Growth order groups

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based on joint work with J. van der Hoeven, E. Kaplan, L. S. Krapp,

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Ordered groups

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A **unary representation** *of* $\mathcal{G}$ *is an embedding* $t.: (\mathcal{G}, \cdot, 1, <) \longrightarrow (\mathrm{Aut}(X, <_X), \circ, \mathrm{id}_X, <_\forall)$ *for some linearly ordered set* X, *where* $<_\forall$ *is the ordering by universal comparison.*

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Theorem [Balzanov-Baldwin-Verbovskiy, 2007]

If $(X,<)$ *is dense or* $(X,t.,<_X)=(\mathcal{G},\mathrm{left.},<)$, *then* $(X,(t_g)_{g\in\mathcal{G}},<)$ *has QE and is o-minimal.*

Two examples

Abelian ordered groups and HAHN's embedding theorem

Each Abelian ordered group embeds in a Hahn ordered group

$$H[\mathfrak{M},\mathbb{R}] = \{ f \in \mathbb{R}^{\mathfrak{M}} : (\operatorname{supp} f) := \{x \in \mathfrak{M} : f(x) \neq 0\} \text{ is reverse well-ordered} \}$$

where $(\mathfrak{M},\prec)$ *is a linearly ordered set. Each* f *is represented as a series* $f = \sum_{\mathfrak{m}\in\mathfrak{M}} f(\mathfrak{m})\,\mathfrak{m}$.

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Strictly increasing affine maps

Let K *be an ordered field. Let* $(G,+,<)$ *be an ordered vector space over* K. *Then* $\mathrm{Aff}_K^+(G) := K^> \times G$ *is an ordered group for the product* $(a,\alpha)\cdot(b,\beta) := (a\,b, a\,.\,\beta+\alpha)$ *and the lexicographic ordering.*

We can represent each (a,α) as the strictly increasing affine map $\mathrm{Aff}_{a,\alpha}\colon \gamma \mapsto a\,.\,\gamma+\alpha$, and the ordering is then $(a,\alpha) < (b,\beta)$ iff $\mathrm{Aff}_{a,\alpha}(\gamma) < \mathrm{Aff}_{b,\beta}(\gamma)$ for sufficiently large $\gamma \in G$.

Fix an ordered group $(\mathcal{G}, \cdot, 1, <)$. Define $\mathrm{Term}(\mathcal{G})$ as the free product of $\mathcal{G}$ and $(y^{\mathbb{Z}}, \cdot) \simeq (\mathbb{Z}, +)$.

Simple task: given

$$t(y) = g_1\, y^{\alpha_1} \cdots g_n\, y^{\alpha_n} \in \mathrm{Term}(\mathcal{G}),$$

understand the theory of equations $t(f) = 1$ for f lying in ordered extensions of $\mathcal{G}$.

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Remark. Given $\mathcal{G}'\supseteq\mathcal{G}$ and $f\in\mathcal{G}'$, the extension $\mathcal{G}\langle f\rangle/\mathcal{G}$ is determined by an ordering on the quotient group $\mathrm{Term}(\mathcal{G})/N_f$ where

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Univariate equations

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Examples:

- $t(y) = f^{-1}\, y^{-1}\, f\, y \quad \longrightarrow \quad$ centralizer extension
- $t(y) = f^{-2}\, y \quad \longrightarrow \quad$ extension with a "square root".
- $t(y) = y\, f\, y^{-1}\, g \quad \longrightarrow \quad$ conjugacy extension

Assume $(\mathcal{G},\cdot,1)$ is Abelian. For f in an Abelian o.g. extension of $\mathcal{G}$, and $t(y)=g_1\,y^{\alpha_1}\cdots g_n\,y^{\alpha_n}\in \mathrm{Term}(\mathcal{G})$, we have $t(f)=1$ if and only if

$$f^{n}=g \tag{1}$$

where $\alpha := \sum_{i=1}^{n}\alpha_i \in \mathbb{Z}$ and $g := g_n^{-1}\cdots g_1^{-1}\in \mathcal{G}$.

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Divisible closure for ordered Abelian groups

There is a universal divisible extension $(\mathcal{G},\cdot,1,<) \longrightarrow (\hat{\mathcal{G}},\hat{\cdot},\hat{1},\hat{<})$.

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In the non-Abelian case, solving

$$g_1\,y^{\alpha_1}\cdots g_n\,y^{\alpha_n} = 1$$

in extensions of $\mathcal{G}$ is highly non-trivial (recall the functional representation of $\mathcal{G}$ ...).

Divisibility, conjugacy, problems

Elementary property of an ordered group $\mathcal{G}$

A) **[Levi, 1942]** *for* $m > 0$, $f^m = g^m \Longrightarrow f = g$.

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Question 1: Is there an ordered group in which any two strictly positive elements are conjugate?

Question 2: Is there a first-order theory of ordered groups which is complete and model complete, and has non-Abelian models? [Pillay-Steinhorn, 1986]: such theory is not o-minimal.

Let $\mathcal{M} = (M, <, \dots)$ be an o-minimal structure.

Write $\mathcal{F}_{\mathcal{M}}$ for the set of germs $[f]$ at $+\infty$ of definable maps $f : M \longrightarrow M$. We have $\mathcal{M} \preccurlyeq \mathcal{F}_{\mathcal{M}}$ for a natural "asymptotic" structure on $\mathcal{H}_{\mathcal{M}}$. We set:

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- For $\mathcal{M}=(R,<,+,\cdot)$, we have $\mathcal{G}_{\mathcal{M}}\supsetneq\mathrm{Aff}^{+}_{R}(R)$. Moreover $\mathcal{F}_{\mathcal{M}}$ has a structure of differential field **[vdDries, 1998]**.

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- For $\mathcal{M} = (R, <, +, \cdot)$, we have $\mathcal{G}_{\mathcal{M}} \supsetneq \mathrm{Aff}^{+}(R)$. Moreover $\mathcal{H}_{\mathcal{M}}$ has a structure of differential field **[vdDries, 1998]**.
- The field $\mathcal{F}_{\mathbb{R}_{\exp}}$ contains all germs of functions that can be obtained as combinations of $\exp$, $\log$, and semialgebraic functions.

A growth axiom

A simple inequation

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More precisely, we should have $f > \mathcal{C}(g)$, where

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Question 3: If $\mathcal{M}$ expands the real ordered field, then must $\mathcal{G}_{\mathcal{M}}$ be a growth order group?

Expansions

Additive expansions in $\mathcal{F}_{\exp}$

Any $f \in \mathcal{F}_{\exp}$ *can be approximated using "additively indecomposable" germs* $\mathfrak{m}_0, \ldots, \mathfrak{m}_n$ *where* $\mathfrak{m}_{i+1} = o_{+\infty}(\mathfrak{m}_i)$ *and* $f - r_0\,\mathfrak{m}_0 - \cdots - r_n\,\mathfrak{m}_n = o_{+\infty}(\mathfrak{m}_n)$.

$$\text{E.g.} \qquad (x \log x)^{\mathrm{inv}} \stackrel{?}{=} \frac{x}{\log x} + \frac{x \log_2 x}{(\log x)^2} + \frac{1}{2}\,\frac{x\,(\log_2 x)^2}{(\log x)^3} + \cdots$$

Additive expansions in $\mathcal{F}_{\exp}$

Any $f \in \mathcal{F}_{\exp}$ *can be approximated using "additively indecomposable" germs* $\mathfrak{m}_0, \ldots, \mathfrak{m}_n$ *where* $\mathfrak{m}_{i+1} = o_{+\infty}(\mathfrak{m}_i)$ *and* $f - r_0\,\mathfrak{m}_0 - \cdots - r_n\,\mathfrak{m}_n = o_{+\infty}(\mathfrak{m}_n)$.

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Compositional expansions in $\mathcal{G}_{\exp}$

Take $f \in \mathcal{G}_{\exp}$. *Can* f *be approximated using "compositionally indecomposable" germs* $f_0, \ldots, f_n$ *where* $f_0 \succ f_1 \succ \cdots \succ f_n$ *and*

$$f \circ (f_n^{[r_n]} \circ \cdots \circ f_1^{[r_1]} \circ f_0^{[r_0]})^{-1} \prec f_n$$

What should the infinite composition

$$\cdots \circ f_n \circ \cdots \circ f_0$$

mean?

Transseries

Transseries [Dahn-Göring, 1987 and Ecalle, 1992]

Field $\mathbb{T}$ *of* **transseries***: generalized series involving* x, $\log x$ *and* e^x *and combinations thereof.*

$$f := \sum_{n=1}^{+\infty} n!\, e^{x/n} + \log_2 x + 7 + x^{-2}\log x + \sum_{p=0}^{+\infty} e^{-x^{p+1}} x^p \quad \textit{is a transseries.}$$

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Remark: The elements $2\,x$ and $x+1$ are conjugate, *via* $(2\,x) \circ \mathrm{e}^{(\log 2)x} = \mathrm{e}^{(\log 2)x} \circ (x+1)$.

We now look at the structure $(\mathbb{T}^{>\mathbb{R}}, x, \circ, <)$.

Theorem [vdDries-Macintyre-Marker, 2001]

This is a right-ordered group, i.e for all f, g, h, $f > g \Rightarrow f \circ h > g \circ h$.

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Since $x + 1 \prec \mathrm{e}^x$ and $f^{[n]} = x + 1$ has a solution $f = x + \frac{1}{n}$ for all $n \in \mathbb{N}^{>}$, we deduce that

$$\mathbb{T}_0^{>\mathbb{R}} := \{f \in \mathbb{T}^{>\mathbb{R}} : f \prec \mathrm{e}^x\}$$

is a divisible growth order group.

Schmeling: defined an extension of $\mathbb{T}$ with formal symbols $\mathrm{e}^x_{\omega^n}, \ell_{\omega^n} x$, $n \in \mathbb{N}$, where $\mathrm{e}^x_1 = \mathrm{e}^x$ and

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A differential ordered field with composition $(\mathbb{L}, +, \times, <, \partial, \circ)$ *where for all* $\mu \in \mathbf{On}$, *we have a symbol* $\ell_{\omega^\mu} x \in \mathbb{L}$ *with* $\ell_1 x = \log x$ *and*

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Finitely nested hyperseries [B.-vdHoeven-Kaplan, 2021]

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Theorem [B., 2022]

The derivation ∂ *and composition law* $\circ$ *on* $\mathbb{L}$ *extend in a natural way to* $\tilde{\mathbb{L}}$.

Using Taylor series arguments, one shows that:

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The structure $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$ *is a growth order group.*

So we have a GOG solution to our **Question 1** on ordered groups.

Ordered products

Let J be a set, write $J^\star := \bigcup_{n\in\mathbb{N}} J^n$, seen as a monoid for the concatenation law. Let $k\langle\langle J\rangle\rangle$ be the local algebra of functions $J^\star \longrightarrow k$ under pointwise sum and Cauchy product. Writing $X_\theta = \chi_{\{\theta\}}$ for all $\theta \in J^\star$, each $P \in k\langle\langle J\rangle\rangle$ is a formal sum

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If A is a local algebra equipped with a notion of transfinite sum and A is closed under sums $(a_{j_1} \cdot a_{j_2} \cdots a_{j_n})_{(j_1, \ldots, j_n) \in J^\star}$ whenever $a = (a_j)_{j \in J}$ is summable, then the products

$$\prod_{(J,<)} a := \sum_{(j_1, \ldots, j_n) \in J^\star} \mathrm{OP}_{(J,<)}(j_1, \ldots, j_n)\, a_{j_1} \cdots a_{j_n}$$

are well-defined.

A group $\mathcal{G}$ is said **strong** if, for all linearly ordered sets $I = (\underline{I}, <)$, we have partial functions $\Pi_I \subseteq \mathcal{G}^{\underline{I}} \rightarrow \mathcal{G}$ defined on subgroups $\operatorname{dom} \Pi_I$ of $\mathcal{G}^{(\underline{I})}$, sending each $\chi_{\{g\}}, g \in \mathcal{G}$ to g, and such that:

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SG1. If $\varphi: I \longrightarrow J$ is an order isomorphism and $g \in \operatorname{dom}\Pi_J$, then $g\circ\varphi \in \operatorname{dom}\Pi_I$ and

$$\Pi_I(g\circ\varphi)=\Pi_J\, g.$$

SG2. If $I=\coprod_{j\in J} I_j$ and $g\in\operatorname{dom}\Pi_I$, then writing $g_j := g \upharpoonright \underline{I_j}$ for all $j\in\underline{J}$, we have

$$g_j \in \operatorname{dom}\Pi_{I_j} \qquad \text{and} \qquad (\Pi_{I_j}\, g_j)_{j\in\underline{J}} \in \operatorname{dom}\Pi_J \qquad \text{and} \qquad \Pi_J((\Pi_{I_j}\, g_j)_{j\in\underline{J}}) = \Pi_I\, g.$$

SG3. If $I=I_1 \amalg I_2$ an $(g,h)\in\operatorname{dom}\Pi_{I_1}\times\operatorname{dom}\Pi_{I_2}$, then $(g\sqcup h)\in\operatorname{dom}\Pi_I$.

SG4. If $g\in\operatorname{dom}\Pi_I$ and $g_0\in\mathcal{G}$, then $g_0\cdot g\cdot g_0^{-1}\in\operatorname{dom}\Pi_I$ and

$$\Pi_I(g_0\cdot g\cdot g_0^{-1}) = g_0\cdot(\Pi_I\, g)\cdot g_0^{-1}.$$

SG5. If $g\in\operatorname{dom}\Pi_I$, then $g^{-1}\in\operatorname{dom}\Pi_{I^*}$ and

$$\Pi_{I^*}\, g^{-1} = (\Pi_I\, g)^{-1}.$$

Let $\mathcal{G}$ be the growth order group of positive infinite elements of $\mathbb{R}[[x^{\mathbb{R}}]]$. This is a strong group.

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This can be generalized to larger subgroups of $\tilde{\mathbb{L}}^{>\mathbb{R}}$.

Thank you!