Some ordered groups of generalized series

by Vincent Bagayoko

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based on joint work with E. Kaplan, J. van der Hoeven and V. Mantova

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Unary representations

A unary representation of $\mathcal{G}$ *is an embedding* $t.:(\mathcal{G},\cdot,1,<)\longrightarrow(\mathrm{Aut}(X,<_X),\circ,\mathrm{id}_X,<_\forall)$ *for some linearly ordered set* X, *where* $<_\forall$ *is the ordering by universal comparison.*

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Theorem [Balzanov-Baldwin-Verbovskiy, 2007]

If $(X,<)$ *is dense or* $(X,t_{\cdot},<_X)=(\mathcal{G},\mathrm{left}_{\cdot},<)$, *then* $(X,(t_g)_{g\in\mathcal{G}},<)$ *has QE and is o-minimal.*

Two examples

Abelian ordered groups and Hahn's embedding theorem

Each Abelian ordered group embeds in a Hahn ordered group

$$H[\mathfrak{M}, \mathbb{R}] = \{f \in \mathbb{R}^{\mathfrak{M}} : (\operatorname{supp} f) := \{x \in \mathfrak{M} : f(x) \neq 0\} \textit{ is reverse well-ordered}\}$$

where $(\mathfrak{M}, \prec)$ *is a linearly ordered set. Each* f *is represented as a series* $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f(\mathfrak{m})\, \mathfrak{m}$.

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Strictly increasing affine maps

Let K *be an ordered field. Let* $(G, +, <)$ *be an ordered vector space over* K. *Then* $\operatorname{Aff}_K^+(G) := K^> \times G$ *is an ordered group for the product* $(a, \alpha) \cdot (b, \beta) := (a\, b, a.\beta + \alpha)$ *and the lexicographic ordering.*

We can represent each (a, α) as the strictly increasing affine map $\operatorname{Aff}_{a,\alpha} : \gamma \mapsto a.\gamma + \alpha$, and the ordering is then $(a, \alpha) < (b, \beta)$ iff $\operatorname{Aff}_{a,\alpha}(\gamma) < \operatorname{Aff}_{b,\beta}(\gamma)$ for sufficiently large $\gamma \in G$.

Fix an o.g. $(\mathcal{G},\cdot,1,<)$. $\mathrm{Term}(\mathcal{G})$: free product of $\mathcal{G}$ and $(y^{\mathbb{Z}},\cdot)\simeq(\mathbb{Z},+)$.

Simple task: given

$$t(y)=g_1\,y^{\alpha_1}\cdots g_n\,y^{\alpha_n}\in\mathrm{Term}(\mathcal{G})$$

understand the theory of equations $t(f)=1$ for f lying in extensions of $\mathcal{G}$.

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Remark. Given $\mathcal{G}'\supseteq\mathcal{G}$ and $f\in\mathcal{G}'$, the extension $\mathcal{G}\langle f\rangle/\mathcal{G}$ is determined by an ordering on the quotient group $\mathrm{Term}(\mathcal{G})/N_f$ where

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Examples:

- $t(y)=f^{-1}\,y^{-1}\,f\,y \longrightarrow$ centralizer extension
- $t(y)=f^{-2}\,y \longrightarrow$ extension with a "square root".
- $t(y)=y\,f\,y^{-1}\,g \longrightarrow$ conjugacy extension

Assume $(\mathcal{G}, \cdot, 1)$ is Abelian. For f in an Abelian o.g. extension of $\mathcal{G}$, and $t(y) = g_1\, y^{\alpha_1} \cdots g_n\, y^{\alpha_n} \in \mathrm{Term}(\mathcal{G})$, we have $t(f) = 1$ if and only if

$$f^n = g \tag{1}$$

where $\alpha := \sum_{i=1}^{n} \alpha_i \in \mathbb{Z}$ and $g := g_n^{-1} \cdots g_1^{-1} \in \mathcal{G}$.

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Divisible closure for prdered Abelian groups

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A classical result

The theory of divisible Abelian ordered groups is the model-completion of the theory of Abelian ordered groups. It has QE and is o-minimal.

Abelian and non-Abelian ordered groups

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In the non-Abelian case, solving

$$g_1\,y^{\alpha_1}\cdots g_n\,y^{\alpha_n} = 1$$

in extensions of $\mathcal{G}$ is highly non-trivial (recall the functional representation of $\mathcal{G}\ldots$).

Elementary property of an ordered group $\mathcal{G}$

A) **[Levi, 1942]** *for* $m > 0$, $f^m = g^m \Longrightarrow f = g$.

B) **[Neumann, 1949]** *for* $m, n > 0$, $[f^m, g^n] = 1 \Longrightarrow [f, g] = 1$.

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Question 10. Is any ordered group contained in a divisible one?

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Question 15. Is there a first-order theory of ordered groups which is complete and model complete, and has non-Abelian models? **[Pillay-Steinhorn, 1986]**: such theory is not o-minimal.

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So any elementary class of ordered groups in $\langle \cdot, 1, < \rangle$ induces an elementary class in $\langle \cdot, < \rangle$ of thus orderable groups.

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Miscellany about free groups

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Theorem [Iwasawa, 1948]

For every ordered group $\mathcal{G}$, there is a an ordering on a free-group L and a convex normal subgroup $N \trianglelefteq L$ such that $\mathcal{G} \simeq L/N$.

Let $\mathcal{M} = (M, <, \ldots)$ be an o-minimal structure.

Write $\mathcal{H}_{\mathcal{M}}$ for the set of germs $[f]$ at $+\infty$ of definable maps $f : M \longrightarrow M$. We have $\mathcal{M} \preccurlyeq \mathcal{H}_{\mathcal{M}}$ for a natural "asymptotic" structure on $\mathcal{H}_{\mathcal{M}}$. We set:

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Ordered groups from o-minimal structures

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- For $\mathcal{M} = (R, <, +, \cdot)$, we have $\mathcal{G}_{\mathcal{M}} \supsetneq \mathrm{Aff}^+_R(R)$. Moreover $\mathcal{H}_{\mathcal{M}}$ has a structure of differential field **[vdDries, 1998]**.

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- For $\mathcal{M} = (R, <, +, \cdot)$, we have $\mathcal{G}_{\mathcal{M}} \supsetneq \mathrm{Aff}^{+}(R)$. Moreover $\mathcal{H}_{\mathcal{M}}$ has a structure of differential field **[vdDries, 1998]**.
- The field $\mathcal{H}_{\mathbb{R}_{\exp}}$ contains all germs of functions that can be obtained as combinations of $\exp$, $\log$, and semialgebraic functions.

A growth axiom

A simple inequation

Let $\mathcal{G}$ *be an ordered group and let* $f, g \in \mathcal{G}$ *with* $f, g > 1$. *How is the inequation*

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How "much faster" must f grow? First of all, we should have $f > g \circ g \circ \cdots \circ g$ for all finite iterations.

More precisely, we should have $f > \mathcal{C}(g)$, where

$$\mathcal{C}(g) = \{h \in \mathcal{G}_{\mathbb{R}_{\exp}} : h \circ g = g \circ h\}.$$

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Second axiom: the relation $g \prec f \Longleftrightarrow \mathcal{C}(g) < \max(f, f^{-1})$ is an ordering which is compatible with $<$. The relation $f \asymp g$ if ($f \not\prec g$ and $g \not\prec f$) is a convex equivalence relation.

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An element $\mathfrak{u} \in \mathcal{G} \setminus \{1\}$ is said **central** if for all $f \asymp \mathfrak{u}$, there is a $\mathfrak{u}_f \in \mathcal{C}(\mathfrak{u})$ such that $f \mathfrak{u}_f^{-1} \prec f$.

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Definition: growth order group

A **growth order group** *is an ordered group* $\mathcal{G}$ *in which* **GOG1**, **GOG2** *and* **GOG3** *hold.*

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Consider the following axioms for an ordered group $\mathcal{G}$:

GOG1. For all $f, g > 1$, we have $f > \mathcal{C}(g) \Longrightarrow f g > g f$.

GOG2. For all $f, g > 1$ with $f > g$ and all $g' \in \mathcal{C}(g)$, there is an $f' \in \mathcal{C}(f)$ with $f' > g'$.

Second axiom: the relation $g \prec f \Longleftrightarrow \mathcal{C}(g) < \max(f, f^{-1})$ is an ordering which is compatible with $<$. The relation $f \asymp g$ if ($f \not\prec g$ and $g \not\prec f$) is a convex equivalence relation.

An element $\mathfrak{u} \in \mathcal{G} \setminus \{1\}$ is said **central** if for all $f \asymp \mathfrak{u}$, there is a $\mathfrak{u}_f \in \mathcal{C}(\mathfrak{u})$ such that $f \mathfrak{u}_f^{-1} \prec f$.

GOG3. For every $f \neq 1$, there is a central $\mathfrak{u}$ with $\mathfrak{u} \asymp f$.

Definition: growth order group

A **growth order group** *is an ordered group $\mathcal{G}$ in which* **GOG1**, **GOG2** *and* **GOG3** *hold.*

All Abelian ordered groups are growth order groups.

Question 22. If $\mathcal{M}$ expands the real ordered field, then must $\mathcal{G}_{\mathcal{M}}$ be a growth order group?

Idea: obtain GOG closed under certain equations $t(y) = 1$ by constructing ordered differential fields of formal series with composition laws:

Differential fields of series and GOG

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Take $\mathbb{R}(x)$ with $x > \mathbb{R}$. For $f, g \in \mathbb{R}(x)$ with $g > \mathbb{R}$, the derivative f' and the compositum $f \circ g$ are well-defined.

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These operations extend to the ordered field $\mathbb{R}[[x^{\mathbb{Z}}]]\supsetneq\mathbb{R}(x)$ of formal Laurent series.

$$\text{If } f=\sum_{k=n}^{-\infty} f_k\,x^k, \text{ then } \quad f\circ g:=\sum_{k=n}^{-\infty} f_k\,g^k \quad \text{and} \quad f'=\sum_{k=n}^{-\infty} k\,f_k\,x^{k-1}.$$

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The operations extend to the field $\mathbb{P}$ of Puiseux series, and $(\mathbb{P}^{>\mathbb{R}}, \circ, x, <)$ is a growth order group.

- Is $\mathbb{P}$ divisible?
- Are any two $f, g > x$ conjugate in $\mathbb{P}^{>\mathbb{R}}$?
- Is $\mathbb{P}$ existentially closed among growth order groups?

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The operations extend to the field $\mathbb{P}$ of Puiseux series, and $(\mathbb{P}^{>\mathbb{R}},\circ,x,<)$ is a growth order group.

- Is $\mathbb{P}$ divisible? No, since x^2 has no functional square root.
- Are any two $f,g>x$ conjugate in $\mathbb{P}^{>\mathbb{R}}$? No, for instance $2x$ and $x+1$ are not conjugate.
- Is $\mathbb{P}$ existentially closed among growth order groups? No, because…

Definition: generalized power series [Hahn, 1907]

Let $(\mathfrak{M}, \cdot, 1, \prec)$ *be a linearly ordered abelian group. Then* $\mathbb{R}[[\mathfrak{M}]]$ *is an ordered field for the Cauchy product*

$$\left(\sum_{\mathfrak{m}\in\mathfrak{M}} f(\mathfrak{m})\,\mathfrak{m}\right)\cdot\left(\sum_{\mathfrak{n}\in\mathfrak{M}} g(\mathfrak{n})\,\mathfrak{n}\right) := \sum_{\mathfrak{m},\mathfrak{n}\in\mathfrak{M}} f(\mathfrak{m})\,g(\mathfrak{n})\,(\mathfrak{m}\,\mathfrak{n}).$$

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Certain *infinite* families can be summed in $\mathbb{R}[[\mathfrak{M}]]$. We get a "formal Banach space":

- if ε is infinitesimal then $\sum_{n\in\mathbb{N}}(-1)^n\,\varepsilon^n = \frac{1}{1+\varepsilon}$.
- we have an implicit function theorem **[vdHoeven, 2006]**
- equations over $\mathbb{R}[[\mathfrak{M}]]$ with approximate solutions in $\mathbb{R}[[\mathfrak{M}]]$ sometime have exact solutions in $\mathbb{R}[[\mathfrak{M}]]$

Transseries [Dahn-Göring, 1987 and Ecalle, 1992]

Field $\mathbb{T}$ *of* **transseries**: *generalized series involving* x, $\log x$ *and* e^x *and combinations thereof.*

$$f := \sum_{n=1}^{+\infty} n!\, \mathrm{e}^{x/n} + \log_2 x + 7 + x^{-2} \log x + \sum_{p=0}^{+\infty} \mathrm{e}^{-x^{p+1}} x^p \quad \textit{is a transseries.}$$

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$$\partial f = \sum_{n=1}^{+\infty} (n-1)!\,\mathrm{e}^{x/n} + \frac{1}{x\log x} - 2\,x^{-3}\log x + x^{-3} + \sum_{p=0}^{+\infty} (p\,x^{p-1} - (p+1)\,x^p)\,\mathrm{e}^{-x^{p+1}}$$

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Remark: The elements $2\,x$ and $x+1$ are conjugate, *via* $(2\,x)\circ\mathrm{e}^{(\log 2)x} = \mathrm{e}^{(\log 2)x}\circ(x+1)$.

We now look at the sructure $(\mathbb{T}^{>\mathbb{R}}, x, \circ, <)$.

Theorem [vdDries-Macintyre-Marker, 2001]

This is a right-ordered group, i.e for all f, g, h, $f > g \Rightarrow f \circ h > g \circ h$.

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The group is not divisible since e^x has no square root. There is no “half exponential” which can be expressed as a combination of exponentials and logarithms. However:

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Any two $f, g > x$ *with* $f, g \prec \mathrm{e}^x$ *are conjugate.*

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Since $x + 1 \prec \mathrm{e}^x$ and $f^{[n]} = x + 1$ has a solution $f = x + \frac{1}{n}$ for all $n \in \mathbb{N}^{>}$, we deduce that

$$\mathbb{T}_0^{>\mathbb{R}} := \{f \in \mathbb{T}^{>\mathbb{R}} : f \prec \mathrm{e}^x\}$$

is a divisible growth order group.

Goal: a group $\mathcal{G} \supseteq \mathbb{T}^{>\mathbb{R}}$ in which e^x is conjugate with $x+1$. $\longrightarrow$ solve the conjugacy equation

$$y \circ (x+1) = \mathrm{e}^x \circ y \tag{5}$$

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(6) is the formal version of ABEL's equation for the exponential function. **[KNESER, 1949]** There is a strictly increasing and analytic solution E of

$$\forall t \geqslant 0, E(t+1) = \mathrm{e}^{E(t)}$$

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Conjucacy and Abel's equation

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Question 25. Can an o-minimal expansion of $(\mathbb{R}, +, \times, <)$ define a transexponential function?

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Question 26. Can an o-minimal expansion of $(\mathbb{R}, +, \times, <)$ define a transexponential function?

Proposition [vdDries-Macintyre-Marker, 2001]

$\mathcal{H}_{\mathbb{R}_{\exp}}$ embeds into $(\mathbb{T}, +, \times, <, \partial, \circ)$.

Can we construct differential fields $\mathbb{H}$ with composition where (8) has solutions, and into which more general $\mathcal{H}_{\mathcal{M}}$ can be embedded?

[Schmeling, 2001] defined an extension $\mathbb{H} \supsetneq \mathbb{T}$ with formal symbols $\mathrm{e}^x_{\omega^n}, \ell_{\omega^n} x$ for all $n \in \mathbb{N}$, where $\mathrm{e}^x_1 = \mathrm{e}^x$ and

$$\begin{aligned} \mathrm{e}^x_{\omega^{n+1}} \circ (x+1) &= \mathrm{e}^x_{\omega^n} \circ \mathrm{e}_{\omega^{n+1}}(x) \qquad (9) \\ \mathrm{e}^x_{\omega^n} \circ (\ell_{\omega^n} x) = (\ell_{\omega^n} x) \circ \mathrm{e}^x_{\omega^n} &= x. \end{aligned}$$

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Logarithmic hyperseries [vdDries-vdHoeven-Kaplan, 2018]

A differential ordered field with composition $(\mathbb{L}, +, \times, <, \partial, \circ)$ *where for all* $\mu \in \mathbf{On}$, *we have a symbol* $\ell_{\omega^\mu} x \in \mathbb{L}$ *with* $\ell_1 x = \log x$ *and*

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I.e. the inverse equations of (10) are valid. But $(\mathbb{L}^{>\mathbb{R}}, \circ, x)$ *is not a group.*

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Theorem [B., 2022]

The derivation ∂ *and composition law* $\circ$ *on* $\mathbb{L}$ *extend in a natural way to* $\tilde{\mathbb{L}}$.

Using Taylor series arguments, one shows that:

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The structure $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, x, <)$ *is a growth order group.*

So we have a **GOG** solution to our Question 14 on ordered groups.

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Fundamental property: if $L, R \subseteq \mathbf{No}$ are subsets with $L < R$, then there is a unique $\sqsubset$-minimal number $\{L \mid R\}$ with $L < \{L \mid R\} < R$.

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Conway inductively defined field arithmetics on $\mathbf{No}$. For $a = \{L_a \mid R_a\}$, $b = \{L_b \mid R_b\}$, we have

$$a + b = \{L_a + b, a + L_b \mid a + R_b, R_a + b\}.$$

The ordered field $(\mathbf{No}, +, \times, <)$ naturally contains the reals $\mathbb{R}$ as well as the ordinals $\mathbf{On}$ with their Hessenberg (commutative) arithmetic.

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We have a simplest positive infinite number ω corresponding to the ordinal ω and the series x.

Idea: in $\mathbf{No}$, asymptotics can be brought to life; for a regular growth rate $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $a = \{L \mid R\} \in \mathbf{No}^{>\mathbb{R}}$, define $f(a)$ as the number

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Numbers as hyperseries

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- We have numbers in $\mathbb{R}(\omega)$ for polynomial asymptotics.

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Numbers as hyperseries

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[B.-vdHoeven, 2022]

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[Work in progress]

There is a natural extension $\circ$ of the composition law on $\tilde{\mathbb{L}}$ to $\mathbf{No}$.

$(\mathbf{No}^{>\mathbb{R}}, \circ, \omega, <)$ is a growth order group with exactly three conjucacy classes.

Let $\mathcal{G}$ be among $\mathbb{T}_0^{>\mathbb{R}}$, $\tilde{\mathbb{L}}^{>\mathbb{R}}$, and $\mathbf{No}^{>\mathbb{R}}$. For $g \in \mathcal{G} \setminus \{x\}$, we have a unique isomorphism

$$\begin{array}{rcl} (\mathbb{R}, +, 0, <) & \longrightarrow & (\mathcal{C}(f), \circ, x, <) \\ r & \longmapsto & f^{[r]} \end{array}$$

with $f^{[1]} = f$. Indeed this holds for $f = x \pm 1$, and is carried over by conjugacy.

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Using the $\{\cdot \mid \cdot\}$ operation, one can define, for all strictly $\prec$-decreasing ordinal indexed sequence $(\mathfrak{u}_\gamma)_{\gamma<\alpha}$ of "$\asymp$-simple" elements of $\mathcal{G}$ and sequences $(r_\gamma)_{\gamma<\alpha} \in (\mathbb{R}^\times)^\alpha$, a transfinite composition

$$\bigodot_{\gamma<\alpha} \mathfrak{u}_\gamma^{[r_\gamma]} = \cdots \circ \mathfrak{u}_\gamma^{[r_\gamma]} \circ \cdots \circ \mathfrak{u}_1^{[r_1]} \circ \mathfrak{u}_0^{[r_0]} \tag{14}$$

Transfinite non-commutative products

Let $\mathcal{G}$ be among $\mathbb{T}_0^{>\mathbb{R}}$, $\tilde{\mathbb{L}}^{>\mathbb{R}}$, and $\mathbf{No}^{>\mathbb{R}}$. For $g \in \mathcal{G} \setminus \{x\}$, we have a unique isomorphism

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[Work in progress]

Every element in $\mathcal{G}$ can be expressed as in (15), in a unique way.

In this sense, the **GOG** $(\mathbf{No}^{>\mathbb{R}}, \circ, \omega, <)$ is a non-commutative Hahn product $\mathbf{No}^{>\mathbb{R}} \simeq \prod_{\mathbf{No}} \mathbb{R}$.

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Question 30. Can this be generalized? Can one define transfinite non-commutative products $\prod_I \mathcal{G}_i$ of Abelian ordered groups $\mathcal{G}_i$ indexed by a linearly ordered set $(I, <)$?

Thank you!