# Hardy fields

This topic would focus on Hardy fields. The main reference would be Maxwell Rosenlicht's 1983 article *Hardy fields* [6], in the beginning of Section 1. Here is a proposition for what you could do.

The Question 1 I added is an intermediate question that should be answered.

## 1 Program

Here is a list of things to do. If you explain the introduction of [6], give a few examples of Hardy fields and prove Theorem 1, then this already qualifies for a talk in the seminar, and for a Bachelorarbeit.

### 1.1 Definition and basic questions

Consider the set

$$\mathcal{C}^{<\infty} := \bigcap_{k \in \mathbb{N}} \bigcup_{a \in \mathbb{R}} \mathcal{C}^k((a, +\infty), \mathbb{R})^{1}$$

of so-called (by me) sufficiently differentiable real-valued functions at $+\infty$. We have an equivalence relation

$$f \equiv g \Longleftrightarrow \exists a \in \mathbb{R} \, (\forall t \in \mathbb{R} \, (t > a \Longrightarrow f(t) = g(t))).$$

In the sequel, I simply write "for $t \gg 1$" to mean "for all large enough $t$".

The quotient set $\mathcal{G}^{<\infty} = \mathcal{C}^{<\infty} / \equiv$ is a ring under pointwise sum and product, whose elements are called germs (at $+\infty$). The germ of $f \in \mathcal{C}^{<\infty}$ is written $[f]$. In fact $\mathcal{G}$ is also a differential ring. That is, we have a derivation operation $' \colon \mathcal{G} \longrightarrow \mathcal{G}$ where for $f \colon \mathcal{C}^1((a, +\infty), \mathbb{R})$ in $\mathcal{C}^1$, we set $[f]' := [f']$.

**Question 1.** Why is the derivation $' \colon \mathcal{G} \longrightarrow \mathcal{G}$ a well-defined on $\mathcal{G}$?

Finally, the set $\mathcal{G}$ is partially ordered by *asymptotic comparison*, i.e. for the strict ordering

$$[f] < [g] \qquad \text{if} \qquad f(t) < g(t) \text{ for all } t \gg 1.$$

In the sequel, I no longer write $[f]$ for the germ fo $f$, but rather $f$. Feel free to always use the notatio $[f]$ if you want things to be perfectly clear.

A *Hardy field* is a subfield of $(\mathcal{G}, +, \times)$ which is closed under derivation (a subfield $\mathcal{H} \subseteq \mathcal{G}^{<\infty}$ with $f' \in \mathcal{H}$ whenever $f \in \mathcal{H}$).

The name Hardy field was introduced by Bourbaki [3], but you don't need to check this reference.

### 1.2 Unpacking the introduction

Rosenlicht's introduction is a nicely written, but very compact summary of the important properties of Hardy fields, along with their definition. There is much to unpack, and it would be a good thing to spend some time doing it, justifying his assertions. So your first goal is to explain what Rosenlicht writes in the first two paragarphs of [6]. You should also give simple examples of Hardy fields.

### 1.3 Real-closure

The main results of [6, Section 1], once reformulated using a characterization of real-closed fields, is

**Theorem 1.** [6, Corollary] *Let $\mathcal{H}$ be a Hardy field. Define $\mathcal{H}^\circ$ as the set of germs $f \in \mathcal{G}^{<\infty}$ for which there is a non-zero polynomial $P \in \mathcal{H}[y]$ with $P(f(t)) = 0$ for all $t \gg 1$. Then $\mathcal{H}^\circ$ is a real-closed Hardy field.*

1. $\mathcal{C}^k((a, +\infty), \mathbb{R})$ is the class of $k$-times continuously differentiable maps $(a, +\infty) \longrightarrow \mathbb{R}$.

Since by defintion $\mathcal{H}^\circ$ is an algebraic extension of $\mathcal{H}$, it follows that up to unique isomorphism over $\mathcal{H}$, the extension $\mathcal{H}^\circ/\mathcal{H}$ is the real closure of $\mathcal{H}$.

In order to prove Theorem 1, Rosenlicht first proves the following:

**Theorem.** [6, Theorem 1] *Let $\mathcal{H}$ be a Hardy field, let $f \in \mathcal{C}^{<\infty}$ and let $P \in \mathcal{H}[X]$ be a non-zero polynomial with $P(f)=0$ as a germ (i.e. $P(f(t))=0$ for all $t \gg 1$). Then the ring $\mathcal{H}[f]$ is a Hardy field.*

**Remark 2.** The ring $\mathcal{G}^{<\infty}$ is *not* a domain, so one cannot deduce direcly from the fact that $f$ is algebraic over $\mathcal{H}$ that $\mathcal{H}[f]$ is a field.

## 2 Further references

For an interesting discussion on Hausdorff fields, Hardy fields, in relations to problems in asymptotic differential algebra, read the beginning (p. 1–6) of Aschenbrenner, van den Dries and van der Hoeven *On numbers, germs and transseries* [1]. This is a survey article (no proofs, only a collection and explanation of results), so it is an easy read.

You can also read the introduction to Hardy's article *Orders of infinity, the 'Infinitärcalcül' of Paul du Bois-Reymond* [4]. This is a very interesting read regarding the concrete questions from which the study of Hardy fields (or the comparability of germs in general) originated.

If you are interested in the connection between Hardy fields and logic, you can also look at the first three chapters of Chris Miller's article *Basics of O-minimality and Hardy Fields* [5]. This will tie in with the o-minimality part of the Fachseminar.

## Bibliography

**[1]** M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. On numbers, germs, and transseries. In *Proc. Int. Cong. of Math. 2018*, volume 1, pages 1–24. Rio de Janeiro, 2018.

**[2]** M. Boshernitzan. An extension of Hardy's class L of "orders of infinity". *Journal d'Analyse Mathématique*, 39:235–255, 1981.

**[3]** N. Bourbaki. *Fonctions d'une variable réelle: Théorie élémentaire*. Eléments de mathématique. Springer Berlin Heidelberg, 2007.

**[4]** G. H. Hardy. *Orders of infinity, the 'Infinitärcalcül' of Paul du Bois-Reymond*. Cambridge University Press edition, 1910.

**[5]** C. Miller. *Basics of o-minimal structures and real analytic geometry*, volume 62, chapter 2, pages 43–69. Springer New York, New York, 2012.

**[6]** M. Rosenlicht. Hardy fields. *Journal of Mathematical Analysis and Applications*, 93(2):297–311, 1983.