotic couple)
- completeness conditions (e.g. spherical completeness)

Is there a *valuation theory* on ordered groups $\mathcal{G}_{\mathcal{M}}$ coming from o-minimal fields?

Growth. Let $\mathcal{M} = (M, +, \cdot, 0, 1, <, \ldots)$ be o-minimal. Let $f, g \in \mathcal{G}_{\mathcal{M}}$. When is

$$f \circ g > g \circ f \ ? \tag{1}$$

- f cannot lie in the centraliser $\mathcal{C}(g)$ of g in $\mathcal{G}_{\mathcal{M}}$
- (1) should hold for sufficiently large f.

Growth. Let $\mathcal{M}=(M,+,\cdot,0,1,<,\ldots)$ be o-minimal. Let $f,g\in\mathcal{G}_{\mathcal{M}}$. When is

$$f\circ g>g\circ f\ ? \tag{2}$$

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- (2) should hold for sufficiently large f.

Valuation. The map $v:g\mapsto \mathrm{ConvexHull}(\mathcal{C}(g))$ should be a measure of the size of elements in $\mathcal{G}$.

Asymptotic expansions. Elements $f\in\mathcal{H}_{\mathcal{M}}$ can have asymptotic expansions

$$f\approx r_0\,\mathfrak{m}_0+\cdots+r_i\,\mathfrak{m}_i+\cdots$$

where $r_i\in\mathbb{R}^{\times}$ and $\mathfrak{m}_0\gg\mathfrak{m}_1\gg\cdots$ lie in a section of the natural valuation.

Growth properties

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Valuation. The map $v : g \mapsto \mathrm{ConvexHull}(\mathcal{C}(g))$ should be a measure of the size of elements in $\mathcal{G}$.

Asymptotic expansions. Likewise, elements $f \in \mathcal{G}_{\mathcal{M}}$ ought to have asymptotic expansions

$$f \approx \mathscr{s}_0^{[r_0]} \circ \cdots \circ \mathscr{s}_i^{[r_i]} \circ \cdots$$

where $r_i \in \mathbb{R}^{\times}$, $\mathscr{s}_i^{[r_i]} \in \mathcal{C}(\mathscr{s}_i)$ and $v(\mathscr{s}_0) < v(\mathscr{s}_1) < \cdots$.

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An element $s \in \mathcal{G}^{>1}$ is **scaling** if $\mathcal{C}(s)$ is Abelian, and for all $f \in \mathcal{G}$ with $v(f) = v(s)$, there is an $s_0 \in \mathcal{C}(s)$ with $v(f s_0^{-1}) < v(f)$.

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For any section $\mathfrak{s}$ of $v : \mathcal{G} \longrightarrow v(\mathcal{G})$ ranging in scaling elements, the family $(\mathcal{C}(\mathfrak{s}(\rho)))_{\rho \in v(\mathcal{G})}$ is a functorial construction.

Strictly increasing affine maps

k: *ordered field;* G: *ordered vector space over* k; $\mathcal{M} := (G, +, 0, <, (g \mapsto c\,g)_{c \in k})$

Then $\mathcal{M}$ *is o-minimal, and has QE and a universal axiomatisation. So* $\mathcal{G}_{\mathcal{M}}$ *is the group* $\mathrm{Aff}_k^+(G) \simeq k^{>0} \ltimes G$ *of strictly increasing affine maps* $G \to G$*. This is a GOG whose value (ordered) set* $\{v(f) : f \in \mathcal{G}_{\mathcal{M}}\}$ *is* 2*.*

Examples

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Theorem 1

Let $\mathcal{M}$ be an o-minimal expansion of the real ordered field. Suppose that each germ in $\mathcal{G}_{\mathcal{M}}$ has a level, i.e.

$$\forall f \in \mathcal{G}_{\mathcal{M}}, \exists e \in \mathbb{Z}, \exists n \in \mathbb{N}, |\log_n \circ f \circ \exp_n - \exp_e| \leqslant 1.$$

Then $\mathcal{G}_{\mathcal{M}}$ is a growth order groups whose centralisers are Archimedean.

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Question: do growth order groups always act on ordered differential fields?

Asymptotic expansions

Three routes in order to make sense of non-commutative asymptotic expansions:

A) Construct ordered groups of formal non-commutative “asymptotic expansions”.

B) Give Kaplansky-like descriptions of extensions of GOGs that do not add new valuations.

C) Propose an axiomatic framework for ordered infinite products in groups.

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! Given a linearly ordered family $(\mathcal{C}_s)_{s\in\mathcal{V}}$ of ordered Abelian groups, defining group laws on the Hahn product $\prod_{s\in\mathcal{V}}\mathcal{C}_s$ is involved from an algebraic standpoint.

It entails more information than just the family $(\mathcal{C}_s)_{s\in\mathcal{V}}$ and also entails making choices.

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! This entails solving inequalities $t(\overline{g}, y) \geqslant 1$ over ordered groups, and even those which do not require new valuations are difficult to study.

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Multiplicability groups

A **multipliability group** is a group $(\mathcal{G}, \cdot, 1)$ together with a family of partial functions

$$\Pi_{(I,<)} : \mathcal{G}^I \longrightarrow \mathcal{G}$$

for all linearly ordered sets $(I, <)$, extending the finite products: if $i_1 < \cdots < i_n$, then

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Work in progress

The group of positive infinite transseries has a natural structure of multipliability group. Moreover, each transseries f can be expressed uniquely as an infinite product

$$f=(\mathrm{e}^x)^{[n_0]}\circ\jmath_1^{[r_1]}\circ\cdots\circ\jmath_\gamma^{[r_\gamma]}\circ\cdots\qquad\gamma<\lambda\in\mathbf{On}$$

for a fixed section of the valuation $\supseteq\jmath_0,\jmath_1,\ldots,$ for $n_0\in\mathbb{Z}$ and $r_1,\ldots,r_\gamma,\ldots\in\mathbb{R}^\times$.

The nearly Abelian case

A growth order group $\mathcal{G}$ is said **nearly Abelian** if for all $f, g \in \mathcal{G} \setminus \{1\}$, we have

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Suppose further that each centraliser in $\mathcal{G}$ is divisible and that valuative balls have the intersection property. Then each function

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Future work: eliminating quantifiers in an extended language.

Thanks!