Three flavors of H-fields

on ordered differential fields of real-valued functions

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Equations in ordered algebraic structures

$(\mathbb{A}, +, <)$: linearly ordered abelian group	$\mathcal{S}$: set of functions $\mathbb{A}^n \longrightarrow \mathbb{A}$ for various $n \in \mathbb{N}$.	$\mathcal{A} := (\mathbb{A}, +, <, \mathcal{S})$: an *ordered algebraic structure*.

$\langle \mathcal{S} \rangle$: intersections of all subsets of $\bigcup_{n \in \mathbb{N}} \mathbb{A}^{\mathbb{A}^n}$ containing $\mathcal{S} \cup \{+\}$, all projections

$$\pi_{i \rightarrow j} \colon \mathbb{A}^n \longrightarrow \mathbb{A}^n \colon (a_0, \ldots, a_{n-1}) \longmapsto (a_0, \ldots, a_{j-1}, a_i, a_{j+1}, \ldots, a_{n-1})$$

and closed under composition. So $\langle S \rangle \equiv$ terms in $\mathcal{A}$ modulo equality as functions.

Constraints imposed by the ordering

1. *How should $\{+\} \cup \mathcal{S}$ behave in relation to the ordering? E.g. for $\mathcal{S} = \{\times, \partial\}$, when should $\partial(f) < \partial(g)$ hold?*

2. *Which equations can be solved in $\mathbb{A}$?*

   For ordered rings, there is no solution to $X^2 + 1 = 0$.

   For ordered differential fields, imposing $\partial(a) > 0$ for all $a > \mathrm{Ker}(\partial)$ implies that

   $$\partial^2(y) + y = 0$$

   has no non-zero solution.

Three particular families of axioms for $\mathcal{A}$

Identities	$\forall \bar{a}(\varphi[\bar{a}] \Longrightarrow t(\bar{a}) = 0)$,	$t \in \langle \mathcal{S} \rangle, \varphi[\bar{x}] \in \mathcal{L}_<$
Inequalities	$\forall \bar{a}(\varphi[\bar{a}] \Longrightarrow t(\bar{a}) > 0)$,	$t \in \langle \mathcal{S} \rangle, \varphi[\bar{x}] \in \mathcal{L}_<$
Equations	$\forall \bar{a}(\varphi[\bar{a}] \Longrightarrow \exists \bar{b}(\psi[\bar{b}] \wedge t(\bar{a}, \bar{b}) = 0))$,	$t \in \langle \mathcal{S} \rangle, \varphi[\bar{x}], \psi[\bar{y}] \in \mathcal{L}_<$

Ordered differential field $(F, +, \times, \partial, <)$

Equalities: *axioms for differential rings.* ***Inequalities:*** *axioms for ordered rings + ?*

Equations: *additive and multiplicative inverses + ?*

A solution: no differential inequalities

Only axioms for ordered fields and differential fields.

Singer - 1978: *Ordered differential fields have a model completion.*

Our framework: Intermediate value theorems, and Hardy type inequalities

IVT: *for* $t \in \langle \mathcal{S} \rangle$ *of arity* 1 *and* $a, c \in \mathcal{A}$ *with* $t(a) < 0 < t(c)$, *there is* $b \in \widetilde{(a, c)}$ *with*

$$t(b) = 0.$$

If real derivation were monotonous (1)

Germs at $+\infty$

We identify two functions f, g in

$$\bigcap_{n\in\mathbb{N}} \bigcup_{r\in\mathbb{R}} \mathcal{C}^n((r,+\infty),\mathbb{R}),$$

if $f(r)=g(r)$ for all sufficiently large $r\in\mathbb{R}$ (written $r\gg 1$).

$\mathcal{G}$: differential ring of equivalence classes (pointwise operations), called **germs** at $+\infty$.

Definition: Hardy fields

Hardy fields *are differential subfields of $\mathcal{G}$ containing $\mathbb{R}$. They are ordered fields for the order*

$$f<g \Longleftrightarrow \forall r\gg 1,\, f(r)<g(r).$$

$\mathcal{H}$ admits a natural valuation ring $\mathcal{O}\subseteq\mathcal{H}$ and corresponding maximal ideal $\mathfrak{o}\subset\mathcal{O}$:

$$\begin{aligned}\mathcal{O} &:= \{f : \exists r\in\mathbb{R}^{>0}, -r<f<r\} = \{f : \lim f\in\mathbb{R}\}, \quad \text{(finite germs)}\\ \mathfrak{o} &:= \{f : \forall r\in\mathbb{R}^{>0}, -r<f<r\} = \{f : \lim f = 0\}. \quad \text{(infinitesimal germs)}\end{aligned}$$

Maximal Hardy fields

A Hardy field $\mathcal{H}$ is said **maximal** *if it has no proper superset which is a Hardy field.*

IVT theorem, order 1 [van den Dries - 2000] *Let $\mathcal{H}$ be a maximal Hardy field. Let*

$$f, h \in \mathcal{H} \text{ and } P \in \mathcal{H}[Y, Y'] \qquad \text{with} \qquad P(f, f') < 0 < P(h, h').$$

Then there is $g \in \widetilde{(f, h)}$ with $P(g, g') = 0$.

Therefore maximal Hardy fields contain / are closed under elementary functions $\mathrm{id}_{\mathbb{R}}$, $\exp_n = \exp \circ \cdots \circ \exp$, $\log_n = \exp_n^{\circ(-1)}$, $\arctan, \ldots$

Properties of $(\mathcal{H}, \mathcal{O}, \mathfrak{o}, \partial, <)$ for a general Hardy field $\mathcal{H}$

- *Let $f > \mathcal{O}$. Then f must be eventually strictly increasing, so $\forall f, f > \mathcal{O} \Longrightarrow f' > 0$.*
- *Let $f \in \mathcal{O}$. Then f has a limit $\lim f \in \mathbb{R}$, so $f - \lim f$ is infinitesimal. So $\mathcal{O} = \mathrm{Ker}(') + \mathfrak{o}$.*
- *Let $f \in \mathfrak{o}$. Then $\lim f = 0$ so we cannot have $\lim f' \in \overline{\mathbb{R}} \setminus \{0\}$. So $\mathfrak{o}' \subseteq \mathfrak{o}$.*

If real derivation were monotonous (3)

Theorem [H. Kneser - 1949]

There is a bijective strictly increasing analytic function $E_\omega\colon \mathbb{R} \longrightarrow \mathbb{R}$ *which solves* **Abel's equation***:*

$$\forall r \gg 1, E_\omega(r+1) = \exp(E_\omega(r)).$$

We have $E_\omega(r) > \exp_n(r)$ for $r \gg 1$, for each $n \in \mathbb{N}$: E_ω is a **hyperexponential** function.

Theorem [Boschernitzan - 1986]

For any such E_ω*, the field* $\mathbb{R}(E_\omega, E'_\omega, E''_\omega, \ldots)$ *is a Hardy field.*

Let $\mathcal{H}$ be maximal with $\mathcal{H} \supseteq \mathbb{R}(E_\omega, E'_\omega, E''_\omega, \ldots)$. For each $n \in \mathbb{N}$, we have $E_\omega > \exp_n$, so

$$\log_n E_\omega > \mathcal{O} \quad \text{and} \quad \frac{1}{\log_n E_\omega} \in \mathfrak{o} \qquad \text{whence}$$

$$(\log_n E_\omega) \cdots (\log E_\omega)\, E_\omega < E'_\omega < E_\omega (\log E_\omega) \cdots (\log_n E_\omega)^2.$$

Idea: Abstractions of Hardy fields as ordered differential fields.

Definition: H-fields (with small derivation) [van den Dries, Aschenbrenner - 2006]

A H-field with small derivation is an ordered differential field (K, ∂) with

H1. $\forall x \in K, x > \mathcal{O} \Longrightarrow \partial(x) > 0$. *($\mathcal{O} = \{x \in K : \exists c \in \mathrm{Ker}(\partial), -c < x < c\}$)*

H2. $\mathcal{O} = \mathrm{Ker}(\partial) + \mathfrak{o}$. *($\mathfrak{o}$: maximal ideal of $\mathcal{O}$)*

H3. $\partial(\mathfrak{o}) \subseteq \mathfrak{o}$.

For instance, (formal) Laurent series with $\partial(\sum_{k=-n}^{+\infty} a_k \varepsilon^k) := \sum_{k=-n}^{+\infty} k\, a_k \varepsilon^{k-1}$ form an H-field with small derivation.

Liouville closure

K is said Liouville-closed if it is real-closed and the following equations (in y) have solutions

$$y' = \xi \quad , \quad y' = y\xi \wedge y > 0 \quad \textit{for each } \xi \in K.$$

A Liouville-closure of K is a minimal H-field extension $K \longrightarrow L$ where L is Liouville-closed.

Any H-field with small derivation has a Liouville-closure. Any Hardy field has a Liouville-closure which is a Hardy field.

Transseries

Transseries [Dahn, Göring - 1987; Ecalle - 1992]

Transseries *are Hahn series involving formal terms* x, $\log x$ *and* e^x *and combinations thereof.*

$$e.g.\quad f := \sum_{n=1}^{+\infty} n!\,\mathrm{e}^{x/n} + \log_2 x + 7 + x^{-2}\log x + \sum_{p=0}^{+\infty} \mathrm{e}^{-x^{p+1}} x^p \quad \textit{is a transseries.}$$

The field $\mathbb{T}$ of transseries is equipped with a formal, termwise derivation $\partial\colon \mathbb{T} \longrightarrow \mathbb{T}$, e.g.

$$\partial(f) = \sum_{n=1}^{+\infty} (n-1)!\,\mathrm{e}^{x/n} + \frac{1}{x\log x} - 2\,x^{-3}\log x + x^{-3} + \sum_{p=0}^{+\infty} (p\,x^{p-1} - (p+1)\,x^p)\,\mathrm{e}^{-x^{p+1}},$$

and a formal composition law $\circ\colon \mathbb{T} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$, e.g.

$$f \circ \log x = \sum_{n=1}^{+\infty} n!\,x^{1/n} + \log_3 x + 7 + (\log x)^{-2}\log_2 x + \sum_{p=0}^{+\infty} \mathrm{e}^{-(\log x)^{p+1}} (\log x)^p.$$

Transseries as germs

Each transseries $f \in \mathbb{T}$ defines a function $\tilde{f}\colon \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}\colon g \longmapsto f \circ g$, which behaves similarly to germs lying in Hardy fields:

Formal Taylor expansions

For all $g \in \mathbb{T}^{>\mathbb{R}}$, for sufficiently small $\delta \in \mathbb{T}$, we have

$$f \circ (g + \delta) = \sum_{k \in \mathbb{N}} \frac{\partial^k(f) \circ g}{k!} \delta^k.$$

So $\widetilde{\partial(f)}$ is the functional derivative of $\tilde{f}$. In that sense, $\tilde{f}$ is a germ at $+\infty$ of a smooth function $\mathbb{T} \longrightarrow \mathbb{T}$.

It follows that $(\mathbb{T}, +, \times, \partial)$ is a H-field with small derivation (same proof as in Hardy fields).

Theorem

The derivation $\partial\colon \mathbb{T} \longrightarrow \mathbb{T}$ is surjective.

There is an exponential function $\exp\colon \mathbb{T} \longrightarrow \mathbb{T}^{>0}$ with

$$\forall h \in \mathbb{T}, \partial(\exp(h)) = \partial(h) \exp(h),$$

so $\mathbb{T}$ is Liouville-closed.

The theory of $(\mathbb{T}, +, \times, \partial, \mathcal{O})$ is known due to work of Aschenbrenner, van den Dries and van der Hoeven:

Theorem [ADH - 2015]

The complete theory $\mathrm{Th}(\mathbb{T}, +, \times, \partial, \mathcal{O})$ *of* $\mathbb{T}$ *is model-complete.*

Theorem [ADH - 2015]

$\mathrm{Th}(\mathbb{T}, +, \times, \partial, \mathcal{O})$ *has QE in a natural language and is decidable.*

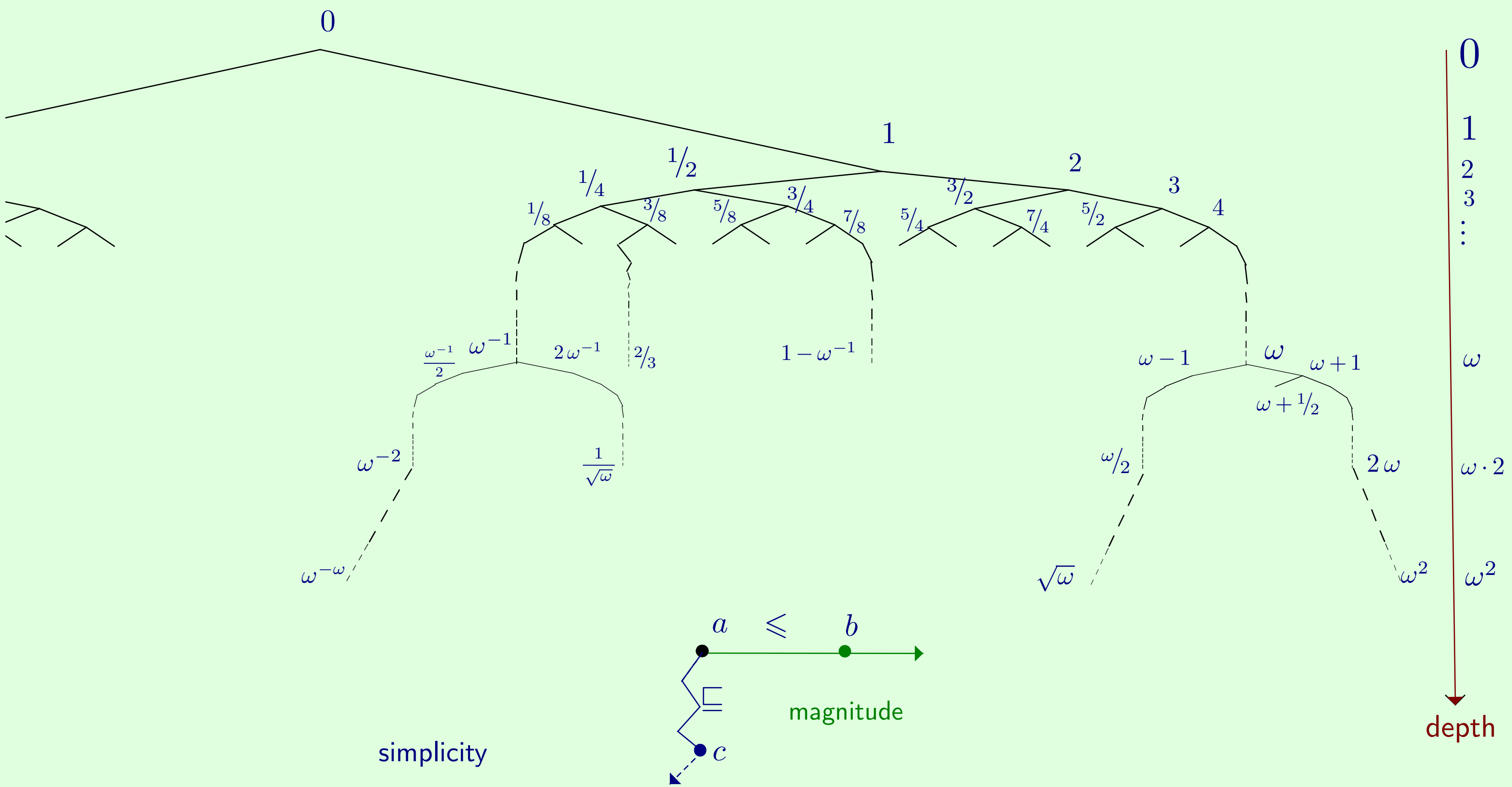
Conjecture (ADH): $\mathrm{Th}(\mathbb{T}, +, \times, \partial, \mathbb{R})$ *is the theory of maximal Hardy fields.*

Theorem [ADH - 2015, based on work of van der Hoeven - 2006]

$\mathrm{Th}(\mathbb{T}, +, \times, \partial, \mathcal{O})$ *is axiomatized by axioms for Liouville-closed* H*-fields with small derivation and the* IVT.

Therefore, the conjecture is equivalent to the conjecture that any maximal Hardy field satisfy the IVT.

Conway's class $\mathbf{No}$ of surreal numbers is an ordered field whose underlying order is the lexicographically ordered complete binary tree $\{-1,1\}^{<\mathbf{On}}$, where depths are arbitrary ordinals.



Simplicity: $a \sqsubseteq b$ if there is a (descending) path from a to b in the tree.

Inductive definitions on $\mathbf{No}$

Fundamental property of $(\mathbf{No}, \leqslant, \sqsubseteq)$

For all sets of numbers L, R *with* $L < R$, *there is a unique* $\sqsubseteq$*-minimal number* $\{L|R\}$ *with*

$$L < \{L|R\} < R.$$

Well-founded order

For $a \in \mathbf{No}$, *we set* $a_L := \{b \in \mathbf{No} : b < a, b \sqsubseteq a\}$, $a_R := \{b \in \mathbf{No} : b > a, b \sqsubseteq a\}$.

So $a = \{a_L | a_R\}$. *The partial order* $(\mathbf{No}, \sqsubseteq)$ *is well-founded* $\longrightarrow$ *inductive definitions.*

Surreal arithmetic [Conway - 1976]

Inductive definition of the sum $a + b$ *of numbers* a, b. *We set*

$$a + b = \{a_L + b, a + b_L | a + b_R, a_R + b\}.$$

Similar equations exist for $-a, a\,b, {}^{a}/_{b}$.

$(\mathbf{No}, +, \times)$ is a real-closed field.

Numbers as transseries

Numbers as Hahn series [Conway - 1976]

There is a subgroup $\mathbf{Mo}$ *of* $(\mathbf{No}^{>0}, \times)$ *which is a section of the natural valuation. Each number* a *can be canonically identified with a unique Hahn series*

$$a \equiv \sum_{\mathfrak{m} \in \mathbf{Mo}} a_{\mathfrak{m}} \mathfrak{m} \quad \text{with monomial group } \mathbf{Mo} \text{ and real coefficients } a_{\mathfrak{m}} \in \mathbb{R}.$$

Exponential [Gonshor - 1986] and transseries [Berarducci, Mantova -2019]

There is a natural isomorphism $\exp\colon (\mathbf{No}, +, <) \longrightarrow (\mathbf{No}^{>0}, \times, <)$*, and* $(\mathbf{No}, +, \times, \exp)$ *is a model of the real exponential field. Write* $\log = \exp^{\circ(-1)}$ *and* E_n, L_n *for* n*-fold iterates of* $\exp$ *and* $\log$*.*

Using the class of "exponents" $\mathbf{\Gamma} := \log(\mathbf{Mo})$*, numbers can be re-presented as transseries*

$$a = \sum_{\gamma \in \mathbf{\Gamma}} a_{[\gamma]} \mathrm{e}^{\gamma}$$

where $a_{[\gamma]} := a_{\mathrm{e}^{\gamma}} \in \mathbb{R}$ *for each* $\gamma \in \mathbf{\Gamma}$*. Iteratingly closing* $\{\omega\}$ *under sums with real coefficients,* $\exp$ *and* $\log$*, we obtain an isomorphic copy of* $\mathbb{T}$ *within* $\mathbf{No}$*, where* x *is identified with* ω*.*

Theorem [Berarducci, Mantova - 2018]

There is a derivation ∂_{BM} on $\mathbf{No}$ such that $(\mathbf{No}, \partial_{\mathrm{BM}})$ is a Liouville-closed H-field with small derivation.

The authors relied on the presentation of numbers $a \in \mathbf{No}$ as transseries $a = \sum_{\gamma \in \boldsymbol{\Gamma}} a_{[\gamma]} \mathrm{e}^{\gamma}$. Imposing $\partial_{\mathrm{BM}}(\sum_{\gamma} a_{[\gamma]} \mathrm{e}^{\gamma}) := \sum_{\gamma} a_{[\gamma]} \partial_{\mathrm{BM}}(\gamma) \mathrm{e}^{\gamma}$, it is enough to define ∂_{BM} on exponents $\gamma \in \boldsymbol{\Gamma}$.

$\boldsymbol{\Gamma} \subseteq \mathbf{No}$ so by induction, this reduces to defining ∂_{BM} on **log-atomics**, i.e. exponents γ with

$$\gamma \in \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot \boldsymbol{\Gamma}}}}}}, \text{ i.e. } \log_n \gamma \in \boldsymbol{\Gamma} \text{ for all } n \in \mathbb{N}.$$

Log-atomics form a proper class of numbers $\rightarrow$ many ways to define ∂_{BM}. ∂_{BM} is “simplest”

Theorem [ADH - 2019]

$(\mathbf{No}, +, \times, \partial_{\mathrm{BM}}, \mathbb{R})$ *is an elementary extension of* $(\mathbb{T}_{\mathrm{LE}}, +, \times, \partial, \mathbb{R})$. *Any H-field with* $\mathrm{Ker}(\partial) = \mathbb{R}$ *embeds into* $(\mathbf{No}, +, \times, \partial_{\mathrm{BM}}, v)$ *as a differential valued field.*

Problem: ∂_{BM} is not compatible with presentations of numbers as functions $\rightarrow$ how to define compatible derivations?

Theorem [with van der Hoeven and Mantova]

There is a surreal function $E_\omega\colon \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}^{>\mathbb{R}}$ *with*

- $E_\omega\colon \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}^{>\mathbb{R}}$ *is strictly increasing and bijective.*
- $E_\omega(a) > \exp_n(a)$ *for all* $a \in \mathbf{No}^{>\mathbb{R}}$ *and* $n \in \mathbb{N}$.
- $E_\omega(a+1) = \exp(E_\omega(a))$ *for all* $a \in \mathbf{No}^{>\mathbb{R}}$.
- E_ω *has Taylor expansions around each number* $a \in \mathbf{No}^{>\mathbb{R}}$. *We have* $E'_\omega = \prod_{n\in\mathbb{N}} \log_n \circ E_\omega$.

This generalizes to even faster growing functions $E_{\omega^\mu}\colon \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}^{>\mathbb{R}}$ for all $\mu \in \mathbf{On}$, with $E_{\omega^{\mu+1}}(a+1) = E_{\omega^\mu}(E_{\omega^{\mu+1}}(a))$ for all $a \in \mathbf{No}^{>\mathbb{R}}$.

Given an ordinal γ in Cantor normal form

$$\gamma = \sum_{i=0}^{p} \omega^{\eta_i} m_i,\ \eta_0 > \cdots > \eta_p,\ m_0, \ldots, m_p \in \mathbb{N},$$

we set $E_\gamma := E_{\omega^{\eta_0}}^{\circ m_0} \circ \cdots \circ E_{\omega^{\eta_p}}^{\circ m_p}$. The functional inverse L_ρ of E_ρ satisfies

$$\forall a \in \mathbf{No}^{>\mathbb{R}}, L'_\rho(a) = \frac{1}{\prod_{\gamma<\omega^\mu} L_\gamma(a)}.$$

Defining a hyperserial derivation ∂

Hyperserial expansions

For $\mathfrak{m} \in \mathbf{Mo}^{\neq 1}$, there are elementary hyperseries f_l, f_r, whose derivatives f_l' and f_r' are known, and $u, \psi \in \mathbf{No}$, such that $\mathfrak{m}$ expands uniquely as

$$\mathfrak{m} = (f_l \circ \psi) \times (f_r \circ u).$$

Defining ∂ on $\mathbf{Mo}^{\neq 1}$: The number $\partial(\mathfrak{m})$ is determined by f_l', f_r', $\partial(\psi)$ and $\partial(u)$. Fixing $\partial(\omega) := 1$, the only ambiguous case is when $\mathfrak{m}$ is an infinite expansions such as

$$\mathfrak{m} = \mathrm{e}^{\sqrt{L_1(\omega)} + \mathrm{e}^{\sqrt{L_2(\omega)}} E_{\omega}^{\left.\sqrt{L_\omega(\omega)} + \mathrm{e}^{\sqrt{L_{\omega 2}(\omega)}} E_{\omega 2}^{\cdot^{\cdot^{\cdot}}}\right\} \mathfrak{m}_2}\Big\} \mathfrak{m}_1}.$$

$$\begin{aligned} \text{We must have} \quad \partial(\mathfrak{m}) &= 1 + \mathrm{e}^{a_1} \left(\frac{1}{2\,\omega\,\sqrt{L_1(\omega)}} + \frac{\mathrm{e}^{\sqrt{L_2(\omega)}}}{2\,\omega\,\sqrt{L_1(\omega)}} \mathfrak{m}_1 + \mathrm{e}^{\sqrt{L_2(\omega)}} \partial(\mathfrak{m}_1) \right) \\ &= \ldots \quad \text{(telescopic sum)} \end{aligned}$$

Work in progress [B.]:

There is a hyperserial derivation ∂ on $\mathbf{No}$ with $(\mathbf{No}, +, \times, \partial, \mathbb{R}) \simeq (\mathbf{No}, +, \times, \partial_{\mathrm{BM}}, \mathbb{R})$.

Thank you!