Hausdorff fields

This topic would focus on Hausdorff fields (or B-fields as per the main reference). Those fields are similar to Hardy fields, except that there is no derivation involved. The main reference would be Michael Boshernitzan's 1981 article *An extension of Hardy's field of "orders of infinity"* [2], in Sections 2–3 or possibly 2–3 + the beginning of Section 4 (namely Theorem 4.1) if you like.

I have to say Boshernitzan's proofs (which precede that of Rosenlicht in [4]) are a bit more complicated than Rosenlicht's ones. But they are quite direct.

1 Program

Here is a proposition for what you could do, along with a brief explanation of what Hausdorff fields are. If you prove Theorems 2 and 3, then this qualifies for a talk in the seminar.

The two questions I added are questions that you can take time for yourself to answer. They are not difficult, but their answers are not contained in [2] or [4].

1.1 Definition and basic questions

Consider the set

$$\mathcal{C} := \bigcup_{a \in \mathbb{R}} \mathcal{C}^0((a, +\infty), \mathbb{R})$$

of continuous real-valued functions defined on neighborhoods of $+\infty$. We have an equivalence relation

$$f \equiv g \Longleftrightarrow \exists a \in \mathbb{R}\, (\forall t \in \mathbb{R}\, (t > a \Longrightarrow f(t) = g(t))).$$

In the sequel, I simply write "for $t \gg 1$" to mean "for all large enough t".

The quotient set $\mathcal{G} = \mathcal{C} / \equiv$ is a ring under pointwise sum and product, whose elements are called germs (at $+\infty$). It is partially ordered by *asymptotic comparison*, i.e. for the strict ordering

$$[f] < [g] \qquad \text{if} \qquad f(t) < g(t) \text{ for all } t \gg 1.$$

A Hausdorff field is a subfield of $(\mathcal{G}, +, \times)$.

Hausdorff fields are called *B-fields* in [2, p. 238, l.12]. A minor detail is that Boshernitzan imposes that B-fields contain the field of real numbers (each real number being identified with the germ of the corresponding constant function). A first important fact on Hausdorff fields is the following:

Proposition 1. [2, Proposition 2.2] *Any Hausdorff field is linearly ordered by asymptotic comparison.*

It would be nice to give simple examples of Hausdorff fields, and in particular of Hausdorff fields which are not Hardy fields.

Question 1. In particular, can you find a germ $f \in \mathcal{G}$ which is not monotonous (on any interval $(a, +\infty)$ for $a \in \mathbb{R}$), but for which $\mathbb{R}(f)$ is a Hausdorff field?

If you can answer Question 1, then you know that contrary to germs in Hardy fields, germs in Hausdorff fields may not be monotonous. So Rosenlicht's argument for the existence of a limit (in $\mathbb{R} \cup \{-\infty, +\infty\}$) for each germ in a Hardy field doesn't apply here.

Question 2. If $f \in \mathcal{G}$ lies in a Hausdorff field, that is, if $\mathbb{R}(f)$ is a Hausdorff field (see [2, Proposition 2.1]), then must f have a limit at $+\infty$ in $\mathbb{R} \cup \{-\infty, +\infty\}$?

1.2 Real-closure

The main results of Boshernitzan regarding these fields, once stated in the language of Hausdorff fields, are as follow:

Theorem 2. [2, Proposition 3.4] *Let $\mathcal{H}$ be a Hausdorff field. Define $\mathcal{H}^\circ$ as the set of germs $f \in \mathcal{G}$ for which there is a non-zero polynomial $P \in \mathcal{H}[y]$ with $P(f(t)) = 0$ for all $t \gg 1$. Then $\mathcal{H}^\circ$ is a Hausdorff field.*

In fact, this was proved earlier by Hausdorff, but Boshernitzan's proof is easier to follow. Secondly, we have:

Theorem 3. [2, Proposition 3.6] *Under the same conditions as in Theorem 2, the field $\mathcal{H}^\circ$ is real-closed.*

Since by defintion $\mathcal{H}^\circ$ is an algebraic extension of $\mathcal{H}$, it follows that up to unique isomorphism over $\mathcal{H}$, the extension $\mathcal{H}^\circ/\mathcal{H}$ is the real closure of $\mathcal{H}$.

If you want to, and if you have the time to do it, you can also prove the following result. This one is interesting because it does not work for Hardy fields.

Theorem 4. [2, Theorem 4.1] *Let $\mathcal{H}$ be a Hausdorff field and let $f \in \mathcal{G}$. Assume that for all $g \in \mathcal{H}^\circ$, we either have $f > g$ or $f < g$ or $f = g$. Then $\mathcal{H}(f)$ is a Hausdorff field.*

2 Further references

Hausdorff fields were introduced by Felix Hausdorff in [3] (which I can't really read, but you can).

For an interesting discussion on Hausdorff fields, Hardy fields, in relations to problems in asymptotic differential algebra, read the beginning (p. 1–6) of Aschenbrenner, van den Dries and van der Hoeven *On numbers, germs and transseries* [1]. This is a survey article (no proofs, only a collection and explanation of results), so it is an easy read.

Bibliography

[1] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. On numbers, germs, and transseries. In *Proc. Int. Cong. of Math. 2018*, volume 1, pages 1–24. Rio de Janeiro, 2018.

[2] M. Boshernitzan. An extension of Hardy's class L of "orders of infinity". *Journal d'Analyse Mathématique*, 39:235–255, 1981.

[3] F. Hausdorff. Untersuchungen über Ordnungstypen, I, II, III. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, Mathematisch-Physikalische Klasse 58:103–169, 1906.

[4] M. Rosenlicht. Hardy fields. *Journal of Mathematical Analysis and Applications*, 93(2):297–311, 1983.