Arithmetics and analytics of surreal numbers*

05-15-2021

Based on joint work with Elliot Kaplan, Vincenzo Mantova, and Joris van der Hoeven.

*. This document has been written using GNU TeXmacs; see www.texmacs.org.

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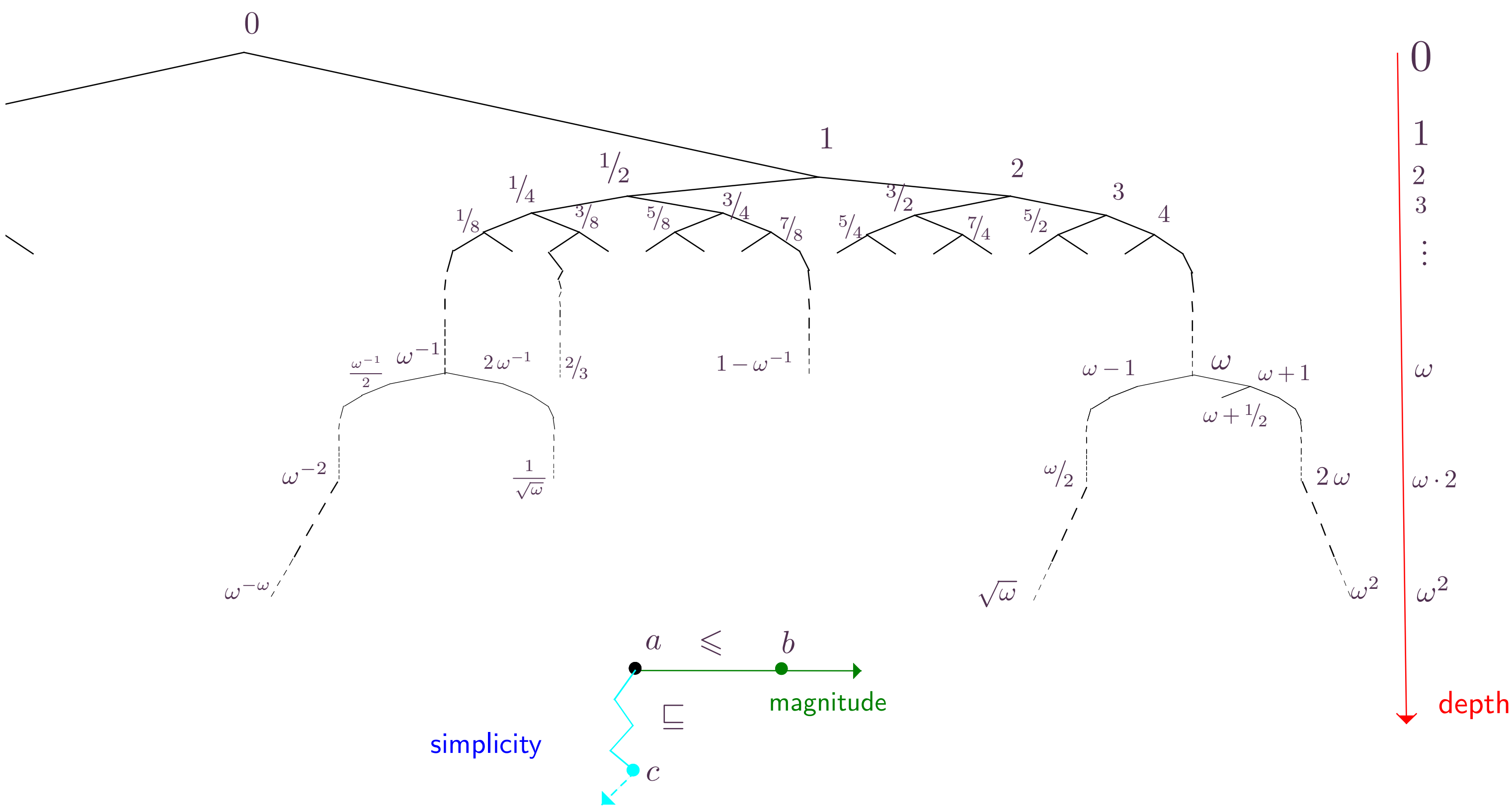

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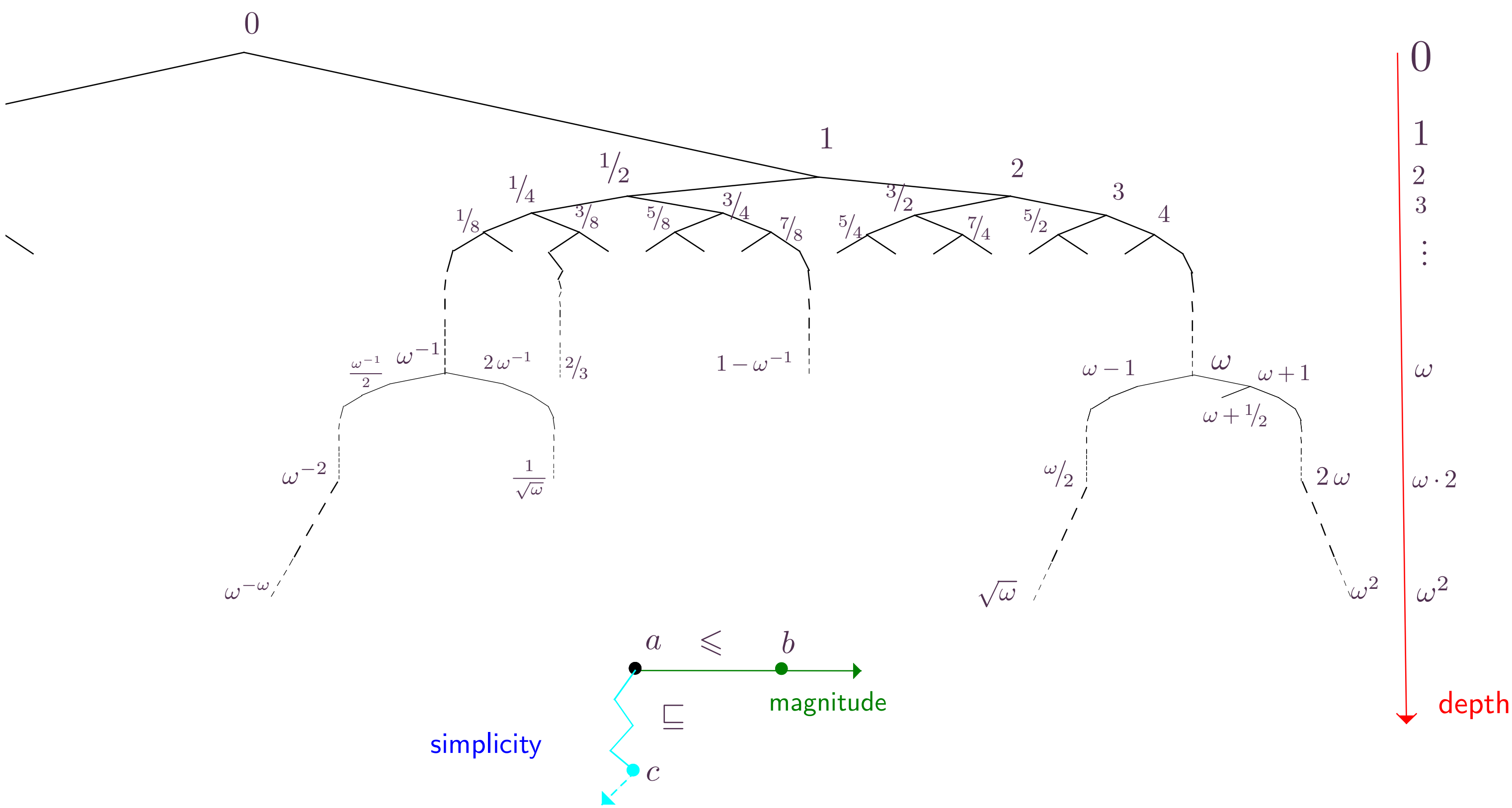

Simplicity: a is **simpler** than b, written $a \sqsubseteq b$, if there is a path from a to b in the tree.

Inductive definitions on $\mathbf{No}$

Fundamental property of $(\mathbf{No}, \leqslant, \sqsubseteq)$

For all sets of numbers L, R with $L < R$, there is a unique $\sqsubseteq$-minimal number $\{L \mid R\}$ with

$$L < \{L \mid R\} < R.$$

In particular $\mathbf{No}$ is not a set (mind the Burali-Forti like paradox $\{\mathbf{No} \mid \varnothing\} > \{\mathbf{No} \mid \varnothing\}$!)

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Well-founded order

For $a \in \mathbf{No}$, we have two sets $a_L := \{b \in \mathbf{No} : b < a, b \sqsubseteq a\}$ and $a_R := \{b \in \mathbf{No} : b > a, b \sqsubseteq a\}$.

So $a = \{a_L \mid a_R\}$. The partial order $(\mathbf{No}, \sqsubseteq)$ is well-founded $\longrightarrow$ inductive definitions.

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Surreal arithmetic [Conway - 1976]

Inductive definition of the sum $a + b$ of numbers a, b:

Induction hypothesis: $a_L + b$, $a + b_L$ *and* $a + b_R$, $a_R + b$ *are defined.*

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By definition, $\underset{<a}{a_L} + b,\ a + \underset{<b}{b_L}$ *and* $a + \underset{>b}{b_R},\ \underset{>a}{a_R} + b$

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We want $\underbrace{a_L + b}_{< a+b}, \underbrace{a + b_L}_{< a+b}$ *and* $\underbrace{a + b_R}_{> a+b}, \underbrace{a_R + b}_{> a+b}$.

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Similar equations exist for $-a, a\,b, {}^{a}/_{b}$. $(\mathbf{No}, +, \times)$ has the same first-order properties as $\mathbb{R}$.

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Strength: $\mathbf{No}$ is spherically complete $\rightarrow$ numbers can be represented as transfinite sums
$\rightarrow$ certain transfinite sums exist on $\mathbf{No}$

Examples

Each ordinal $\alpha \in \mathbf{On}$ is inductively identified with $\{\beta : \beta < \alpha \mid \varnothing\}$, so

$$0 = \{\varnothing \mid \varnothing\} \qquad 1 = \{0 \mid \varnothing\} \qquad 2 = \{0, 1 \mid \varnothing\} \qquad \omega = \{\mathbb{N} \mid \varnothing\}.$$

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$$1 + 1 \;=\; \{0 \mid \varnothing\} + \{0 \mid \varnothing\}$$

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We beat *Principia Mathematica* ! Since $\mathbf{No}$ is a strong real-closed field, we have exotic numbers

$$\varepsilon_0 + \frac{\omega^\omega}{\pi\,(\omega^{3/2} - 1)^2 - 1} + 2\,\omega^{\frac{1}{2}} + 3\,\omega^{\frac{1}{3}} + \cdots \qquad ..$$

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$$x \longmapsto \exp_\omega(x) + \frac{\exp(x)}{\pi\,(x^{3/2} - 1)^2 - 1} + 2\,x^{\frac{1}{2}} + 3\,x^{\frac{1}{3}} + \cdot$$

Omnific integers

Conway defined the strong subring $\mathbf{Oz}$ *of* **omnific integers**. *We have*

$$\mathbf{Oz} = \{x \in \mathbf{No} : \ x = \{x - 1 \mid x + 1\} \ \}.$$

The additive group $\langle \mathbf{On} \rangle$ *generated by ordinals is contained in* $\mathbf{Oz}$, *and* $\mathbf{Oz}$ *is an* integer part*: for all* $x \in \mathbf{No}$, *there is a unique* $z \in \mathbf{Oz}$ *with* $z \leqslant x < z + 1$. *In fact* $\mathbf{No} = \operatorname{Frac}(\mathbf{Oz})$.

An integer part

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Some open questions.

- Factorization theorem in $\mathbf{Oz}$? (partial results in $\mathbb{R}[[\omega^{\mathbb{R}^{>0}}]] + \mathbb{Z}$ by **Mantova**)
- Can one characterize prime numbers in $\mathbf{Oz}$?
- Given $z \in \mathbf{Oz}$, how does the order type $\pi(z) \in \mathbf{On}$ of $\{y : y \sqsubseteq z \text{ and } y \text{ is prime}\}$ behave?
- Can we define a subring $\mathbf{Z}$ of $\mathbf{No}$ such that $(\mathbf{No}, \mathbf{Z})$ and $(\mathbb{R}, \mathbb{Z})$ have the same properties?

  \- Gödel asks: how would we even know?

G. H. Hardy's dream

Imagine there existed a scale containing "regular" and "simple" functions, building blocks, such that any regular function can be asymptotically (at $+\infty$) described by combining those blocks.

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Real-analytic functions: monomials x^n as building blocks.

$$
\begin{aligned}
1 + x^{-1} + x^{-2} + \cdots &\equiv \frac{x}{x-1} \\
&\equiv \exp \\
\sum_{n \in \mathbb{N}} n!\, x^{-n} &\equiv \mathrm{e}^{\frac{1}{x}}
\end{aligned}
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Transseriable functions: $\exp$ and $\log$ as building blocks.

$$\begin{aligned} 1 + x^{-1} + x^{-2} + \cdots + e^{\frac{1}{x^2}} &\equiv \frac{x}{x-1} + e^{\frac{1}{x^2}} \\ \log x + \log_2 x + \log_3 x + \cdots &\equiv ? \end{aligned}$$

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Ecalle's growth scale, Grevey $\alpha<\omega^\omega$ functions: e.g. very slowly growing functions L with

$$\log L' \approx \log x+\log_2 x+\log_3 x+\cdots$$

But $L+\delta$ for $\delta=O(1)$ also satisfies this. How to single out "simple" building blocks?

Hardy's dream surrealized

id
$\mathrm{id}+1$
$\mathrm{id}+2$
$\vdots$
$2\,\mathrm{id}$
$2\,\mathrm{id}+1$
$\vdots$
$3\,\mathrm{id}$
$\vdots$
id^2
$\vdots$
$\exp$
$\exp\circ\exp$
$\vdots$

A simple scale.

id
$\mathrm{id}+1$
$\vdots$
$2\,\mathrm{id}$
$\vdots$
id^2
$\vdots$
$\exp$

(discarding iterates)

Hardy's dream surrealized

id
$\vdots$
$\mathrm{id} + \frac{1}{\mathrm{id}}$
$\vdots$
$\mathrm{id} + \frac{\log}{\mathrm{id}}$
$\vdots$
$\mathrm{id} + 1$
$\vdots$
$2\,\mathrm{id}$
$\vdots$
id^2
$\vdots$
$\exp \circ \log^2$
$\vdots$
$\exp_2 \circ (\log_2^2)$
$\vdots$
$\exp$

(adding intermediate asymptotics)

Hardy's dream surrealized

id	
$\vdots$	
$\mathrm{id} + \frac{1}{\mathrm{id}}$	
$\vdots$	L
$\mathrm{id} + 1$	
$\vdots$	
$2\,\mathrm{id}$	
$3\,\mathrm{id}$	
$\vdots$	
$?$	$\cdots$ $\cdots$
$\vdots$	
$\mathrm{id}^{3/2}$	
id^2	
$\vdots$	
$\exp \circ (\log^2)$	R
$\vdots$	
$\exp$	

Looking for a building block between L and R.

ω
$\vdots$
$\omega + \omega^{-1}$
$\vdots$
$\omega + 1$
$\vdots$
$2\,\omega$
$3\,\omega$
$\vdots$
$\left\{ \mathbb{N}\,\omega \mid \omega^{1+\frac{1}{\mathbb{N}+1}} \right\}$
$\vdots$
$\omega^{3/2}$
ω^2
$\vdots$
$\omega^{\omega^{2/\omega}}$
$\vdots$
ω^{ω}

A portion of the surreal scale containing $\{L \mid R\}$.

Hardy's dream surrealized

ω
$\vdots$
$\omega+\omega^{-1}$
$\vdots$
$\omega+1$
$\vdots$
$2\,\omega$
$\vdots$
$\omega^{1+\omega^{-1}}$
$\vdots$
ω^2
$\vdots$
$\omega^{\omega^{2/\omega}}$
$\vdots$
ω^{ω}

…but those are no longer real-valued functions.

Surreal-valued ones then? What kind?

A **field of real functions** is a *field* $\mathcal{H}$ of germs at $+\infty$ of real-valued functions, closed under derivation and composition. **E.g.** rational functions (not meromorphic functions: they may have zeroes at $+\infty$), or germs of functions definable using $+$, $\times$ and $\exp$ and $\log$.

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For $f \in \mathcal{H}$, we must have $f \neq 0$ at $+\infty$, hence by continuity $f > 0$ at $+\infty$ or $f < 0$ at $+\infty$. So $\mathcal{H}$ is an ordered field. Moreover f is either constant or strictly monotonous (since $f' \in \mathcal{H}$).

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As a differential ordered field

$\mathcal{H}$ is in an ***$\boldsymbol{H}$-field with small derivation*** *as per Aschenbrenner and van den Dries.*

This means that we have $\partial(f) > 0$ whenever $f > \mathbb{R}$ and $\lim \partial(f) = 0$ if $\lim f = 0$.

With the compositional structure $\circ\colon \mathcal{H} \times \mathcal{H}^{>\mathbb{R}} \longrightarrow \mathcal{H}$ (besides associativity)

- *Each* $f \longmapsto f \circ g\colon \mathcal{H} \longrightarrow \mathcal{H}$ *for a fixed* $g \in \mathcal{H}^{>\mathbb{R}}$ *is an endomorphism of ordered rings.*
- *Each* $g \longmapsto f \circ g\colon \mathcal{H}^{>\mathbb{R}} \longrightarrow \mathcal{H}$ *for a fixed* $f \in \mathcal{H}$ *is monotonous.*
- *Each* $f \in \mathcal{H}$ *has Taylor expansions around each* $g \in \mathcal{H}^{>\mathbb{R}}$*: when* $\delta \prec \frac{f \circ g}{f' \circ g}$*, we have*

$$f \circ (g + \delta) - \sum_{k=0}^{n} \frac{f^{(k)} \circ g}{k!} \delta^k \prec (f^{(n+1)} \circ g)\, \delta^{n+1}, \qquad n \in \mathbb{N}.$$

Definition

A **field of surreal functions** *(henceforth FSF) is a strong subfield $\mathcal{F} \subseteq \mathbf{No}$ together with a strong derivation $\partial \colon \mathcal{F} \to \mathcal{F}$ such that $(\mathcal{F}, \partial)$ is an H-field and an associative map $\circ \colon \mathcal{F} \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$ with $\mathcal{F} \circ \mathcal{F}^{>\mathbb{R}} \subseteq \mathcal{F}$ such that*

- *Each $\mathcal{F} \longrightarrow \mathbf{No}; f \mapsto f \circ x$ for fixed $x \in \mathbf{No}^{>\mathbb{R}}$ is a strong morphism of ordered rings.*
- *Each $\mathbf{No}^{>\mathbb{R}} \to \mathbf{No}; x \mapsto f \circ x$ for fixed $f \in \mathcal{F}^{>\mathbb{R}}$ is strictly increasing.*
- *For all $f \in \mathcal{F}$, $x, \delta \in \mathbf{No}$ with $x > \mathbb{R}$, $\delta \prec x$ and $\delta \prec \frac{\mathfrak{m} \circ x}{\partial(\mathfrak{m}) \circ x}$ for all $\mathfrak{m} \in \operatorname{supp} f$, we have*

$$f \circ (x + \delta) = \sum_{k \in \mathbb{N}} \frac{\partial^k(f) \circ x}{k!} \delta^k.$$

Closure under functional equations

Other interesting properties which are *not* imposed:

Neutral element: An $\mathrm{id} \in \mathcal{F}^{>\mathbb{R}}$ with $$\begin{aligned} \mathrm{id} \circ x &= x \ \forall x \in \mathbf{No}^{>\mathbb{R}} \\ f \circ \mathrm{id} &= f \ \forall f \in \mathcal{F}. \end{aligned}$$
Inverses: For each $f \in \mathcal{F}^{>\mathbb{R}}$, an $f^{\mathrm{inv}} \in \mathcal{F}^{>\mathbb{R}}$ with $$f \circ f^{\mathrm{inv}} = f^{\mathrm{inv}} \circ f = \mathrm{id}.$$
Conjugates: For all $f, g > \mathrm{id}$, an $h \in \mathcal{F}^{>\mathbb{R}}$ with $$h \circ f = g \circ h.$$

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We'll consider FSFs $\mathcal{F}$ such that for each $f \in \mathcal{F}$, all strictly simpler numbers $a \sqsubset f$ lie in $\mathcal{F}$. In other words $\mathcal{F}$ will be intial (= downward closed) in $(\mathbf{No}, \sqsubseteq)$. Let us start:

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Cuts in real numbers $\{L \mid R\}$ where $L, R \subseteq \mathbb{R}$ always lie in $\mathbb{R} \cup (\mathbb{R} \pm \omega^{\pm 1})$, so we first have to deal with the number $\omega = \{\mathbb{N} \mid \varnothing\}$ as a function.

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Since ω is the simplest number with $\omega > n$ for all $n \in \mathbb{N}$, we may ask ourselves:

What is the simplest function $f > n$ for all $n \in \mathbb{N}$?

Laurent series

ω as the identity function

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$$\omega \circ x \sqsupseteq \{\mathbb{N}, \omega \circ (x_L \cap \mathbf{No}^{>\mathbb{R}}) \mid \omega \circ x_R\} = \{\mathbb{N}, x_L \cap \mathbf{No}^{>\mathbb{R}} \mid x_R\} = \{x_L \mid x_R\} = x.$$

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$\longrightarrow$ each Laurent series $a = \sum_{n \in \mathbb{Z}} a_n\,\omega^n \in \mathbb{R}[[\omega^{\mathbb{Z}}]]$ in ω should act as $x \mapsto \sum_{n \in \mathbb{Z}} a_n\,x^n$.

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Now what about the number $\omega^\omega = \{\omega^n : n \in \mathbb{N} \mid \varnothing\}$?

What is the simplest non polynomially bounded function?

Exponential [Gonshor - 1986]

For $a \in \mathbf{No}$ *and* $n \in \mathbb{N}$, *set* $[a]_n := \sum_{k \leqslant n} \frac{a^k}{k!}$. *The inductive equation*

$$\exp(a) := \left\{ \exp(a_L)\,[a - a_L]_{\mathbb{N}}\,,\ \exp(a_R)\,[a - a_R]_{2\mathbb{N}+1} \;\middle|\; \frac{\exp(a_R)}{[a_R - a]_{\mathbb{N}}}, \frac{\exp(a_L)}{[a_L - a]_{2\mathbb{N}+1}} \right\}$$

defines an isomorphism $(\mathbf{No}, +, <) \rightarrow (\mathbf{No}^{>0}, \times, <)$.

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We should have $\partial(\omega^{\omega}) > \partial(\omega^{\mathbb{N}}) = \mathbb{N}\,\omega^{\mathbb{N}-1}$ by H-fields axioms. So $\{\mathbb{N}\,\omega^{\mathbb{N}-1} \mid \varnothing\} = \omega^{\omega}$ is the simplest value for $\partial(\omega^{\omega})$. It follows that $\exp$ is the simplest function for ω^{ω}.

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Theorem [Berarducci and Mantova - 2019, Schmeling - 2001]

The function $\circ : \mathbb{R}[[\omega^{\mathbb{Z}}]] \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *extends uniquely as a function* $\circ : \mathbb{T} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *which is compatible with* $\exp$, $\log$ *and transfinite sums.*

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We have $\exp_n(\omega) = \omega^{\cdot^{\cdot^{\omega}}}$ (n times) for all $n \in \mathbb{N}$. So what about $\varepsilon_0 = \{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \mid \varnothing\}$?

What is the simplest transexponential function on $\mathbf{No}$?

Further properties of a FSF with $\exp(\omega), \log(\omega) \in \mathcal{F}$

a) If $f > \exp_{\mathbb{N}}$, then $\forall n, f\,(\log \omega) \circ f \cdots (\log_n \omega) \circ f < \partial(f) < f\,(\log \omega) \circ f \cdots (\log_n \omega)^2 \circ f$.

b) If $f > \mathrm{id} + \mathbb{R}$, then $f \circ (\mathrm{id} + r) > f + \mathbb{R}$ *for all* $r \in \mathbb{R}^{>0}$,

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Value of $\partial(\varepsilon_0)$: By a), the simplest value for $\partial(\varepsilon_0)$ is $\prod_{n \in \mathbb{N}} \log_n(\varepsilon_0) = \mathrm{e}^{\sum_{n \in \mathbb{N}} \log_{n+1}(\varepsilon_0)}$.

Value of $\varepsilon_0 \circ (\omega + 1)$: We have $\log_{\mathbb{N}}(\varepsilon_0) > \omega + \mathbb{R}$. So b) yields $\varepsilon_0 \circ (\omega + 1) > \exp_n(\log_n(\varepsilon_0) + 1)$ for all $n \in \mathbb{N}$. But $\{\exp_n(\log_n(\varepsilon_0) + 1) : n \in \mathbb{N} \mid \varnothing\} = \exp(\varepsilon_0)$. Thus ε_0 should satisfy

$$\varepsilon_0 \circ (x + 1) = \exp(\varepsilon_0 \circ x) \qquad \forall x \in \mathbf{No}^{>\mathbb{R}} \tag{2}$$

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Theorem [B. van der Hoeven, Mantova - 2020]

There is a simplest strictly increasing solution $\exp_\omega : \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}^{>,\succ}$ *of (3) with*

$$\exp_\omega(x + \delta) = \sum_{k \in \mathbb{N}} \frac{\exp_\omega^{(k)}(x)}{k!} \delta^k$$

for all $x \in \mathbf{No}^{>\mathbb{R}}$ *and* $\delta \prec \frac{\exp_\omega(x)}{\prod_{n \in \mathbb{N}} \log_n(\exp_\omega(x))}$.

Theorem [Kneser - 1949]

There is a strictly increasing analytic map $E_\omega \colon [a, +\infty) \longrightarrow \mathbb{R}$ *which solves* **Abel's equation***:*

$$\forall r > a, E_\omega(r+1) = \exp(E_\omega(r)). \quad (4)$$

We have $E_\omega(r) > \exp_n(r)$ for $r \gg 1$, for all $n \in \mathbb{N}$. We call E_ω a **hyperexponential** function.

Real hyperexponentiation

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Ecalle and Schmeling (among others) studied this type of functions, along with other very fast growing / slowly growing functions $E_{\omega^\mu}, L_{\omega^\mu}$ such that for $r \gg 1$, we have

$$\begin{aligned} E_{\omega^{\mu+1}}(r+1) &= E_{\omega^\mu}(E_{\omega^{\mu+1}}(r)), \\ L_{\omega^{\mu+1}}(L_{\omega^\mu}(r)) &= L_{\omega^{\mu+1}}(r) - 1, \\ L_{\omega^\mu}(E_{\omega^\mu}(r)) = E_{\omega^\mu}(L_{\omega^\mu}(r)) &= r. \end{aligned}$$

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On $\mathbb{R}$, one is limited to small values of μ (e.g. finite). But on $\mathbf{No}$, we can take any $\mu \in \mathbf{On}$!

Logarithmic hyperseries [van den Dries, van der Hoeven, Kaplan - 2018]

There is a strong subfield $\mathbb{L} \subseteq \mathbf{No}$ *of* **logarithmic hyperseries**, *built upon transfinite products*

$$\mathfrak{l} \equiv \prod_{\gamma < \mu} \ell_\gamma^{\mathfrak{l}_\gamma} \qquad (\text{formally } \mathfrak{l} \text{ is a map } \mu \longrightarrow \mathbb{R}; \gamma \longmapsto \mathfrak{l}_\gamma),$$

equipped with a strong derivation $\partial: \mathbb{L} \longrightarrow \mathbb{L}$ *and a composition law* $\circ: \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ *with:*

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Each function $f \longmapsto \ell_{\omega^\mu} \circ f$ behaves similarly to the real function L_{ω^μ} (e.g. for $\mu \in \mathbb{N}$). For all ordinals γ, μ, we have generative identities:

$$\partial(\ell_\gamma) = \prod_{\rho < \gamma} \ell_\rho^{-1} \qquad \partial(\ell_\omega) = \prod_{n \in \mathbb{N}} \ell_n^{-1}$$

$$\ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} - 1,$$

$$\ell_{\omega^{\mu_p} m_p + \cdots + \omega^{\mu_0} m_0} = \ell_{\omega^{\mu_0}}^{\circ m_0} \circ \cdots \circ \ell_{\omega^{\mu_p}}^{\circ m_p}.$$

Definition

A **hyperserial field** *is a strong field* $\mathbb{T}$ *together with an action* $\circ \colon \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$ *of* $\mathbb{L}$ *on* $\mathbb{T}$ *by monotonous, Taylor expandable functions (with a few axiomatic properties).*

In particular $\mathbb{L}$ is a hyperserial field. Say that $\mathbb{T}$ is *closed* if for each $\mu \in \mathbf{On}$, the function $\mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}^{>\mathbb{R}}; s \mapsto \ell_{\omega^\mu} \circ s$ is bijective.

Hyperserial fields

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Theorem [B., van der Hoeven, Kaplan - 2021]

There is a closed extension $\iota\colon \mathbb{T} \longrightarrow \tilde{\mathbb{T}}$ *which is initial among closed extensions of* $\mathbb{T}$*.*

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Hyperserial fields

Definition

A **hyperserial field** *is a strong field* $\mathbb{T}$ *together with an action* $\circ\colon \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$ *of* $\mathbb{L}$ *on* $\mathbb{T}$ *by monotonous, Taylor expandable functions (with a few axiomatic properties).*

In particular $\mathbb{L}$ is a hyperserial field. Say that $\mathbb{T}$ is *closed* if for each $\mu \in \mathbf{On}$, the function $\mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}^{>\mathbb{R}}; s \mapsto \ell_{\omega^\mu} \circ s$ is bijective.

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Work in progress [B.]

The composition $\mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$ *extends uniquely into a composition* $\tilde{\mathbb{L}} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{T}}$*.*

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Work in progress [B.]

$(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, <)$ *is a linearly bi-ordered group in which any two positive elements are conjugate.*

Back to $\mathbf{No}$

Theorem [B., van der Hoeven - 2021]

There is a composition law $\circ\colon \mathbb{L} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *for which* $(\mathbf{No}, \circ)$ *is a closed hyperserial field.*

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The extension $\circ\colon \tilde{\mathbb{L}} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ yields a FSF containing such functions as

$\exp_{\omega^\mu}, \log_{\omega^\mu} \quad \mu \in \mathbf{On}$: "transfinite iterates" of $\exp$ and $\log$

$\exp_r := \exp_\omega \circ (\log_\omega + r), r \in \mathbb{R}$: fractional iterates of $\exp$, e.g. $\exp_{1/2}$ is the *unique* solution of $\phi \circ \phi = \exp$, and $\{\exp_r : r \in \mathbb{R}\}$ is the set of solutions of $\phi \circ \exp = \exp \circ \phi$.

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Theorem [B., van der Hoeven - 2021]

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$$\begin{array}{rcl} \exp_{\omega^\mu}, \log_{\omega^\mu} \qquad \mu \in \mathbf{On} & : & \text{“transfinite iterates” of } \exp \text{ and } \log \\ \exp_r := \exp_\omega \circ (\log_\omega + r), r \in \mathbb{R} & : & \text{fractional iterates of } \exp, \text{ e.g. } \exp_{1/2} \text{ is the} \\ & & \textit{unique} \text{ solution of } \phi \circ \phi = \exp, \text{ and } \{\exp_r : r \in \mathbb{R}\} \\ & & \text{is the set of solutions of } \phi \circ \exp = \exp \circ \phi. \end{array}$$

We have $\tilde{\mathbb{L}} \subsetneq \mathbf{No}$. Indeed **[B., van der Hoeven - 2019]**, there are numbers $a \in \mathbf{No}$ with

$$a = \sqrt{\omega} + \mathrm{e}^{\sqrt{\log \omega} + \mathrm{e}^{\sqrt{\log_2 \omega} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}} \notin \tilde{\mathbb{L}}.$$

Those are good candidate solutions to the functional equation

$$\phi = \sqrt{\omega} + \exp(\phi \circ \log \omega).$$

Thank you!